MOCK TEST

FOR

ISI 2006

Exam on 7th May 2006

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The Indian Statistical Institute (ISI), Kolkata, is considered as one of the foremost centres in the world for training and research in statistics and the related sciences. The B.Stat (Hons) degree program, the flagship programme of the institute, offers comprehensive instruction in the theory, method and application of statistics, in addition to several areas of Mathematics and some basic areas of computer science.

Each candidate applying for admission to this programme has to take a selection test comprising Objective type and Short-answer type questions in mathematics at the Higher Secondary level (10 + 2 year's programme).

The selection tests consists of

(1) A multiple choice type test having about 30 questions, and

(2) A short-answer type test having about 10 questions.

Questions will be set on the following and related topics.

Algebra : Sets, operations on sets, prime numbers, factorization of integers and divisibility, rational and irrational numbers, permutations and combinations. Binomial theorem, logarithms, theory of quadratic equations, polynomial and remainder theorem, arithmetic and geometric progressions, inequalities involving A.M., G.M., and H.M., complex numbers.

Geometry : Plane geometry of class X level. Geometry of 2 dimensions with cartesian and polar co-ordinates. Concept of a locus, equation of a line, angle between two lines, distance from a point to a line. Areas of a triangle, equations of a circle, parabola, ellipse and hyperbola and equations of their tangents and normals, mensuration.

Trigonometry : Measures of angles, trigonometric and inverse trigonometric functions, trigonometric identities including addition formulae, solutions of trigonometric equations. Properties of triangles, heights and distances.

Calculus : Functions, one-one functions, onto functions, limits and continuity, derivatives and methods of differentiation, slope and curve, tangents and normals, maxima and minima, use of calculus in sketching graph of functions, methods of integration, definite and indefinite integrals, evaluation of area using integrals.

Logical Reasoning : Consistency of statements.

In response to growing demand from students preparing for the ISI, we bring to you the first Mock ISI paper, which closely simulates the real exam. There is more to follow in the coming months.

MULTIPLE CHOICE TEST

1. Let x, y, z be non-zero real numbers. The set of all possible values of the expression

$$\frac{|x+y|}{|x|+|y|}+\frac{|y+z|}{|y|+|z|}+\frac{|z+x|}{|z|+|x|} \text{ is}$$

(a) [0.5] (b) [3,4]

(c) [1, 3] (d) [−1, 2]

2. The number $N = 12321$ is a perfect square, viz. $(111)^2$, in base 10. In what other bases is it a perfect square?

(a) exactly two bases $b \geq 7$

(b) exactly three bases $b \geq 4$

(c) for no base b other than 10

(d) any base $b \geq 4$.

3. Let $\alpha = \sqrt[3]{45+29\sqrt{2}} + \sqrt[3]{45-29\sqrt{2}}$. Then which of the following statements is true?

(a) α is an irrational number and $5 \leq \alpha \leq 7$

(b) α is an irrational number and $5 < \alpha < 7$

(c) α is a rational number and $5 < \alpha < 6$

(d) α is an irrational number and $5 \leq \alpha \leq 6$

4. The positive solution (x, y) of the system of equations $\frac{a^2}{x^2}-\frac{b^2}{y^2} = 8(y^4 - x^4)$ and $ax - by = x^4 - y^4$, $(a, b > 0)$ is, (assume $x \neq y$)

(a) $\left(\frac{\sqrt[3]{a+b}+\sqrt[3]{a-b}}{2}, \frac{\sqrt[3]{a+b}-\sqrt[3]{a-b}}{2}\right)$

(b) $\left(\frac{\sqrt[3]{a+b}+\sqrt[3]{a-b}}{4}, \frac{\sqrt[3]{a+b}-\sqrt[3]{a-b}}{4}\right)$

(c) $\left(\frac{\sqrt[3]{a+b}-\sqrt[3]{a-b}}{4}, \frac{\sqrt[3]{a+b}+\sqrt[3]{a-b}}{4}\right)$

(d) $\left(\frac{\sqrt[3]{a+b}-\sqrt[3]{a-b}}{2}, \frac{\sqrt[3]{a+b}+\sqrt[3]{a-b}}{2}\right)$

5. The remainder $R(x)$, when the plynomial x^{100} is divided by $x^2 - 3x + 2$ is

(a) $(2^{100} - 1)x - 2(2^{99} - 1)$
(b) $2^{100}x - 2(2^{100} - 1)$
(c) $(2^{100} - 1)x + 2(2^{99} - 1)$
(d) $(2^{100} + 1)x - 2(2^{99} + 1)$

6. Three straight lines are drawn through a point P lying inside a triangle ABC, parallel to its sides. The areas of the resulting triangles are 1, 4 and 9 sq.cm. Then the area of triangle ABC is (in sq. cm)

(a) 25 (b) 36 (c) 49 (d) 144.

7. Let $N = 2^{744} - 1$. Then about the divisors of N which of the following statement is true?

(a) $2^{248} - 2^{124} + 1$ is a divsion of N but $2^{93} + 2^{47} + 1$ is not
(b) $2^{93} + 2^{47} + 1$ is a divisor of N but $2^{248} - 2^{124} + 1$ is not
(c) $2^{248} - 2^{124} + 1$ and $2^{93} + 2^{47} + 1$ are both divisors of N
(d) Neither $2^{248} - 2^{124} + 1$ nor $2^{93} + 2^{47} + 1$ is a divisor of N.

8. Let $a_1, a_2, \ldots a_N$ be a sequence of real numbers such that the sum of every 5 consecutive terms is positive whereas the sum of every 9 consecutive term is negative. The sequence can have at most

(a) 5×9 terms (b) 14 terms
(c) 13 terms (d) 12 terms.

9. A polynomial $f(x)$ with real coefficient satisfies the functional equation $f(x)\cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$. If $f(2) = 33$, then $f(3)$ is .

(a) 244 (b) 1024 (c) 81 (d) 1023.

10. $ABCD$ is a square and P is a point inside the square such that $PA = 6\sqrt{2}$ cm, $PB = 13$ cm and $PD = 5$ cm. Then the angle $\angle APD$ is

(a) 75° (b) 135° (c) 120° (d) 112°.

11. Repunits are the numbers that contain only the digit 1 in their writing, namely numbers of the form 111....1. The 123-digit repunit $\underbrace{111\ldots1}_{123\text{ times}}$ when divided by 271 leaves a remainder of

(a) 123 (b) 110 (c) 101 (d) 111.

12. In triangle ABC, the ratio of side BC to AC is $2+\sqrt{3}$ and $\angle C = 60°$. The measures of angle A and B are respectively

(a) 45° and 75° (b) 75° and 45°
(c) 105° and 15° (d) 15° and 105°.

13. The minimum value of

$$\frac{\sec^4\alpha}{\tan^2\beta}+\frac{\sec^4\beta}{\tan^2\alpha},\ \alpha, \beta \neq \frac{n\pi}{2},\ n \in Z, \text{ is}$$

(a) 4 (b) 8 (c) 1 (d) 16.

14. Let a and b be non-zero integers with $|a| \leq 100$, $|b| \leq 100$. Then about the bound of $|a\sqrt{2} + b\sqrt{3}|$ which of the following is true?

(a) $|a\sqrt{2}+b\sqrt{3}| \geq \frac{1}{350}$ (b) $|a\sqrt{2}+b\sqrt{3}| \leq \frac{1}{450}$
(c) $|a\sqrt{2}+b\sqrt{3}| < \frac{1}{450}$ (d) $|a\sqrt{2}+b\sqrt{3}| < \frac{1}{550}$.

15. If $f(x) = \frac{e^{2x-1}}{1+e^{2x-1}}$ then the value of

$$f\left(\frac{1}{2006}\right)+f\left(\frac{2}{2006}\right)+\ldots f\left(\frac{2005}{2006}\right) \text{ is}$$

(a) 1002.5 (b) 1001.5 (c) 1003 (d) 1004.

16. There are N boxes, each containing at most r balls. If the number of boxes containing at least i balls is N_i for $i = 1, 2, \ldots r$, then the total number of balls contained in these N boxes is

(a) exactly equal to $N_1 + N_2 + \ldots + N_r$
(b) is strictly larger than $N_1 + N_2 + \ldots + N_r$
(c) is strictly smaller than $N_1 + N_2 + \ldots + N_r$
(d) cannot be determined from the given information.

17. Let $n = \underline{|2006} + 1$. Then the number of primes among $n + 1, n + 2, \ldots n + 2005$ is

(a) 2 (b) 1 (c) 0 (d) > 5.

18. Let $P(x)$ be a polynomial of degree 11 such that $P(x) = \frac{1}{1+x}$, for $x = 0, 1, 2, \ldots 11$. The value of $P(12)$ is

(a) 1/13 (b) 1
(c) 0 (d) cannot be determined.

19. $\sqrt[3]{\sqrt[3]{2}-1}$ is equivalent to

(a) $\sqrt[3]{\frac{1}{9}}+\sqrt[3]{\frac{2}{9}}-\sqrt[3]{\frac{4}{9}}$ (b) $\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}}$

(c) $\sqrt[3]{\frac{1}{9}}+\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}}$ (d) $-\sqrt[3]{\frac{1}{9}}+\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}}$.

20. If α, β, γ are the roots of the equation $x^3 + 2x^2 + 3x + 3 = 0$, then the value of

$$\left(\frac{\alpha}{\alpha+3}\right)^3+\left(\frac{\beta}{\beta+1}\right)^3+\left(\frac{\gamma}{\gamma+1}\right)^3 \text{ is}$$

(a) 14 (b) 44 (c) 45 (d) 15.

21. If the sequence $\{a_n\}$, satisfies the recurrence, $a_{n+1} = 3a_n - 2a_{n-1}$, $n \geq 2$, $a_0 = 2$, $a_1 = 3$, then a_{2007} is
(a) $2^{2006} + 1$ (b) $2^{2007} - 1$
(c) $2^{2007} + 1$ (d) $2^{2006} - 1$.

22. If A, B and C are the angles of a triangle and e^{iA}, e^{iB}, e^{iC} are in arithmetic progression, then the triangle is
(a) right angled but not isosceles
(b) isosceles but not right angled
(c) equilateral (d) right angled isosceles.

23. Statistics show that 20% of smokers get lung cancer and 80% of lung cancer patients are smokers. If 30% of the population smokes, then the percentage of population having lung cancer is:
(a) 16 (b) 7.5 (c) 8 (d) 25.

24. Let $f(x)$ be a function such that
$$f(x-1)+f(x+1) = \sqrt{2}f(x).$$
Then the period of $f(x)$ is
(a) 8 (b) 6 (c) 10 (d) 4.

25. The solutions in integers of the equation $x^2 + xy = y^2 + xz$ can be expressed as, (where n, a, b are arbitrary integers)
(a) $x = na^2$, $y = nab$, $z = n(a^2 + ab - b^2)$
(b) $x = na^2$, $y = -nab$, $z = n(a^2 + ab - b^2)$
(c) $x = na^2$, $y = nab$, $z = n(a^2 - ab + b^2)$
(d) $x = -na^2$, $y = nab$, $z = n(a^2 + ab + b^2)$.

26. The neighbouring sides AB and BC of a square $ABCD$ of side a units are tangents to a circle. The vertex D of the square lies on the circumference of the circle. The radius of the circle is
(a) $a(2\sqrt{2}-1)$ (b) $2a(\sqrt{2}-1)$
(c) $a(2-\sqrt{2})$ (d) $a(2+\sqrt{2})$

27. The sum of all distinct four digit numbers that can be formed using the digits 1, 2, 3, 4 and 5, each digit appearing at most once is
(a) 399960 (b) 396990
(c) 399600 (d) 369960.

28. For what values of d is the product of two numbers of the form $x^2 - dy^2$ and $u^2 - dv^2$ is also of the same form? ($d \neq$ a square)
(a) $d \geq 10$ (b) $d = 2$ only
(c) $d = 2$ and 5 (d) for any d.

29. The minimum value of the expression
$$x^3(x^3 + 1)(x^3 + 2)(x^3 + 3) \text{ is, } (x \in \mathbb{R})$$
(a) 1 (b) -1
(c) 4 (d) none of these.

30. How many strings of 6 digits are there which uses only the digits 0, 1 or 2 and in which digit 2, whenever it appears, it always does so after 1?
(a) 250 (b) 242 (c) 256 (d) 224.

SHORT ANSWER TYPE TEST

31. Let $a_1, a_2, ..., a_n$ be n numbers such that each a_i is either 1 or -1. If $a_1a_2a_3a_4 + a_2a_3a_4a_5 + ... + a_na_1a_2a_3 = 0$, then prove that 4 divides n.

32. Let a, b be integers. Then show that the polynomial $(x-a)^2(x-b)^2 + 1$ is not the product of two polynomials with integral coefficients.

33. Let n be a positive integer. Find all pairs (x, y) such that $x^2(x^2 + y) = y^{n+1}$.

34. Let $f: [a, b] \to R$ be a continuous positive function, differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that $\dfrac{f(b)}{f(a)} = e^{(b-a)\frac{f'(c)}{f(c)}}$.

35. Prove that the number 1280000401 is composite.

36. Solve in real numbers the system for a, b, c and d.
$$a + b = 8,\ ab + c + d = 23,\ ad + bc = 28,\ cd = 12.$$

37. In the trapezoid $PQRS$, $PQ \parallel RS$, $PQ = 4$ cm, $RS = 10$ cm. Also the lines PR and QS intersect at right triangles, and that lines PS and QR when extended to point N, form an angle of 45°. Find the area of the trapezoid $PQRS$.

38. Find the minimum value of
$$|\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x|$$
for real numbers x.

39. Consider the squares of an 8 × 8 chessboard filled

with the numbers 1 to 64 as in the given figure. If we choose 8 squares with the property that is exactly one from each row and exactly one from each column, and add up the numbers in the chosen squares, show that the sum obtained is always 260.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

40. Let A be any set of 19 distinct integers chosen from the arithemtic progression 1, 4, 7, ... 1000. Prove that there must be two distinct integers in A, whose sum is 104.

SOLUTION

1. **(c)** : Let $f(x,y,z)=\dfrac{|x+y|}{|x|+|y|}+\dfrac{|y+z|}{|y|+|z|}+\dfrac{|z+x|}{|z|+|x|}$

Two of the three number, say x and y, must have the same sign, therefore $|x+y|=|x|+|y|$

$$f(x,y,z)=1+\frac{|y+z|}{|y|+|z|}+\frac{|z+x|}{|z|+|x|}$$

on the other hand, as $|a+b|\le|a|+|b|$, we have

$$f(x,y,z)=1+\frac{|y+z|}{|y|+|z|}+\frac{|z+x|}{|z|+|x|}\le 1+1+1=3$$

$$f(1,1,-1)=\frac{|1+1|}{|1|+|1|}+\frac{|1-1|}{|1|+|-1|}+\frac{|-1+1|}{|-1|+|1|}$$

$$=1+0+0=1$$

$$f(1,1,1)=\frac{|1+1|}{|1|+|1|}+\frac{|1+1|}{|1|+|1|}+\frac{|1+1|}{|1|+|1|}=1+1+1=3$$

and for each $t\in(1,3)$, take $x=1$, $y=z=\dfrac{t-3}{t+1}$

Then $f(x,y,z)=f\left(1,\dfrac{t-3}{t+1},\dfrac{t-3}{t+1}\right)$

$$=\frac{\left|1+\frac{t-3}{t+1}\right|}{|1|+\left|\frac{t-3}{t+1}\right|}+\frac{\left|\frac{t-3}{t+1}+\frac{t-3}{t+1}\right|}{\left|\frac{t-3}{t+1}\right|+\left|\frac{t-3}{t+1}\right|}+\frac{\left|\frac{t-3}{t+1}+1\right|}{\left|\frac{t-3}{t+1}\right|+|1|}$$

$$=2\cdot\frac{\left|1+\frac{t-3}{t+1}\right|}{1+\left|\frac{3-t}{t+1}\right|}+\frac{2\left|\frac{t-3}{t+1}\right|}{2\left|\frac{t-3}{t+1}\right|}$$

$$=2\cdot\frac{\left|\frac{2t-2}{t+1}\right|}{1+\frac{3-t}{t+1}}+1=2\cdot\frac{\frac{2}{t+1}|t-1|}{\frac{4}{t+1}}+1$$

$$=(t-1)+1=t$$

The $f(x,y,-z)$ realizes all possible values in the interval $1\le t\le 3$.

2. **(d)** : Let us consider the number in any base b

$N=(12321)_b=b^4+2b^3+3b^2+2b+1,\ (b\ge 4)$

The above expression reminds us of reciprocal equation. We try to factorize it by paring terms equidistant from beginning and end.

$$N=(b^4+1)+2b(b^2+1)+3b^2$$
$$=(b^2+1)^2-2b^2+2b(b^2+1)+3b^2$$
$$=(b^2+1)^2+2b(b^2+1)+b^2=(b^2+b+1)^2$$

The number N is a perfect square in any base $b\ge 4$ (Note that we must have $b\ge 4$ as digits 1, 2, 3 have been used up).

3. **(b)** : Let $\alpha=\sqrt[3]{45+29\sqrt{2}}+\sqrt[3]{45-29\sqrt{2}}$ ...(i)

on cubing both sides

$$\alpha^3=(45+29\sqrt{2})+(45-29\sqrt{2})+$$
$$3\sqrt[3]{(45+29\sqrt{2})(45-29\sqrt{2})}\{(45+29\sqrt{2})+(45-29\sqrt{2})\}$$
$$=90+3\sqrt[3]{2025-1682}\cdot\alpha\qquad\text{(using (i))}$$
$$=90+3\cdot 7\cdot\alpha=90+21\alpha$$
$$\Rightarrow\alpha^3-21\alpha-90=0\Rightarrow(\alpha-6)(\alpha^2+6\alpha+15)=0$$

The equation $\alpha^2+6\alpha+15=0$ has no real roots; hence $\alpha=6$ i.e. $\sqrt[3]{45+29\sqrt{2}}+\sqrt[3]{45-29\sqrt{2}}\neq 6$.

Thus α is a rational number.

4. **(a)** : The system is $\dfrac{a^2}{x^2}-\dfrac{b^2}{y^2}=8(y^4-x^4)$ ...(i)

$ax-by=x^4-y^4,\qquad a,b>0$ ...(ii)

(*Idea* : The system, when seen as a pair of equations in x and y is of much higher degree, than when seen as a pair of equations in a and b. The trick is to solve the system for a, b instead of x, y)

Multiplying (i) by x^4 yields

$$a^2x^2=\frac{b^2x^4}{y^2}+8x^4(y^4-x^4)\qquad\text{...(A)}$$

from (ii) we have $a^2x^2=(by+x^4-y^4)^2$

$$=b^2y^2+2by(x^4-y^4)+(x^4-y^4)^2\qquad\text{...(B)}$$

from (A) and (B)

$$\frac{b^2x^4}{y^2}+8x^4(y^4-x^4)=b^2y^2+2by(x^4-y^4)+(x^4-y^4)^2$$

$$\Rightarrow\frac{b^2(x^4-y^4)}{y^2}-2by(x^4-y^4)$$
$$+8x^4(y^4-x^4)-(x^4-y^4)^2=0$$

$$\Rightarrow\frac{b^2(x^4-y^4)}{y^2}-2by(x^4-y^4)-(x^4-y^4)(9x^4-y^4)=0$$

If $x \neq y$, then dividing by $x^4 - y^4$, we have

$b^2 - 2by^3 - y^2(9x^4 - y^4) = 0$

$\Rightarrow \quad b^2 - 2by^3 - y^2(3x^2 - y^2)(3x^2 + y^2) = 0$

$\Rightarrow \quad b^2 - 2by^3 + (y^3 - 3x^2y)(y^3 + 3x^2y) = 0$

$\Rightarrow \quad b^2 - b\{(y^3 - 3x^2y) + (y^3 + 3x^2y)\}$
$$+ (y^3 - 3x^2y)(y^3 + 3x^2y) = 0$$

if $b = y^3 - 3x^2y$, then (ii) gives $a = x^3 - 3xy^2$.

But $a, b > 0 \Rightarrow x^2 > 3y^2$ and $y^2 > 3x^2 > 9y^2$, a contradiction

$\therefore\ b = y^3 + 3x^2y$. Then $a = x^3 + 3xy^2$

Now $a + b = (x + y)^3$, $(a - b) = (x - y)^3$

Thus $x + y = \sqrt[3]{a+b}$ and $x - y = \sqrt[3]{a-b}$, yielding

$$(x, y) = \left(\frac{\sqrt[3]{a+b}+\sqrt[3]{a-b}}{2}, \frac{\sqrt[3]{a+b}-\sqrt[3]{a-b}}{2}\right).$$

5. **(a) :** By division algorithm

$$x^{100} = (x^2 - 3x + 2)g(x) + ax + b \qquad ...(1)$$

where $ax + b$ is the remainder obtained when x^{100} is divided by the $x^2 - 3x + 2$, a polynomial of degree 2.

(1) can be recast as

$x^{100} = (x - 1)(x - 2)g(x) + ax + b$

put $x = 1 \Rightarrow 1 = a + b$

put $x = 2 \Rightarrow 2^{100} = 2a + b$

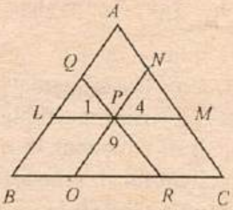
On subtraction $a = 2^{100} - 1$

then $b = 1 - a = 1 - (2^{100} - 1) = 2 - 2^{100} = 2(1 - 2^{99})$

Thus $R|x| = ax + b = (2^{100} - 1)x + 2(1 - 2^{99})$

$= (2^{100} - 1)x - 2(2^{99} - 1)$.

6. **(b) :** Let the line through P, parallel to BC meet AB and AC at L and M respectively. Similarly for lines parallel to AB and AC. Note that triangle PLQ, PNM and PRO are similar to the triangle ABC. Let S be the area of the triangle ABC.



then $\dfrac{1}{S} = \dfrac{LP^2}{BC^2}, \dfrac{4}{S} = \dfrac{PM^2}{BC^2}, \dfrac{9}{S} = \dfrac{OR^2}{BC^2}$

$$\Rightarrow LP = \sqrt{\frac{1}{S}}BC,\ PM = \sqrt{\frac{4}{S}}BC \text{ and } OR = \sqrt{\frac{9}{S}}BC \quad ...(i)$$

Also $BC = BO + OR + RC = (LP) + (OR) + PM$

$$\Rightarrow BC = \sqrt{\frac{1}{S}}BC + \sqrt{\frac{9}{S}}BC + \sqrt{\frac{4}{S}}BC$$

$$\Rightarrow BC = \left(\frac{\sqrt{1}+\sqrt{9}+\sqrt{4}}{\sqrt{S}}\right)BC$$

$$\Rightarrow \sqrt{S} = \sqrt{1}+\sqrt{9}+\sqrt{4} = 1+3+2 = 6 \ \therefore\ S = 36 \text{ sq.cm.}$$

7. **(c) :** $N = 2^{744} - 1 = 2^{93\times 8} - 1 = (2^{93})^8 - 1$

$= (2^{93} - 1)(2^{93} + 1)\{(2^{93})^2 + 1\}\{(2^{93})^4 + 1\}$

Note that $x^8 - 1 = (x^4 - 1)(x^4 + 1)$

$= (x^2 - 1)(x^2 + 1)(x^4 + 1)$

Now $(2^{93})^2 + 1 = (2^{93} + 1)^2 - 2 \cdot 2^{93}$

$= (2^{93} + 1)^2 - 2^{94} = (2^{93} + 1)^2 - (2^{47})^2$

$= (2^{93} + 2^{47} + 1)(2^{93} - 2^{47} + 1)$

Again $(2^{93})^4 + 1 = 2^{93\times 4} + 1 = 2^{3\times 31\times 4} + 1 = (2^{124})^3 + 1$

$= (2^{124} + 1)(2^{248} - 2^{124} + 1)$

Thus $2^{744} - 1 = (2^{93} - 1)(2^{93} + 1)(2^{93} + 2^{47} + 1)$

$(2^{93} - 2^{47} + 1)(2^{124} + 1)(2^{248} - 2^{124} + 1)$.

8. **(d) :** We will show that the square can't have more than 12 terms. We will use the method of contradiction. Consider the first 13 terms of the sequence

$a_1 + a_2 + a_3 + a_4 + a_5 > 0$

$a_2 + a_3 + a_4 + a_5 + a_6 > 0$

....

$a_9 + a_{10} + a_{11} + a_{12} + a_{13} > 0$

Adding vertically, we get

$(a_1 + a_2 + ... + a_9) + (a_2 + a_3 + ... + a_{10}) + ...$

$+ (a_5 + a_6 + ... + a_{13}) > 0 \quad ....(1)$

But since sum of every 9 consecutive term is negative, the sum on the left hand side of (1) must be negative, a contradiction. Hence the sequence can have 12 terms at the most.

9. **(a) :** $f(x)\cdot f\left(\dfrac{1}{x}\right) = f(x) + \dfrac{1}{f(x)}$

Transform the above relation as

$$f(x)f\left(\frac{1}{x}\right) - \left\{f(x) + \frac{1}{f(x)}\right\} = 0$$

$$\Rightarrow 1 - \left\{f(x) + f\left(\frac{1}{x}\right)\right\} + f(x)f\left(\frac{1}{x}\right) = 1$$

$$\Rightarrow (1 - f(x))\left(1 - f\left(\frac{1}{x}\right)\right) = 1$$

$$\Rightarrow (f(x) - 1)\left(f\left(\frac{1}{x}\right) - 1\right) = 1 \qquad ...(1)$$

The only polynomials that satisfy equation (1) are $f(x) - 1 = \pm x^n \Rightarrow f(x) = 1 \pm x^n$

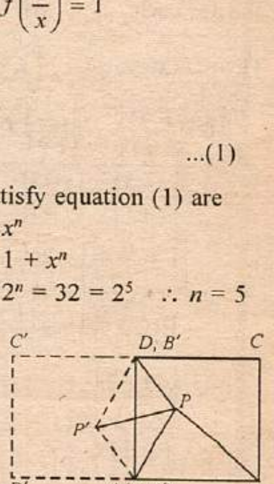
as $f(2) = 33$, we take $f(x) = 1 + x^n$

$\Rightarrow \quad f(2) = 1 + 2^n = 33 \Rightarrow 2^n = 32 = 2^5 \quad \therefore\ n = 5$

Hence $f(3) = 1 + 3^5 = 244$.

10. **(b) :** The idea behind the problem is rotation through a suitable angle.

Rotate the square about A in the anticlockwise direction by 90°. Then B goes to $B'(= D)$, C goes to C', P goes to P'.



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We have $P'A \perp PA$, as the rotation is through 90° anticlockwise.

$P'A = PA = 6\sqrt{2} \quad \therefore \angle APP' = 45°$

$P'B' = PB = 13$ cm

The triangle PAP', $PP' = \sqrt{PA^2 + P'A^2} = \sqrt{2}PA$

$= \sqrt{2} \times 6\sqrt{2} = 12$ cm.

Again in triangle $PP'D = PP'^2 + PD^2 = 12^2 + 5^2 = 13^2$
$= PB^2 = P'B'^2 \qquad \therefore \angle P'PD = 90°$

Thus $\angle APD = \angle APP' + \angle P'PD = 45° + 90° = 135°$.

11. (d) : $\underbrace{111....1}_{123 \text{ times}} = \underbrace{111....1000}_{1201's} + 111 \qquad ...(A)$

Now $\quad 11111 = 41 \times 271$

Now note that

$\underbrace{111....1}_{1201's} = 11111 \times \underbrace{10001....0001}_{241's} + 111 \qquad ...(B)$

From (A) and (B)

$\underbrace{111...1}_{123 \text{ times}} = 11111 \times 100001....00001 \times 1000 + 111$

$= 41 \times 271 \times 100001 .... 00001 \times 1000 + 111$

Thus $\underbrace{111...11}_{123 \text{ times}}$ when divided by 271 leaves a remainder of 111.

12. (c) : Given $\frac{a}{b} = 2 + \sqrt{3}$

Consider $\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$

$$= \frac{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}} = \tan\frac{A-B}{2}\cot\frac{A+B}{2}$$

$$= \tan\frac{A-B}{2}\cot\left(\frac{\pi - C}{2}\right) = \tan\left(\frac{A-B}{2}\right)\tan\frac{C}{2} \quad ...(1)$$

Now $\frac{a-b}{a+b} = \frac{\frac{a}{b} - 1}{\frac{a}{b} + 1} = \tan\left(\frac{A-B}{2}\right)\tan\frac{C}{2}$

$$\Rightarrow \frac{2+\sqrt{3}-1}{2+\sqrt{3}+1} = \tan\left(\frac{A-B}{2}\right)\cdot\frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{1+\sqrt{3}}{3+\sqrt{3}} = \tan\left(\frac{A-B}{2}\right)\cdot\frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \tan\left(\frac{A-B}{2}\right)\cdot\frac{1}{\sqrt{3}} \Rightarrow \tan\left(\frac{A-B}{2}\right) = 1$$

$$\Rightarrow \tan\left(\frac{A-B}{2}\right) = 1 \Rightarrow \frac{A-B}{2} = 45° \Rightarrow A - B = 90°$$

Also $A + B = 125°$. They both yield $A = 105°$, $B = 15°$.

13. (b) : The problem which looks like a trigonometrical maxima/min problem can be reduced to an inequality in algebra by means of suitable substitutions.

Let $x = \tan^2\alpha$, $y = \tan^2\beta$

Then with

$$\frac{\sec^4\alpha}{\tan^2\beta} + \frac{\sec^4\beta}{\tan^2\alpha} = \frac{(1+x)^2}{y} + \frac{(1+y)^2}{x} \text{ with } x, y \ge 0$$

$$= \frac{1+2x+x^2}{y} + \frac{1+2y+y^2}{x}$$

$$= \left(\frac{1}{y} + 2\frac{x}{y} + \frac{x^2}{y}\right) + \left(\frac{1}{x} + \frac{2y}{x} + \frac{y^2}{x}\right)$$

$$= \left(\frac{x^2}{y} + \frac{1}{y} + \frac{y^2}{x} + \frac{1}{x}\right) + 2\left(\frac{x}{y} + \frac{y}{x}\right) \qquad ...(A)$$

From A.M. - G.M. theorem

$$\frac{\frac{x^2}{y} + \frac{1}{y} + \frac{y^2}{x} + \frac{1}{x}}{4} \ge \sqrt[4]{\left(\frac{x^2}{y}\right)\left(\frac{1}{y}\right)\left(\frac{y^2}{x}\right)\left(\frac{1}{x}\right)}$$

$$\Rightarrow \frac{x^2}{y} + \frac{1}{y} + \frac{y^2}{x} + \frac{1}{x} \ge 4$$

Similarly $\frac{x}{y} + \frac{y}{x} \ge 2$

$\therefore$ From (A) minimum value of the sought for expression is $4 + 2 \times 2 = 8$ and equality is attained when $x = y = 1$, i.e. $\alpha = 2\pi k \pm \frac{\pi}{4}$, $\beta = 2\pi k \pm \frac{\pi}{4}$, for integers k.

14. (a) : At first sight the problem looks intractable. However, using the idea of conjugate and modulus function, the problem is solved easily.

Consider $\left|a\sqrt{2} + b\sqrt{3}\right| = \frac{|(a\sqrt{2}+b\sqrt{3})(a\sqrt{2}-b\sqrt{3})|}{|a\sqrt{2}-b\sqrt{3}|}$

$$= \frac{|2a^2 - 3b^2|}{|a\sqrt{2} - b\sqrt{3}|} \qquad ...(A)$$

$a\sqrt{2} - b\sqrt{3} \ne 0$ for then $\frac{a}{b}$ would be $\sqrt{\frac{3}{2}}$, i.e. a rational number being equal to the irrational number. Impossible.

Now a and b are integer and so is $2a^2 - 3b^2$

$\therefore |2a^2 - 3b^2| \ge 1 \qquad ...(B)$

Also $|a\sqrt{2} - b\sqrt{3}| \le |a\sqrt{2}| + |b\sqrt{3}| = |a|\sqrt{2} + |b|\sqrt{3}$

$= 100\sqrt{2} + 100\sqrt{3} = 100(\sqrt{2}+\sqrt{3}) < 350 \qquad ...(C)$

It follows from (A), (B) and (C)

$$\frac{|2a^2 - 3b^2|}{|a\sqrt{2} - b\sqrt{3}|} \ge \frac{1}{350} \text{ i.e. } |a\sqrt{2} + b\sqrt{3}| \ge \frac{1}{350}.$$

15. **(a)** : Rewrite $f(x)$ as

$$f(x)=\frac{e^{2x-1}}{1+e^{2x-1}}=\frac{e\cdot e^{2x-1}}{e+e\cdot e^{2x-1}}$$

$$=\frac{e^{2x}}{e+e^{2x}}=\frac{\frac{e^{2x}}{e^x}}{\frac{e}{e^x}+\frac{e^{2x}}{e^x}}=\frac{e^x}{e^{1-x}+e^x}$$

$$f(1-x)=\frac{e^{1-x}}{e^x+e^{1-x}}$$

So $f(x)+f(1-x)=\frac{e^x}{e^{1-x}+e^x}+\frac{e^{1-x}}{e^x+e^{1-x}}=1$

Now $f\left(\frac{1}{2006}\right)+f\left(\frac{2}{2006}\right)+\dots f\left(\frac{2005}{2006}\right)$

$$=\left\{f\left(\frac{1}{2006}\right)+f\left(\frac{2005}{2006}\right)\right\}+\left\{f\left(\frac{2}{2006}\right)+f\left(\frac{2004}{2006}\right)\right\}$$

$$+\dots\left\{f\left(\frac{1002}{2006}\right)+f\left(\frac{1004}{2006}\right)\right\}+f\left(\frac{1003}{2006}\right)$$

$$=1+1+1+\dots \text{ to 1002 terms }+f\left(\frac{1}{2}\right)$$

$$=1002+\frac{1}{2}\quad\left(\because f\left(\frac{1}{2}\right)=\frac{1}{2},\text{ by putting values}\right)$$

$$=1002.5$$

16. **(a)** : Number of boxes having exactly i balls = Number of boxes having at least i balls – Number of boxes having at least $(i+1)$ balls, $(1\le i\le r-1)$

$\therefore$ Number of boxes having exactly 1 ball $=N_1-N_2$

Number of boxes having exactly 2 balls $=N_2-N_3$

....

Number of boxes having exactly $(r-1)$ balls $=N_{r-1}-N_r$

Number of boxes having exactly r balls $=N_r$, for a box can hold a maximum of r balls.

Total Number of balls $=(N_1-N_2)+2(N_2-N_3)+3(N_3-N_4)+\dots(r-1)(N_{r-1}-N_r)+r\cdot N_r$

$=N_1-N_2+2N_2-2N_3+3N_3-3N_4+\dots(r-2)N_{r-1}+(r-1)N_{r-1}-(r-1)N_r+N_r$

$=N_1+N_2+N_3+\dots+N_{r-1}+N_r$

Remark : This was a question on logical reasoning. Two important things are to be answered while having a go at this problem. 1> How many boxes contain exactly i balls? 2> How many balls does box N_r contain?

17. **(c)** : We embed the problem in a more general one.

Let $n=\underline{|51}+1=\underline{|k}+1,\ k=51$ (say)

We consider the numbers $\underline{|k}+2, \underline{|k}+3, \dots \underline{|k}+51$.

As $\underline{|k}=1\times2\times\dots\times51$

$\therefore 2|\underline{|k},\ 3|\underline{|k},\ 5|\underline{|k}$

$\therefore 2|\underline{|k}+2,\ 3|\underline{|k}+3,\ 4|\underline{|k}+4,\dots 51|\underline{|k}+51$

Thus the numbers $\underline{|k}+2, \underline{|k}+3, \underline{|k}+4,\dots \underline{|k}+51$ are all composite.

18. **(c)** : In such problems we use factor theorem in an innovative way. Set up the polynomial

$g(x)=(x+1)P(x)-1$, then from the hypothesis

$g(0)=g(1)=g(2)\dots=g(11)=0$

But $g(x)$ is a polynomial of degree 12, so because 0, 1, 2, ... 11 are its zeroes, we must have by factor theorem $g(x)=a(x)(x-1)(x-2)\dots(x-11)$, where a is a constant to be determined. ...(1)

Also $g(x)=(x+1)P(x)-1\Rightarrow g(-1)=-1$

and from (1)

$$g(-1)=a(-1)(-2)(-3)\dots(-12)=-1$$

$$\Rightarrow a\underline{|12}=(-1)\quad\therefore a=-\frac{1}{\underline{|12}}$$

Thus $(x+1)P(x)-1=-\frac{1}{\underline{|12}}(x)(x-1)(x-2)\dots(x-11)$

setting $x=12$ in the above relation

$$(12+1)P(12)-1=-\frac{1}{\underline{|12}}\cdot12\times11\times10\times\dots1=-\frac{\underline{|12}}{\underline{|12}}$$

$$\Rightarrow 13P(12)-1=-1\Rightarrow 13P(12)=0\quad\therefore P(12)=0.$$

19. **(b)** : Set $\sqrt[3]{2}=a\Rightarrow a^3=2$

Now $(1-a+a^2)^2=1+a^2+a^4-2a+2a^2-2a^3$

$=1+a^2+a^3\cdot a-2a+2a^2-2a^3$

$=1+a^2+2a-2a+2a^2-4$

$=3a^2-3=3(a^2-1)$

$(1-a+a^2)^3=(1-a+a^2)^2(1-a+a^2)$

$=3(a^2-1)(1-a+a^2)$

$=3(a-1)(a+1)(1-a+a^2)$

$=3(a-1)(1+a^3)$

$=3(a-1)(1+2)=9(a-1)$ ...(1)

From (1) $9\left(\sqrt[3]{2}-1\right)=\left(1-\sqrt[3]{2}+\sqrt[3]{4}\right)^3$

Taking cube roots $\sqrt[3]{\sqrt[3]{2}-1}=\sqrt[3]{\frac{1}{9}}-\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{4}{9}}$.

20. **(b)** : Given $\alpha,\ \beta,\ \gamma$ are the roots of $x^3+2x^2+3x+3=0$, then we have to form the equation whose roots are $\frac{\alpha}{\alpha+1},\ \frac{\beta}{\beta+1},\ \frac{\gamma}{\gamma+1}$

Let $y=\frac{\alpha}{\alpha+1}=\frac{x}{x+1}$ express x in terms of

$$y\Rightarrow x=\frac{y}{1-y}$$

So that equation is

$$\left(\frac{y}{1-y}\right)^3+2\left(\frac{y}{1-y}\right)^2+3\left(\frac{y}{1-y}\right)+3=0$$

$\Rightarrow y^3+2y^2(1-y)+3y(1-y)^2+3(1-y)^3=0$

$\Rightarrow y^3+2y^2-2y^3+3y(1-2y+y^2)+3(1-3y+3y^2-y^3)=0$

$\Rightarrow -y^3+5y^2-6y+3=0 \Rightarrow y^3-5y^2+6y-3=0$ ...(2)

Let $\frac{\alpha}{\alpha+1}=\alpha'$, $\frac{\beta}{\beta+1}=\beta'$, $\frac{\gamma}{\gamma+1}=\gamma'$.

Thus α', β', γ' are the roots of (2)

$\therefore \sum\alpha'=5,\ \sum\alpha'\beta'=6,\ \sum\alpha'\beta'\gamma'=\alpha'\beta'\gamma'=3$

Now use the identity

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$
$$=(a+b+c)\{(a+b+c)^2-3(ab+bc+ca)\}$$
$$=(a+b+c)^3-3(a+b+c)(ab+bc+ca)$$

$\therefore a^3+b^3+c^3=(a+b+c)^3-$
$3(ab+bc+ca)(a+b+c)+3abc$

Thus

$$\alpha'^3+\beta'^3+\gamma'^3=\left(\sum\alpha'\right)^3-3\left(\sum\alpha'\right)\left(\sum\alpha'\beta'\right)+3\alpha'\beta'\gamma'$$
$$=5^3-3(6)(5)+3\times3$$
$$=5(5^2-18)+9=5\times7+9=44.$$

21. (c) : Although there are standard methods of solving linear recurrence relation of a given order, we try to exploit the idea of geometric progression to solve our problem.

$a_{n+1}=3a_n-2a_{n-1}$, $n\ge 2$ can be recast as

$a_{n+1}-a_n=2a_n-2a_{n-1}=2(a_n-a_{n-1})$

Let $b_n=a_{n+1}-a_n$, $n\ge 1$

thus $b_n=2b_{n-1}$, $n\ge 1$

Thus b_n is a G.P. of common ratio 2

$\therefore b_n=2^{n-1}b_1=2^{n-1}(a_2-a_1)$
$=2^n(a_1-a_0)=2^n(3-2)=2^n$

$$\therefore \left.\begin{array}{l} a_{n+1}-a_n=2^n\\ a_n-a_{n-1}=2^{n-1}\\ \dots\dots\dots\dots\\ a_2-a_1=2\end{array}\right\}$$

Adding vertically, $a_{n+1}-a_1=2+2^2+\ldots+2^n=2(2^n-1)$

$\Rightarrow a_{n+1}=2^{n+1}-2+a_1=2^{n+1}-2+3=2^{n+1}+1$

Thus $a_n=2^n+1$.

22. (c) : *1st solution*

e^{iA}, e^{iB}, e^{iC} are in A.P.

$\Rightarrow e^{iA}+e^{iC}=2(e^{iB})$

$\Rightarrow (\cos A+i\sin A)+(\cos C+i\sin C)=2(\cos B+i\sin B)$

$\Rightarrow (\cos A+\cos C)+i(\sin A+\sin C)=2\cos B+i(2\sin B)$

equating real and imaginary parts

$\cos A+\cos C=2\cos B$

$$\Rightarrow 2\cos\frac{A+C}{2}\cos\frac{A-C}{2}=2\cos B$$

$$\Rightarrow 2\cos\left(\frac{\pi}{2}-\frac{B}{2}\right)\cos\frac{A-C}{2}=\cos B \quad \text{...(A)}$$

$$\sin\frac{B}{2}\cos\frac{A-C}{2}=\cos B$$

again $\sin A+\sin C=2\sin B$

$$\Rightarrow 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}=2\sin B$$

$$\Rightarrow 2\sin\left(\frac{\pi}{2}-\frac{B}{2}\right)\cdot\cos\frac{A-C}{2}=2\sin\frac{B}{2}\cos\frac{B}{2}$$

$$\Rightarrow \cos\frac{B}{2}\cdot\cos\frac{A-C}{2}=2\sin\frac{B}{2}\cos\frac{B}{2}$$

$$\Rightarrow \cos\frac{A-C}{2}=2\sin\frac{B}{2} \quad \text{...(B)} \quad \left(\because \cos\frac{B}{2}\neq 0\right)$$

From (A) and (B)

$$\left(\sin\frac{B}{2}\right)\left(2\sin\frac{B}{2}\right)=\cos B$$

$$\Rightarrow 2\sin^2\frac{B}{2}=\cos B=1-2\sin^2\frac{B}{2}\Rightarrow 4\sin^2\frac{B}{2}=1$$

$$\Rightarrow \sin^2\frac{B}{2}=\frac{1}{4}\Rightarrow \sin\frac{B}{2}=\frac{1}{2}\ \Rightarrow \frac{B}{2}=30^\circ\ \therefore\ B=60^\circ$$

From (B) $\cos\frac{A-C}{2}=2\times\frac{1}{2}=1$

$$\Rightarrow \frac{A-C}{2}=0\ \therefore A=C=60^\circ \quad (\because B=60^\circ)$$

Thus the triangle is equilateral.

2nd Solution

Instead of dealing with two separate equations, we exploit the result (to be derived) to deal with them at one stroke.

$$e^{ix}+e^{iy}=e^{i\left(\frac{x+y}{2}\right)}\left\{e^{i\left(\frac{x-y}{2}\right)}+e^{-i\left(\frac{x-y}{2}\right)}\right\}$$
$$=e^{i\left(\frac{x+y}{2}\right)}\left\{2\cos\frac{x-y}{2}\right\}=2\cos\frac{x-y}{2}e^{i\left(\frac{x+y}{2}\right)}$$

e^{iA}, e^{iB}, e^{iC} are in A.P.

$$\Rightarrow e^{iA}+e^{iC}=2e^{iB};\quad 2\cos\frac{A-C}{2}\cdot e^{i\left(\frac{A+C}{2}\right)}=2e^{iB}$$

$$\Rightarrow \cos\frac{A-C}{2}\cdot e^{i\left(\frac{\pi}{2}-\frac{B}{2}\right)}=e^{iB}$$

$$\Rightarrow \cos\frac{A-C}{2}\cdot e^{-\frac{iB}{2}}\cdot i=e^{iB}$$

$$\Rightarrow i\cos\frac{A-C}{2}=e^{i\frac{3B}{2}}\Rightarrow i\cos\frac{A-C}{2}=\cos\frac{3B}{2}+i\sin\frac{3B}{2}$$

$$\Rightarrow \cos\frac{3B}{2}=0 \text{ and } \sin\frac{3B}{2}=\cos\frac{A-C}{2}$$

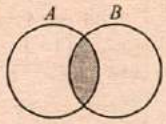
$$\Rightarrow \frac{3B}{2}=\frac{\pi}{2}\ \therefore B=\frac{\pi}{3}=60^\circ.$$

$$\cos\frac{3B}{2} = 0 \Rightarrow \sin\frac{3B}{2} = 1$$

$$\Rightarrow \cos\frac{A-C}{2} = 1 \Rightarrow A - C = 0 \quad \therefore A = C$$

Thus $A = B = C = 60^\circ$.

23. **(b)** : Let A be the set of smokers and B the set of lung cancer patients. Then shaded area = the number of people who are both smokers and lung cancer patients.



$$= \frac{20A}{100} = \frac{80B}{100} \Rightarrow A = 4B \qquad ...(1)$$

again $A = \frac{30}{100} \times P$, where P is the population ...(2)

From (1) and (2)

$$\frac{30}{100}P = 4B \Rightarrow B = \frac{30}{100 \times 4}P = \frac{7.5}{100}P$$

Thus 7.5% of population suffers from lung cancer.

24. **(a)** : Rather than changing $x+1$ to x etc. and hoping that the answer will somehow come out, we indicate to you the natural way to solve the problems of this type. Change the lowest argument, $x-1$ to x, the equation reads

$$f(x) + f(x+2) = \sqrt{2}\, f(x+1) \qquad ...(1)$$

chang x to $x+2$ to obtain

$$f(x+2) + f(x+4) = \sqrt{2}\, f(x+3) \qquad ...(2)$$

Adding (1) and (2)

$$f(x) + 2f(x+2) + f(x+4) = \sqrt{2}\{f(x+1) + f(x+3)\}$$

$$= \sqrt{2} \cdot \left\{\sqrt{2} f(x+2)\right\} \qquad \text{using (1)}$$

$$= 2f(x+2)$$

$$\Rightarrow \quad f(x) + 2f(x+2) + f(x+4) = 2f(x+2)$$

$$\Rightarrow \quad f(x) + f(x+4) = 0 \qquad ...(3)$$

Now change x to $x+4$

$$\Rightarrow \quad f(x+4) + f(x+8) = 0 \qquad ....(4)$$

(3) and (4) on subtraction give $f(x) - f(x+8) = 0$

$$\Rightarrow \quad f(x) = f(x+8)$$

Thus the function is periodic with period 8.

Note : The reader is advised to refer to Concept Booster XII, page 60, of the August issue of the magazine for a detailed discussion.

25. **(a)** : The given equation can be rewritten as

$$x^2 + xy = y^2 + xz \Rightarrow x^2 - xz = y^2 - xy$$

$$\Rightarrow \quad x(x-z) = y(y-x) \qquad ...(1)$$

Let $d = gcd\,(x, y)$ Then $x = da, y = db$ with $gcd\,(a, b) = 1$

(1) now becomes

$da(x-z) = db(y-x) \Rightarrow a(x-z) = b(y-x)$

as $(a, b) = 1$, we have $x - z = kb$ and $y - x = ka$ for some integer k

But $gcd\,(a, b-a) = gcd\,(a, b) = 1$,

Thus it follows that $(b-a)$ divides k

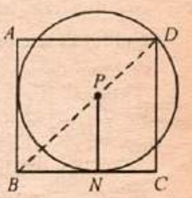
Let $k = n(b-a)$, we obtain $d = na$

Hence solution becomes $x = na^2, y = nab$,

$z = n(a^2 + ab - b^2)$ where n, a, b are arbitrary integers.

26. **(c)** : Suppose P is the centre of the circle. The point P lies on the bisector of $\angle ABC$, i.e., on the diagonal BD of the square.

$\therefore \angle PBC = 45^\circ, PN = PD = R$ = radius of the circle



From triangle PNB

$$\sin 45^\circ = \frac{PN}{PB} \Rightarrow \frac{1}{\sqrt{2}} = \frac{R}{PB} \quad \therefore \ PB = R\sqrt{2}$$

BD = diagonal of the square = $a\sqrt{2}$

Also $BD = BP + PD = R\sqrt{2} + R = R(\sqrt{2}+1)$

$$\Rightarrow a\sqrt{2} = R(\sqrt{2}+1) \Rightarrow R = \frac{a\sqrt{2}}{\sqrt{2}+1}$$

$$= a\sqrt{2}(\sqrt{2}-1) = a(2-\sqrt{2}).$$

27. **(a)** : Let us calculate the number of numbers whose last digit is x, where $x = 1, 2, 3, 4$ or 5.

$$\boxed{}\boxed{}\boxed{}\boxed{x}$$

Since each digit can appear at most once, the ten's place can be filled in 4 ways, hundred's place can be filled in 3 ways and thousand's place can be filled in 2 ways.

Thus the number of numbers whose last digit is

$$x = 4 \times 3 \times 2 = 4!$$

So, the digits in the unit place of all the 120 numbers add upto $4!\,(1+2+3+4+5) = 24 \times 15 = 360$

Similarly the numbers at ten's place add up to 360 and so on. Hence the sum of all numbers is

$$360(1 + 10 + 10^2 + 10^3) = 360 \times \frac{10^4 - 1}{9} = 40 \times (10^4 - 1)$$

$$= 4 \times 10^5 - 40 = 399960.$$

28. **(d)** : *1st Solution :* In fact the product of two number of the form $x^2 - dy^2$ and $u^2 - dv^2$ is of the same form for any d, where d is not a perfect square.

$$(x^2 - dy^2)(u^2 - dv^2) = (x - y\sqrt{d})(x + y\sqrt{d})(u - v\sqrt{d})(u + v\sqrt{d})$$

$$= \{(x - y\sqrt{d})(u + v\sqrt{d})\}\{(x + y\sqrt{d})(u - v\sqrt{d})\}$$

$$= (ux + vx\sqrt{d} - uy\sqrt{d} - vyd)(xu - xv\sqrt{d} + uy\sqrt{d} - vyd)$$

$$= [(ux - vyd) + (vx - uy)\sqrt{d}][(xu - vyd) + (uy - vx)\sqrt{d}]$$

$$= [(ux - dvy) + (vx - uy)\sqrt{d}][(ux - dvy) - \sqrt{d}(vx - uy)]$$

$$= (ux - dvy)^2 - d(vx - uy)^2$$

2nd Solution (using matrices)

Consider $\begin{pmatrix} x & yd \\ y & x \end{pmatrix}$, it's determination is $x^2 - dy^2$

$$\begin{pmatrix} x & yd \\ y & x \end{pmatrix}\begin{pmatrix} u & vd \\ v & u \end{pmatrix} = \begin{pmatrix} ux+dvy & d(vx+uy) \\ xv+uy & ux+dvy \end{pmatrix}$$

As $\det(AB) = (\det A)(\det B)$, i.e., the determinant of the product is the product of determinants.

We have $(x^2 - dy^2)(u^2 - vd^2) = (ux + dvy)^2 - d(xv + yu)^2$.

29. **(b)** : Let $u = x^3$

then $x^3(x^3 + 1)(x^3 + 2)(x^3 + 3) = u(u + 1)(u + 2)(u + 3)$

$= u(u + 3)(u + 1)(u + 2) = (u^2 + 3u)(u^2 + 3u + 2)$

$= \{(u^2 + 3u + 1) - 1\}\{(u^2 + 3u + 1) + 1\}$

$= (u^2 + 3u + 1)^2 - 1 = (x^6 + 3x^3 + 1)^2 - 1$

Thus the minimum value of the expression is -1. But we must ensure that it's attained too. For that we note that the minimum value is attained where $x^6 + 3x^3 + 1 = 0$, i.e., for the real roots of $x^6 + 3x^3 + 1 = 0$.

30. **(c)** : We are going to introduce you to a powerful technique of counting i.e. recursion.

Let us denote by u_n the number of n-digit string made up of 0, 1 or 2 and satisfying the condition of the problem.

u_2, the number of 2-digit string, can be obtained by direct calculation. $u_2 = 8$, viz., 00, 01, 02, 10, 11, 12, 22, 20.

Let $x = x_1, x_2 \dots x_n$ be a sequence belonging to u_n. We can have two mutually exclusive cases.

(1) If x starts wth 2, i.e. $x_1 = 2$, then each of $x_2, x_3, \dots$ can be 0 or 2. So there are 2^{n-1} such sequences. (Recall that 1 cannot appear to the right of 2).

(2) If x start with 0 or 1, i.e. $x_1 = 0$ or 7, then $x_2 x_3, x_n$ is a sequence of $(n - 1)$ digits satisfying the condition of the problem. so there are $2u_{n-1}$ such sequences.

Thus $u_n = 2^{n-1} + 2u_{n-1}, n \ge 2$

Also $u_2 = 8$

$u_3 = 2^2 + 2u_2 = 4 + 2 \times 8 = 20$

$u_4 = 2^3 + 2u_3 = 8 + 2 \times 20 = 48$

$u_5 = 2^4 + 2u_4 = 16 + 2 \times 48 = 112$

$u_6 = 2^5 + 2u_5 = 32 + 2 \times 112 = 256$

31. 1st Solution

Let $b_k = a_k a_{k+1} a_{k+2} a_{k+3}$ for $k = 1, 2, \dots n$

Also let $a_{n+1} = a_1, a_{n+2} = a_2, a_{n+3} = a_3$

The given condition

$a_1a_2a_3a_4 + a_2a_3a_4a_5 + \dots + a_na_1a_2a_3 = 0$ reduces to

$b_1 + b_2 + \dots + b_n = 0$ ...(A)

As $a_i = 1$ or -1, so $b_i = 1$ or -1 too

suppose that among the b_i's, there are n_1 numbers equal to 1 and n_2 numbers equal to -1

then $n_1 + n_2 = n$ and from (A)

$1 \times n_1 + (-1) \times n_2 = 0 \Rightarrow n_1 = n_2$

Thus $n_1 = n_2 = n/2$

So n is even $\therefore\ n = 2l$, say ...(B)

Again $b_1b_2 \dots b_n = (a_1a_2a_3a_4)(a_2a_3a_4a_5) \dots (a_na_1a_2a_3)$

$= a_1^4\, a_2^4 \dots a_n^4 = (a_1^2\, a_2^2 a_3^2 \dots a_n^2)^2$

$= (1 \cdot 1 \cdot 1 \dots 1)^2 = 1$ ...(C)

Also $b_1b_2 \dots b_n = 1^{n_1} \times (-1)^{n_2}$

$= (-1)^{n_2} = (-1)^{n/2} = (-1)^l$ ....(D)

From (C) and (D), $(-1)^l = 1 \Rightarrow l =$ even $\therefore\ l = 2m$

then (B) gives $n = 2 \times 2m = 4m$

Thus $4 \mid n$, i.e. 4 divides n.

2nd Solution

It can also be solved in an elegant way by using the "idea of invariance". If we replace any a_i by $-a_i$, then the sum $S = a_1a_2a_3a_4 + a_2a_3a_4a_5 + \dots + a_na_1a_2a_3$ does'nt change mod 4 because four cyclically adjacent terms change their sign. It can be seen like this

(1) If two terms are positive and two negative, then by changing a_i to $-a_i$, the sum doesn't change.

(2) If one or three terms have the same sign, the sum changes by ± 4.

(3) If all the four terms are of the same sing, then S changes by $\pm$ 8.

Initially we have $S = 0$ which gives $S \equiv 0 \pmod 4$. Now, changing the sign of one a_i at a time, we change each negative sign in a positive sign. This doesn't change S mod 4. At the end also we must have $S \equiv 0 \pmod 4$, but having changed the sign $S = n$. $\therefore\ 4 \mid n$.

32. Suppose that $(x - a)^2(x - b)^2 + 1$ can be written as a product of two polynomials with integral coefficients.

Let $(x - a)^2(x - b)^2 + 1 = f(x)g(x)$ ...(1)

We have $f(a) = f(b) = g(a) = g(b) = 1$ ...(A)

From (A) it is seen that both $f(x) - 1$ and $g(x) - 1$ are divisible by $(x - a)(x - b)$

Assume that $f(x) - 1 = (x - a)(x - b)$ and

$g(x) - 1 = (x - a)(x - b)$

This gives $f(x)g(x) = \{(x - a)(x - b) + 1\}^2$

$= (x - a)^2(x - b)^2 + 2(x - a)(x - b) + 1$ ...(2)

But then (1) and (2) yield

$(x - a)(x - b) \equiv 0$, i.e. $(x - a)(x - b)$ is identically zero. Obviously we reach a contradiction.

Hence $(x - a)^2(x - b)^2 + 1$ can't be written as a product of two polynomials with integer coefficients.

33. Transform the equation after multiplying by 4 as below

$x^4 + x^2y = y^{n+1} \Rightarrow 4x^4 + 4x^2y = 4y^{n+1}$

$\Rightarrow \quad y^2 + 4x^4 + 4x^2y = y^2 + 4y^{n+1}$

$\Rightarrow \quad (2x^2 + y)^2 = y^2(1 + 4y^{n-1}) \qquad ...(1)$

From (1) it follows that $1 + 4y^{n-1}$ is an odd square, so $1 + 4y^{n-1} = (2k + 1)^2 \Rightarrow 1 + 4y^{n-1} = 4k^2 + 4k + 1$

$\Rightarrow \quad y^{n-1} = k^2 + k = k(k + 1)$

Since k and $k + 1$ are relatively prime integers, each of them must be the $(n - 1)$th power of some integer. This is possible only when $n = 2$, thus giving $y = k(k + 1)$.

Now from (1),

$2x^2 + k(k + 1) = k(k + 1)(2k + 1)$

$\Rightarrow \quad 2x^2 = k(k + 1)(2k + 1) - k(k + 1) = k(k + 1) \times 2k$

$x^2 = k^2(k + 1)$

Then $k + 1$ should be a square, let $k + 1 = t^2$. Then

$x^2 = (t^2 - 1)^2t^2 \qquad \therefore x = (t^2 - 1)t = t^3 - t$

and then $y = k(k + 1) = (t^2 - 1)t^2 = t^4 - t^2$

$\therefore (x, y) = (t^3 - t, t^4 - t^2)$ describes the solution of the problem.

34. (When solving problem on mean value theorem, we offen have to consturct auxiliary functions on which the theorem can be applied. Ingenuity is needed to consturct such functions. Offer the result to be proved is a key to constructing such functions).

As f is positive, $\ln f(x)$ is well defined,

we get $g(x) = \ln f(x)$

then $g(x)$ continuous where $f(x)$ is continuous and differentiable where $f(x)$ is. So $g(x)$ satisfies the hypothesis of mean-value theorem on $[a, b]$, so there exists $C \in (a, b)$ such that

$$\frac{g(b) - g(a)}{b - a} = g'(c) \Rightarrow \frac{\ln f(b) - \ln f(a)}{b - a} = \frac{f'(c)}{f(c)}$$

$$\Rightarrow \ln\frac{f(b)}{f(a)} = (b - a)\frac{f'(c)}{f(c)}$$

By exponentiation

$$\frac{f(b)}{f(a)} = e^{(b-a)\frac{f'(c)}{f(c)}}.$$

35. Here we use a technism based on factorization of polynomials.

Write $1280000401 = (20)^7 + (20)^2 + 1 = x^7 + x^2 + 1$, where $x = 20$

Claim : $x^2 + x + 1$ divides $x^7 + x^2 + 1$

Proof: We have $(x^2 + x + 1) = (x - \omega)(x - \omega^2)$, where ω is the complex cube root of unity

Now $\omega^7 + \omega^2 + 1 = (\omega^6)\omega + \omega^2 + 1 = \omega + \omega^2 + 1 = 0$

also $(\omega^2)^7 + (\omega^2)^2 + 1 = \omega^{14} + \omega^4 + 1$

$= (\omega^3)^4 \cdot \omega^2 + (\omega^3) \cdot \omega + 1 = \omega^2 + \omega + 1 = 0$

so both $(x - \omega)$ and $(x - \omega^2)$ are factor of $x^7 + x^2 + 1$

Thus $x^2 + x + 1$ divides $x^7 + x^2 + 1$

Hence $20^2 + 20 + 1 = 421$ is a factor of 1280000401.

36. At first there appears to be no pattern to the problem. But to get a feel of the solution we multiply two polynomials

$$(x^2 + ax + c)(x^2 + bx + d) = x^4 + (a + b)x^3 + (ab + c + d)x^2 + (ad + bc)x + cd$$

So we observe that the system resembles the coefficients when these two polynomials are multiplied.

Consider the polynomial $f(x) = x^4 + 8x^3 + 23x^2 + 28x + 12$

$f(-1) = 1 - 8 + 23 - 28 + 12 = 36 - 36 = 0$

Also $f(-2) = 16 - 64 + 92 - 56 + 12 = 120 - 120 = 0$

$f'(x) = 4x^3 + 24x^2 + 46x + 28$

$f'(-2) = -32 + 96 - 92 + 28 = -128 + 124 = 0$

Thus $-1, -2, -2$ are three roots. As product of roots = 12, the fourth root is -3.

$$\therefore f(x) = x^4 + 8x^3 + 23x^2 + 28x + 12 = (x + 1)(x + 2)^2(x + 3)$$

$f(x)$ can be written in two ways as a product of quadratic polynomials.

$f(x) = \{(x + 1)(x + 2)\}\{(x + 2)(x + 3)\}$

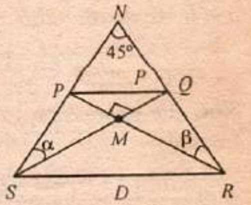
$= (x^2 + 3x + 2)(x^2 + 5x + 6)$

or $\quad f(x) = \{(x + 1)(x + 3)\}\{(x + 2)(x + 2)\}$

$= (x^2 + 4x + 3)(x^2 + 4x + 4)$

Consequently the solutions for (a, b, c, d) are

$(3, 5, 2, 6), (5, 3, 6, 2), (4, 4, 3, 4), (4, 4, 4, 3)$.



37. Let PR and QS meet at M. As $PQ \parallel RS$, triangle PMQ and SMR are similar with

$$\frac{PQ}{SR} = \frac{4}{10} = \frac{2}{5}$$

Let $PM = 2x$ and $QM = 2y$, then $MR = 5x$ and $MS = 5y$

As $\angle PMQ = 90°$

$$\text{area of the trapezoid} = \frac{1}{2} \times PR \times QS = \frac{1}{2} \times 7x \times 7y = \frac{49xy}{2} \qquad ...(1)$$

Let $\alpha = \angle PSM$ and $\beta = \angle QRM$

In right triangles PMS and QMR

$$\tan\alpha = \frac{PM}{PS} = \frac{2x}{5y}, \ \tan\beta = \frac{QM}{MR} = \frac{2y}{5x}$$

Now $\angle RMS + \angle MSR + \angle HRS = 180°$ (from ΔRMS)...(2)

Also $\angle SNR + \angle NSR + \angle NRS = 180°$ (from ΔNRS) ...(3)

Subtracting (2) from (3)

$\angle SNR + \angle NSM + \angle NRM - \angle RMS = 0$

$\Rightarrow \angle RMS = 45° + \alpha + \beta$

$\Rightarrow 90° = 45° + \alpha + \beta \therefore \alpha + \beta = 45°$

Now by $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

we have $\tan 45° = 1 = \dfrac{(2x/5y)+(2y/5x)}{1-(2x/5y)\cdot(2y/5x)}$

$$= \frac{10(x^2+y^2)}{21xy} \Rightarrow xy = \frac{10(x^2+y^2)}{21} \quad ...(4)$$

In triangle PMQ, we have $PQ^2 = PM^2 + MQ^2$

$\Rightarrow 16 = (2x)^2 + (2y)^2 \Rightarrow x^2 + y^2 = 4 \quad ....(5)$

from (4) and (5)

$xy = \dfrac{10}{21} \times 4 = \dfrac{40}{21}$

From (1) area of trapezoid $= \dfrac{49}{2}xy = \dfrac{49}{2} \times \dfrac{40}{21}$

$$= \frac{7\times 40}{2\times 3} = \frac{7\times 20}{3} = \frac{140}{3} \text{ sq.cm.}$$

38. Let $S = |\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x|$

Put $a = \sin x$, $b = \cos x$

Then $S = \left|a + b + \dfrac{a}{b} + \dfrac{b}{a} + \dfrac{1}{a} + \dfrac{1}{b}\right|$

$$= \left|\frac{ab(a+b)+a^2+b^2+a+b}{ab}\right| = \left|\frac{ab(a+b)+1+a+b}{ab}\right| ...(1)$$

Let $c = a + b = \cos x + \sin x$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right) = \sqrt{2}\sin\left(\frac{\pi}{4}+x\right)$$

So $C \in [-\sqrt{2}, \sqrt{2}]$

$a + b = c \Rightarrow a^2 + 2ab + b^2 = c^2 \Rightarrow 1 + 2ab = c^2$

$$\therefore\ ab = \frac{c^2-1}{2}$$

Now the expression to be minimized is equivalent to, from (1)

$$S = \left|\frac{2ab(a+b)+2+2(a+b)}{2ab}\right|$$

$$= \left|\frac{(c^2-1)c+2+2c}{c^2-1}\right| = \left|\frac{c^3+c+2}{c^2-1}\right| = \left|\frac{c(c^2-1)+2c+2}{c^2-1}\right|$$

$$= \left|c + \frac{2}{c-1}\right| = \left|c - 1 + \frac{2}{c-1} + 1\right| \quad ...(2)$$

As $c \in \left[-\sqrt{2}, \sqrt{2}\right]$, we consider two cases

If $c - 1 > 0$ then by AM-GM inequality

$$(c-1) + \frac{2}{(c-1)} > 2\sqrt{2}$$

If $c - 1 < 0$ then

$$(c-1) + \frac{2}{(c-1)} = -\left\{(1-c) + \frac{2}{(1-c)}\right\} \le -2\sqrt{2}$$

with equality holding if $1 - c = \dfrac{2}{1-c}$ i.e. $(1-c)^2 = 2$

$\Rightarrow 1 - c = \sqrt{2}\ \therefore\ c = 1 - \sqrt{2}$

The minimum value from (2) is then $\left|-2\sqrt{2}+1\right| = 2\sqrt{2} - 1$, attained when $c = 1 - \sqrt{2}$, i.e. $\sqrt{2}\sin\left(\dfrac{\pi}{4} + x\right) = 1 - \sqrt{2}$.

39. Denote by a_{ij} the number that is in the ith row and jth column. The entries of the chessboard are given by

$a_{ij} = 8(i-1) + j\ \forall\ i, j = 1, 2, ... 8$

From each row and each column exactly one number is chosen

Let a_1j_1, a_2j_2, a_3j_3, $\therefore$ a_8j_8 be the numbers chosen where $j_1, j_2, ... j_8$ is just a permutation of 1, 2, ... 8.

Sum of all number $= a_1j_1 + a_2j_2 + ... + a_8j_8$

$$= \sum_{k=1}^{8} a_k j_k = \sum_{k=1}^{8} \{8(k-1) + j_k\}$$

$$= \sum_{k=1}^{8} 8(k-1) + \sum_{k=1}^{8} j_k = 8\sum_{k=1}^{8}(k-1) + \sum_{k=1}^{8} j_k$$

$$= 8(1 + 2 + ... + 7) + (1 + 2 + ... + 8)$$

$$= 8 \times \frac{7\times 8}{2} + \frac{8\times 9}{2} = 224 + 36 = 260.$$

40. There are $\dfrac{100-1}{3} + 1 = 34$ elements in the the progression 1, 4, 7, ... 100.

We make pairs as follows

(4, 100), (7, 97), (10, 94), ... (49, 55)

Number of pairs $= \dfrac{49-4}{3} + 1 = 16$ pairs

We are giving to prove that we can choose a set of eighteen distinct numbers form the A.P. such that no two of them add up to 104. But once a nineteenth member is chosen, we will end up with two elements whose sum is 104.

Taking one of the number from each of the pairs, we can have 10 numbers and including 1 and 52 with these sixteen numbers, we now have 18 numbers. But no pair of numbers from these 18 numbers add upto 104, because just one number is selected from each pair and the other number of the pair, which would have made the total 104, is not selected.

So we can choose 18 numbers, such that no two of them add up to 104.

For getting 19 numbers, we will be forced to choose one of the 16 not chosen numbers, but then this number will add up with the number which was paired with it and they will add up to 104.

So, once a set of 19 distinct integers is chosen, we cannot escape the fast that some two of them add up to 104. ■

CONCEPTS & ANALYSIS

Domain of Parameters and Variables

By : Prof. S. S. Dahiya

In Mathematics the solution of a problem is considerable whereas the solution alone is not considerable, more important is the domain of parameters and the variables involved in the problem without which you may lead to wrong answer thereby giving you -1 marks instead of $+3$ marks.

The following examples are given for clarifications.

Example-1 : Find values of a for which

$$\int_{a/4}^{3a/4} \frac{x^n}{x^n+(a-x)^n}dx = \frac{2}{\pi}\int_0^{\infty} \frac{1}{x^2+a^2}dx$$

Evaluating the integerals we get $\frac{a}{4} = \frac{2}{\pi}\left(\frac{\pi}{2a}\right)$

$a^2 = 4, a = \pm 2$

If you select the answer value of a is $+2$ or -2, your answer is wrong

For values of parameter a, we consider

$\int_{a/4}^{3a/4} \frac{x^n}{x^n+(a-x)^n}dx$, here $\frac{3a}{4} \le x \le \frac{a}{4}$ for $a > 0$ and $\frac{a}{4} \le x \le \frac{3a}{4}$ for $a < 0$

For $a = -2$ we get $-\frac{1}{2} \le x \le -\frac{3}{2}$ for which $\frac{x^n}{x^n+(a-x)^n}$ may become discontinuous everywhere because value of n is not specified and hence may be any real number, if $n = \frac{25}{14}$, $x^{\frac{25}{14}}$ is imaginary because x is negative. $\therefore a = 2$

Example-2 : Variable chord AB of circle $x^2 + y^2 - 2x - 2y = 0$ is such that $\angle AOB$ is bisected by x-axis (O is origin)

The locus of mid-point of related variable chord AB is $x + y = 2$.

Your answer $x + y = 2$ is wrong because $x + y = 2$ is straight line extending to exterior part of the circle and part of a line which belongs to exterior of the circle cannot be chord of the circle. Therefore you are to consider the segment of the line playing the part of locus which is

$x + y = 2,\ 1 < x < 2$ or $0 < y < 1$

Example-3 : Find locus of point $P(x, y)$ such that sum of its distances from origin and from line $x = 2$ is always 4 units.

The locus subjected by the student is

$$y^2 = \begin{cases} 4(x+1), & \text{for } x \le 2 \\ 12(3-x), & \text{for } x \ge 2 \end{cases}$$

If this answer is analysed the further conclusions are $x + 1 \ge 0$, as well as $3 - x \ge 0$

Therefore the proper format of answer is

$$y^2 = \begin{cases} 4(x+1), & \text{for } -1 \le x \le 2 \\ 12(3-x), & \text{for } 2 \le x \le 3 \\ \text{No locus} & \text{for } x \in R-[-1,\ 3] \end{cases}$$

Example-4 : If $\theta = \tan^{-1}\sqrt{\frac{a\lambda}{bc}} + \tan^{-1}\sqrt{\frac{b\lambda}{ca}} + \tan^{-1}\sqrt{\frac{c\lambda}{ab}}$ then find value of θ given that $\lambda = a + b + c$, where a, b, c non zero real numbers.

Using various method, we conclude $\tan\theta = 0$ and $\frac{-3\pi}{2} < \theta < \frac{3\pi}{2}$ hence we get $\theta = \pi$, In fact possible values of θ are 0 and π, because $\lambda = 0$ when $a = 6$, $b = -4$, $c = -2$ we get $\theta = 0$

If $\frac{a\lambda}{bc}, \frac{b\lambda}{ca}, \frac{c\lambda}{ab}$ all are positive then $\theta = \pi$

Hence our answer is $\theta = 0$ or $\theta = \pi$.

Example-5 : Solve for x,

$\tan(3x) - \tan(2x) = 1 + \tan(3x)\tan(2x)$

Note that $3x \ne (2k-1)\frac{\pi}{2},\ 2x \ne (2k-1)\frac{\pi}{2}$, $k \in I$

On solving the equation we get $x = n\pi + \frac{\pi}{4}$, $n \in$ integers which is rejected because $2x \ne (2k-1)\frac{\pi}{2}$. Hence the answer we get is no solution.

Example-6 : If $\theta = \tan^{-1}\left(\frac{rx}{yz}\right) + \tan^{-1}\left(\frac{ry}{zx}\right) + \tan^{-1}\left(\frac{rz}{xy}\right)$ where x, y, z are non-zero real numbers and $r^2 = x^2 + y^2 + z^2$ then find values of θ.

Using various methods we conclude that $\tan\theta = 0$ and further $\frac{-3\pi}{2} < \theta < \frac{3\pi}{2}$. The conclusion that $\theta = 0$ or π

contd. on page no. 66

2. $1+\frac{3}{\lfloor 1}+\frac{5}{\lfloor 2}+...=\sum_{n=0}^{\infty}\frac{2n+1}{\lfloor n}=2\sum_{n=1}^{\infty}\frac{1}{\lfloor n-1}+\sum_{n=0}^{\infty}\frac{1}{\lfloor n}=3e$

3. Domain is $\{1, 2, 3, 4\}$, Range is $\{a, b, c\}$
No value of domain corresponds to a, hence one value to b, three value to c
No value of domain corresponds to a, hence two value to b, three value to c
No value of domain corresponds to a, hence three value to b, three value to c

Number of ways is ${}^4C_1+{}^4C_2+{}^4C_3=14$
No value of domain corresponds to a and b, hence all the values to c, 1 way
Total number of ways $= 3\times 14+3\times 1=45$.

4. $3^x+4^x+5^x=6^x$, graph of $y_1=3^x+4^x+5^x$ and $y_2=6^x$ may be considered for $x=1$, $y_1>y_2$;
for $x=2$, $y_1>y_2$, for $x=3$, $y_1=y_2$,
for $x=4$, $y_1<y_2$ the only root is $x=3$.

5. $a+b=4$ where a and b are length of two diagonals. Area of convex quadrilateral is

$\frac{1}{2}\left|\vec{d_1}\times\vec{d_2}\right|=\frac{1}{2}\cdot ab\sin\theta$ $\therefore$ $\frac{1}{2}ab\sin\theta=2$ or $ab=\frac{4}{\sin\theta}$,

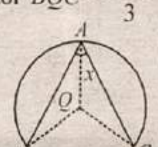
because $a+b=4$, hence $ab\le 4$

Therefore $\frac{4}{\sin\theta}\le 4$ or $\sin\theta\ge 1$
$\sin\theta>1$ is impossible, $\sin\theta=1$ gives $\theta=\pi/2$, diagonals are perpendicular.

6. In ΔABC, $1+\frac{r}{R}=\cos A+\cos B+\cos C$, angles of right angled triangle are $\frac{\pi}{2},\theta,\ \frac{\pi}{2}-\theta$ where θ is actue.
$1+\frac{r}{R}=\cos\theta+\sin\theta\le\sqrt{2}$ therefore $\frac{R}{r}\ge\left(\sqrt{2}+1\right)$.

7. Area ΔAQB + area ΔAQC + area sector $BQC=\frac{\pi r^2}{3}$

$r^2\sin(\pi-x)+\frac{1}{2}\left(r^2\right)(2x)=\frac{\pi r^2}{3}$



$x+\sin x=\frac{\pi}{3}$ or $x+\sin x-\frac{\pi}{3}=0$

Hence $f(x)=\lambda\left(x+\sin x-\frac{\pi}{3}\right)$, λ is non zero real no.

8. $\sin(mx+c)$ is periodic with period $\frac{2\pi}{m}$, $\frac{2\pi}{m}=\pi$ gives $m=2$
$\sin(2x+c)=0$ when x is 2, hence $c=-4$
$\therefore$ $g(x)=3+5\sin(2x-4)$.

9. Choice (c) is correct for all real values of x, y, z.

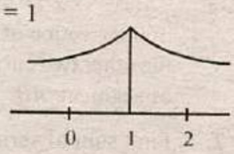
10. $24=3\times 2\times 2\times 2=(2+1)(1+1)(1+1)(1+1)$
Hence prime factors are 2, 2, 3, 5, 7,
number $=(2\times 2)\times 3\times 5\times 7=420$
Because $7>2\times 3$,
hence replace factor 7 by (2×3)
$\therefore$ break $24=4\times 3\times 2=(3+1)(2+1)(1+1)$
Number $=(2\times 2\times 2)\times(3\times 3)\times 5=360$
Answer is (c) i.e. 360.

11. (a) **12. (a)**

13. $\lambda=\frac{4}{81}, D=b^2-4ac=1-16\lambda=1-\frac{64}{81}>0$, real roots

14. Set of prime numbers
$\subset\{6k-1, 6k+1$ where $k\in N\}$ greater than 3
$k\in N$, $6k-1$ is composite of units place of k is 1 or 6 otherwise it is prime
$k\in N$, $6k+1$ is composite of units place of k is 4 or 9 otherwise it is prime
Three pirme number is A.P. are in the form $6k-1$, $6k-1+6$, $6k-1+12$ form
Three pirme number is A.P. are in the form $6k+1$, $6k+1+6$, $6k+1+12$ form
Hence common difference is 6 or multiple of 6

15. $e^{-(x-1)^2}$ is maximum 1 at $x=1$
Therefore $a-1=0$,
$a+1=2$ gives $a=1$
which is mid point of interval $[a-1, a+1]$, hence $a=1$



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or $-\pi$, here it may be noted that θ cannot be zero because either the three values $\frac{rx}{yz}$, $\frac{ry}{zx}$, $\frac{rz}{xy}$ all are positive or all the three are negative, hence $\theta=\pi$ or $\theta=-\pi$.

Example-7 : Find conditions on a and b for which

$\frac{\sec^4(\theta)}{a}+\frac{\tan^4(\theta)}{b}=\frac{1}{a+b}$ where $\theta\in R-(2k-1)\frac{\pi}{2}$,

k is an integer. $a\ne 0$, $b\ne 0$, $a+b\ne 0$.
Given that $b(a+b)\sec^4(\theta)+a(a+b)\tan^4(\theta)=ab$
Using $\sec^4(\theta)+\tan^4(\theta)=1+2\sec^2(\theta)\tan^2(\theta)$
we get $b\sec^2(\theta)+a\tan^2(\theta)=0$

$\therefore$ $\sec^2(\theta)=\frac{a}{a+b}$ and $\tan^2(\theta)=\frac{-b}{a+b}$. $\therefore$ $ab<0$

Further $\sin^2(\theta)=\frac{-b}{a}$, hence $|b|<|a|$
Therefore the answer $ab<0$ is wrong, the proper answer is $ab<0$ and $|b|<|a|$

MATHS FORUM

by Prof. S.S. Dahiya

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1. Graphs of $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \operatorname{cosec}(x)$ are drawn on same axes system for $x \in \left(0, \frac{\pi}{2}\right)$, A vertical line is drawn through point of intersection of $y = \cos x$ and $y = \tan x$ intersecting the other two curves at points A and B. Find length of segment AB.

2. Find sum of series
$$1 + \frac{3}{\lfloor 1} + \frac{5}{\lfloor 2} + \frac{7}{\lfloor 3} + \frac{9}{\lfloor 4} + \ldots\ \text{infinite terms.}$$

3. Find number of into functions $f : A \to B$ where $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$.

4. Find number of roots of equation $3^x + 4^x + 5^x = 6^x$.

5. Sum of diagonals of trapezium is 4 units and its area is 2 square units, find angle between the diagonals. *[Pratham Saxena, Jaipur]*

6. Saurav Talukar has sent two constructions, right angled triangles are formed, by actual measurements, in first triangle $R/r = 2.5$, in second construction $\frac{R}{r} = 2.6$, explain reasons for this ratio to be 2.5, 2.6 etc. *[Saurav Talukar, Assam]*

7. A is point on circumference of a circle, chords AB and AC divide circle area in three equal parts, $f(x) = 0$ is an equation whose root is angle BAC, find $f(x)$.

8. Find function $g(x)$ such that
(a) domain is $(-\infty, \infty)$ (b) range is $[-2, 8]$
(c) period is π (d) $g(2) = 3$.
[K. Deepak, Kota]

9. $\{x\} = x - [x]$ where $[\lambda]$ is integeral part of λ then which of the following is correct
(a) $[x + y + z] = [x] + [y] + [z]$
(b) $\{x + y + z\} = \{x + y\} + \{z\}$
(c) $[x + y + z] = [x + y] + [z + \{x + y\}]$
(d) None of these.

10. Smallest natural number which has 24 divisors is
(a) 420 (b) 240
(c) 360 (d) None of these.

11. $0 \le x \le \frac{\pi}{2}$, which of the following is true
(a) $\sin(\cos x) < \cos(\sin x)$
(b) $\sin(\cos x) > \cos(\sin x)$
(c) $\sin(\cos x) \ge \cos(\sin x)$
(d) None of these.

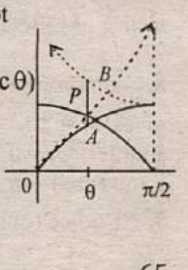
12. a, b, c are non zero distinct real numbers, If $a + b + c = 0$ then value of
$$\frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ac} + \frac{c^2}{2c^2 + ab}\ \text{is}$$
(a) 1 (b) -1
(c) 0 (d) None of these.

13. a, b, c are positive and $a + b + c = 1$
If $\lambda = \min\{a^3 + a^2bc,\ b^3 + b^2ac,\ c^3 + c^2ab\}$, then prove that root of $x^3 \pm x + 4\lambda = 0$ are real.

14. If three prime number (greater than 3) are in A.P. then their common difference is divisible by
(a) 2 but not by 3 (b) 3 but not by 2
(c) 2 and 3 both (d) None of these.

15. For maximum value of $\int_{a-1}^{a+1} e^{-(x-1)^2}\,dx$ value of 'a' is
(a) 0 (b) 1
(c) -1 (d) None of these.
[Ranganathan, Vijaywara]

SOLUTIONS



1. Point P is $(\theta, \cos\theta)$ where θ is root of $\cos\theta = \tan\theta$ ....(i)
Point A is $(\theta, \sin\theta)$, Point B is $(\theta, \operatorname{cosec}\theta)$
$\therefore\ AB = \operatorname{cosec}\theta - \sin\theta$
$$= \frac{1 - \sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta} = 1\ \text{[Using (i)]}$$

ISI 2006

2006 121

MOCK TEST FOR ISI 2006

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(For details of exam refer to April issue of Mathematics Today).

By **Alok kumar**, B.Tech, IIT Kanpur

MULTIPLE CHOICE TEST

1. The number of triplets (x, y, z) satisfying the system of equations

$x+y=\sqrt{4z-1},\ y+z=\sqrt{4x-1},\ z+x=\sqrt{4y-1}$ is

(a) 4 (b) 3 (c) 1 (d) 0.

2. Consider the sequence a_n given by $a_1 = 1/2$, $a_{n+1}=a_n^2+a_n$. Let $S=\frac{1}{a_1+1}+\frac{1}{a_2+1}+\ldots+\frac{1}{a_{2006}+1}$.

then $[S]$, the greatest integer less than or equal to sum S, is

(a) 2 (b) 3 (c) 4 (d) 1.

3. Two circles with radii 2 and 8 cm touch each other externally. Also drawn to these two circles are three common tangents intersecting each other at three points. The area of the triangle (in sq. cm) formed by joining these three intersection points is

(a) 16 (b) 64/3 (c) 256/3 (d) 24.

4. In an examination, 41% candidate failed in Music and 32% failed in Mathematics. Also, 13% failed in both the subjects. If the number of candidates who passed in Music alone is 399, then the total number of candidates is

(a) 2100 (b) 2000 (c) 2400 (d) 5200.

5. The three sides of a right-angled triangles are in G.P. The tangents of the two acute angles are

(a) $\frac{\sqrt{\sqrt{5}+1}}{2}, \frac{2}{\sqrt{1+\sqrt{5}}}$ (b) $\frac{\sqrt{1+\sqrt{5}}}{2}, \sqrt{\frac{2}{1+\sqrt{5}}}$

(c) $\sqrt{\frac{\sqrt{5}-1}{2}}, \sqrt{\frac{2}{\sqrt{5}+1}}$ (d) $\frac{2}{\sqrt{\sqrt{5}-1}}, \frac{\sqrt{\sqrt{5}-1}}{2}$

6. Consider the two Arithmetic progressions 61, 21, 26, ... and 21, 25, 29, 33, ... The sum of the first 100 common terms of the two progressions is

(a) 101010 (b) 110010 (c) 101100 (d) 110100

7. Let $x_n=\frac{n}{(n-1)^{4/3}+n^{4/3}+(n+1)^{4/3}}$. Then the sum $S=x_1+x_2+\ldots.+x_{999}$ satisfies

(a) $S>200$ (b) $S>100$ (c) $S>60$ (d) $S<50$

8. Consider the two numbers $A=2^{2^{25}-2}+1$, $B=2^{2^{16}-2}+1$. Then about the numbers A and B which of the following statements is true?

(a) A is prime but B is composite

(b) A is composite but B is prime

(c) A and B are both composite

(d) A and B are both prime.

9. Let $g: R \to R$ be a function satisfying the two conditions

(i) $g(x+y)+g(x-y)=2g(x)\ \forall\ g(y)\ x, y \in R$

(ii) there exists a such that $g(a)=-1$, $a>0$

Then which of the following statement is false?

(a) $g(x)=g(x+4a)$ (b) $g(x+a)=g(x+9a)$

(c) $g(x)=g(x+16a)$ (d) $g(x)\neq g(x+2a)$.

10. The polynomial $x^6-\frac{11}{8}x^3+1$, when divided by $x^2+\frac{x}{2}+1$, leaves a remainder

(a) $\frac{1}{4}x+\frac{5}{8}$ (b) $\frac{2}{3}x+\frac{4}{5}$

(c) $\frac{1}{4}x-\frac{5}{6}$ (d) 0.

11. The mth term of an arithmetic progression is x and the nth term is y. Then the sum of the first $(m+n)$terms is

(a) $\frac{(m+n)}{2}\left[x+y+\frac{x-y}{m-n}\right]$

(b) $\frac{(m+n)}{2}\left[x+y+\frac{y-x}{m-n}\right]$

(c) $\frac{(m+n)}{2}\left[x+y+\frac{2(y-x)}{m-n}\right]$

(d) $\frac{(m+n)}{2}\left[x+y+\frac{2(x-y)}{m-n}\right]$

12. ABC is a right-angled triangle, $AC = 6$ cm and $\angle BAC=\pi/3$. I is the incentre of the triangle ABC, then BI is (in cm)

(a) $2(\sqrt{6}-\sqrt{3})$ (b) $2(\sqrt{6}-\sqrt{2})$

(c) $2(\sqrt{6}-2)$ (d) $2(2\sqrt{6}-\sqrt{3})$

13. Let a, b, c, d be distinct positive integers such that $a^5 + b^5 = c^5 + d^5$. Then of the following statements which one is false?
(a) $|a-c| + |b-d| \geq 5$ (b) $|a-c| + |b-d| \geq 4$
(c) $|a-c| + |b-d| \geq 1$ (d) $|a-c| + |b-d| < 3$

14. The sum $S = \cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7}$ equals
(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

15. The value(s) of λ for which the equation
$|x-58| + |x-65| + |x+58| + |x+65| = 123\lambda$
has atleast one real solution is given by
(a) $\lambda < 1$ (b) $\lambda < 2$ (c) $\lambda \geq 2$ (d) $\lambda \leq 1$

16. If $\dfrac{x^2+y^2-u^2-v^2}{x-y+u-v} = \dfrac{x^2-y^2-u^2+v^2}{x+y+u+v}$, then which of the following is true?
(a) $\dfrac{xy-uv}{x-y+u-v} = \dfrac{uy-vx}{x+y+u+v}$
(b) $\dfrac{xy-uv}{x-y+u-v} = \dfrac{vx-uy}{x+y+u+v}$
(c) $\dfrac{2(xy-uv)}{x-y+u-v} = \dfrac{uy-vx}{x+y+u+v}$
(d) $\dfrac{2(uv-xy)}{x-y+u-v} = \dfrac{uy-vx}{x+y+u+v}$.

17. The number of solutions of the following systems of equations $\{x\} + y + (z) = 3.1$, $x + [y] + \{z\} = 2.4$, $[x] + \{y\} + z = 1.3$ is, (where $[x]$ and $\{x\}$ denote the integral and fractional part of x respectively)
(a) exactly two (b) infinite
(c) exactly one (d) none.

18. Let $x, y, z > 0$ and denoted by R the expression
$$R = \frac{x}{2x+y+z} + \frac{y}{x+2y+z} + \frac{z}{x+y+2z}$$
Then R satisfies
(a) $R > 2$ (b) $R \leq 3/4$ (c) $R > 1$ (d) $R > 5/4$.

19. Let $f_n(x) = \frac{1}{n}(\sin^n x + \cos^n x)$ for $n = 1, 2, \ldots$ Then $f_4\left(\frac{3\pi}{8}\right) - f_6\left(\frac{3\pi}{8}\right)$ equals
(a) $\frac{2}{\sqrt{2}+1}$ (b) $\frac{1}{12}$ (c) $\frac{4}{\sqrt{3}-1}$ (d) $\frac{1}{8}$

20. A triangle ABC has $A - B = 120°$. And the circumradius is 8 times the inradius. The value of $\cos C$ equals
(a) $\frac{7}{8}$ (b) $\frac{15}{16}$ (c) $\frac{3}{4}$ (d) $\frac{2}{3}$

21. If $\sin\alpha \cos\beta = -1/2$, then the range of all possible values of $\cos\alpha \sin\beta$ is the interval
(a) $[-1/2, 0]$ (b) $[0, 1/2]$
(c) $(-1/2, 1/2)$ (d) $[-1/2, 1/2]$.

22. Let x and y be real numbers in the interval $[0, \pi/2]$. Then $\sin^6 x + 3\sin^2 x \cos^2 y + \cos^6 y = 1$ holds iff
(a) $x + y = \pi/2$ (b) $x - y = \pi/4$
(c) $x = y$ (d) $x = 2y$.

23. Let $f(a, b, c) = \dfrac{a^2}{2a^2+bc} + \dfrac{b^2}{2b^2+ca} + \dfrac{c^2}{2c^2+ab}$. Then the value of $f(121, -341, 220)$ is
(a) 2 (b) -1 (c) 0 (d) 1.

24. Let n be the sum of three cubes, $n = 1001^3 + 1011^3 + 1000^3$. Then n divided by 6 leaves a remainder of
(a) 3 (b) 2 (c) 1 (d) 0.

25. Suppose $f(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c, d are constants. If $f(1) = 10, f(2) = 20, f(3) = 30$, then the value of $\dfrac{f(10)+f(-6)}{8}$ is
(a) 1013 (b) 1012 (c) 1011 (d) 1010.

26. The highest power of 5 in the number $\underline{|2934}$ is given by
(a) $\dfrac{2934 - \text{Sum of the digits appearing in base 5 representation of 2934}}{6}$
(b) $\dfrac{2934 - \text{Sum of the digits appearing in base 5 representation of 2934}}{4}$
(c) $\dfrac{1467 - \text{Sum of the digits appearing in base 5 representation of 2934}}{4}$
(d) $\dfrac{5868 - \text{Sum of the digits appearing in base 5 representation of 2934}}{6}$

27. How many distinct real solutions does the equation $(x^2-3x+3)^2 - 3(x^2-3x+3) + 3 = x$ have ?
(a) exactly one (b) exactly two
(c) exactly three (d) all four.

28. For how may integral values of n, $(n > 11)$ is the expression $n^2 - 19n + 89$ a perfect square?
(a) exactly 7 (b) exactly 3
(c) exactly 5 (d) none of these.

29. The equation of the lowest degree with integral coefficients and having one solution $x_1 = 1 + \sqrt{2} + \sqrt{3}$ is
(a) $x^4 - 4x^3 + 4x^2 - 16x - 8 = 0$
(b) $x^4 + 4x^3 + 4x^2 - 16x + 8 = 0$

(c) $x^4 - 4x^3 - 4x^2 + 16x - 8 = 0$
(d) $x^4 + 4x^3 - 4x^2 + 16x + 8 = 0$.

30. The perimeter of a cyclic quadrilateral $PQRS$ is 80 cm. Two sides PS and QR are extended to meet at a point M, where $MS = 40$ cm and $MR = 20$ cm. If a tangent drawn at a point P, meets QR at point N, where $NP = 6$cm, $NQ = 2$cm, the area of the quadrilateral $PQRS$ (in sq.cm), if $RS = 6$ cm, is

(a) $4\sqrt{510}$ (b) $8\sqrt{340}$ (c) $4\sqrt{340}$ (d) $8\sqrt{510}$

SHORT ANSWER TYPE TEST

31. Let a, b and c be given positive numbers. Determine, with problem, all positive real numbers x, y and z such that

$$x + y + z = a + b + c$$
$$4xyz - (a^2x + b^2y + c^2z) = abc$$

32. Let $f : [a, b] \to R$ be a continuous function, differentiable on (a, b). Also assume that the function doesn't vanish on (a, b). Show that there exists $c \in (a, b)$ such that $\dfrac{f'(c)}{f(c)} = \dfrac{1}{a-c} + \dfrac{1}{b-c}$

33. For any positive integer k, prove that

$$2\left(\sqrt{k+1} - \sqrt{k}\right) < \frac{1}{\sqrt{k}} < 2\left(\sqrt{k} - \sqrt{k-1}\right)$$

Also compute the integral part of the sum

$$S = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{10,000}}.$$

34. For a regular heptagon $A_1A_2A_3A_4A_5A_6A_7$, prove that

$$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}.$$

35. Let ABC be a triangle such that

$$\left(\cot\frac{A}{2}\right)^2 + \left(2\cot\frac{B}{2}\right)^2 + \left(3\cot\frac{C}{2}\right)^2 = \left(\frac{63}{7r}\right)^2,$$

where s and r denotes its semiperimeter and its radius, respectively. Show that the triangle ABC is similar to a triangle with side lengths 13, 40 and 45.

36. Suppose n is a positive integer. Find the roots of the polynomial

$$P_n(x) = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!} + ... + \frac{x(x+1)...(x+n-1)}{n!}.$$

37. Suppose x, y, z be real numbers with $x \geq y \geq z \geq \frac{\pi}{\sqrt{2}}$ such that $x + y + z = \frac{\pi}{2}$. Find out the maximum and minimum values of the product $P = \cos x \sin y \cos z$ and also the point at which they are attained.

38. Determine with proof, whether the number $N = 99899$ is prime or composite.

39. Find, with proof, real numbers a, b, c such that

$$|ax + by + cz| + |bx + cy + az| + |cx + ay + bz| = |x| + |y| + |z|$$ holds for all real numbers x, y, z.

40. Prove that, for any triangle with sides a, b, c and area A, $a^2 + b^2 + c^2 \geq 4\sqrt{3}A$.

SOLUTION

1. **(c)** : We have
$$x + y = \sqrt{4z-1} \quad ...(i)$$
$$y + z = \sqrt{4x-1} \quad ...(ii)$$
$$z + x = \sqrt{4y-1} \quad ...(iii)$$

Adding all of them, we get

$$\Rightarrow 2x + 2y + 2z - \sqrt{4x-1} - \sqrt{4y-1} - \sqrt{4z-1} = 0$$
$$\Rightarrow x + y + z - \sqrt{x - \frac{1}{4}} - \sqrt{y - \frac{1}{4}} - \sqrt{z - \frac{1}{4}} = 0 \quad ...(A)$$

Note that $x - \frac{1}{4} - \sqrt{x - \frac{1}{4}} + \frac{1}{4} = \left(\sqrt{x - \frac{1}{4}} - \frac{1}{2}\right)^2 \quad ...(B)$

$$\left(\sqrt{x - \frac{1}{4}} - \frac{1}{2}\right)^2 + \left(\sqrt{y - \frac{1}{4}} - \frac{1}{2}\right)^2 + \left(\sqrt{z - \frac{1}{4}} - \frac{1}{2}\right)^2 = 0$$

(From (A) and (B))

so each of these squares must equal zero, giving

$$\sqrt{x - \frac{1}{4}} = \frac{1}{2}, \sqrt{y - \frac{1}{4}} = \frac{1}{2}, \sqrt{z - \frac{1}{4}} = \frac{1}{2}$$

$\Rightarrow x = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ similarly $y = z = \frac{1}{2}$

Thus $(x, y, z) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the only solution.

2. **(d)** : $a_{n+1} = a_n^2 + a_n = a_n(a_n + 1)$

$$\Rightarrow \frac{1}{a_{n+1}} = \frac{1}{a_n(a_n+1)} = \frac{1}{a_n} - \frac{1}{a_n+1} \Rightarrow \frac{1}{a_n+1} = \frac{1}{a_n} - \frac{1}{a_{n+1}} \quad ...(i)$$

Now $S = \dfrac{1}{a_1+1} + \dfrac{1}{a_2+1} + ... + \dfrac{1}{a_{2006}+1}$

$= \left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) + ... + \left(\frac{1}{a_{2006}} - \frac{1}{a_{2007}}\right)$, using (i)

$= \frac{1}{a_1} - \frac{1}{a_{2007}}$ ...(ii) Again $a_1 = 1/2$

$$\Rightarrow a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4},\ a_3 = \frac{3}{4} + \frac{9}{16} = \frac{21}{16} > 1$$

$a_4 > 1$ .... $a_{2007} > 1$ ...(iii) $\therefore 0 < \frac{1}{a_{2007}} < 1$

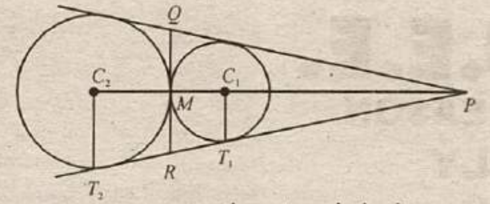
so let $\frac{1}{a_{2007}} < 1 - f,\ 0 < f < 1$

From (ii), $S = \frac{1}{a_1} - \frac{1}{a_{2007}} = 2 - (1 - f) = 1 + f, 0 < f < 1$

$\therefore [S] = 1$.

3. **(b)** : Let C_1 and C_2 be the centre of the circles with radii r_1 and r_2 respectively. Also, it is given that the two

circles touch each other externally. Let this point be called M.



Three common tangent to these two circles intersect each other at P, Q and R.

We have triangles PT_1C_1 and PT_2C_2 as similar

$$\Rightarrow \frac{PC_1}{PC_2} = \frac{C_1T_1}{C_2T_2} = \frac{r_1}{r_2}$$

$$\Rightarrow PC_1 = \frac{r_1}{r_2}PC_2 = \frac{r_1}{r_2}(PC_1 + r_1 + r_2)$$

$$(\because\ PC_2 = PC_1 + C_1C_2 = PC_1 + r_1 + r_2)$$

$$\Rightarrow PC_1\left(1 - \frac{r_1}{r_2}\right) = \frac{r_1}{r_2}(r_1 + r_2)$$

$$\Rightarrow PC_1 = \frac{r_1(r_1+r_2)}{r_2 - r_1} = \frac{r_1(1+\lambda)}{1-\lambda},\ \lambda = \frac{r_1}{r_2}$$

Now $PM = PC_1 + MC_1 = \dfrac{r_1(1+\lambda)}{1-\lambda} + r_1 = \dfrac{2r_1}{1-\lambda}$

Again $PT_1^2 = PC_1^2 - C_1T_1^2$

$$= r_1^2\frac{(1+\lambda)^2}{(1-\lambda)^2} - r_1^2 = r_1^2 \cdot \frac{4\lambda}{(1-\lambda)^2} \quad \therefore\ PT_1 = \frac{2r_1\sqrt{\lambda}}{1-\lambda}$$

Again ΔPT_1C and ΔPMR are similar

$$\Rightarrow \frac{PT_1}{C_1T_1} = \frac{PM}{MR} \Rightarrow MR = PM \times \frac{C_1T_1}{PT_1}$$

$$= \frac{2r_1}{1-\lambda} \times r_1 \times \frac{(1-\lambda)}{2r_1\sqrt{\lambda}} = \frac{r_1}{\sqrt{\lambda}}$$

area $(PQR) = 2$ area $(PMR) = 2 \times \frac{1}{2} \times PM \times MR$

$$= \frac{2r_1}{1-\lambda} \times \frac{r_1}{\sqrt{\lambda}} = \frac{2r_1^2}{(1-\lambda)\sqrt{\lambda}}$$

We have $r_1 = 2,\ \ \lambda = \dfrac{r_1}{r_2} = \dfrac{2}{8} = \dfrac{1}{4}$

$$\therefore \text{ area } (PQR) = \frac{2 \times 4}{\left(1 - \frac{1}{4}\right) \times \frac{1}{2}} = \frac{8}{\frac{3}{4} \times \frac{1}{2}} = \frac{64}{3} \text{ sq.cm.}$$

4. **(a)** : Number of candidates passed in music alone
= Number of candidates failed in Maths alone
= Number of candidates failed in Maths
– Number of candidates failed in both subjects
= (32 – 13) = 19%

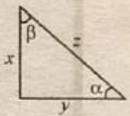
But the number of candidates who passed in Music alone is 399

$\therefore$ Total number of candidates

$$= \frac{399}{19} \times 100 = 21 \times 100 = 2100.$$

(Remark : The piece of data about the candidates failed in music is superfluous).

5. **(b)** : Let the hypotenuse of the triangle be z and the legs be x and y.



Then $z^2 \neq xy$ as $z > x$, y so either $x^2 = yz$ or $y^2 = xz$

Let us take $x^2 = yz$, $x^2 = yz$ and $x^2 + y^2 = z^2$ give

$yz + y^2 = z^2 \Rightarrow y^2 + yz - z^2 = 0$

$$\Rightarrow \left(\frac{y}{z}\right)^2 + \left(\frac{y}{z}\right) - 1 = 0 \ (\because\ z \neq 0 \text{ division is allowed})$$

$$\frac{y}{z} = \frac{-1 \pm \sqrt{1+4}}{2} \quad \therefore\ \frac{y}{z} = \frac{-1+\sqrt{5}}{2}$$

$$\cos\alpha = \frac{-1+\sqrt{5}}{2},$$

$$\sin\alpha = \sqrt{1 - \frac{(\sqrt{5}-1)^2}{4}} = \sqrt{1 - \frac{5+1-2\sqrt{5}}{4}} = \sqrt{\frac{\sqrt{5}-1}{2}}$$

$$\tan\alpha = \frac{\sqrt{\sqrt{5}-1}}{\sqrt{2}} \times \frac{2}{\sqrt{5}-1} = \frac{\sqrt{2}}{\sqrt{\sqrt{5}-1}} = \sqrt{\frac{1+\sqrt{5}}{2}}$$

$$\tan\beta = \tan(90° - \alpha) = \cot\alpha = \sqrt{\frac{2}{1+\sqrt{5}}}.$$

6. **(c)** : Let $A = \{16, 21, 26, 31, ...\}$ be the A.P. with first term 16 and common difference 5.

Its mth term is

$$a_m = 16 + (m-1)5 = 5m + 11,\ m = 1, 2, ...$$

Again let $B = \{21, 25, 29, 33, ...\}$ be the A.P. with first term 21 and common difference 4.

Its nth term is

$$b_n = 21 + (n-1) \times 4 = 4n + 17,\ n = 1, 2, ...$$

To see the common term we equate a_m and b_n

$a_m = b_n \Rightarrow 5m + 11 = 4n + 17$ ...(1)

The solution in integers of (i) will determine the common terms.

(i) $\Rightarrow 5m - 4n = 6$

Note that $(m, n) = (2, 1)$ is a solution. Rewrite

$5m - 4n = 6 = 5 \times 2 - 4 \times 1 \Rightarrow 5(m-2) = 4(n-1)$

From the equation $5 \mid 4(n-1)$ but $5 \nmid 4 \ \therefore\ 5 \mid n-1$

Then $n - 1 = 5t \Rightarrow n = 5t + 1$, $t = 0, 1, ...$

Then $m - 1 = 4t \Rightarrow m = 4t + 1$, $t = 0, 1, ...$

So the common terms are a_{4t+2} (or b_{5t+1}), $t = 0, 1, ...$

i.e., they are $a_2, a_6, a_{10}, ...$ or $b_1, b_6, b_{11}, ...$

$a_{4t+2} = 5(4t+2) + 11 = 20t + 21$

$a_{4(t+1)+2} = 20(t+1) + 21$

Thus $a_{4(t+1)+2} - a_{4t+2} = 20$, giving us that $a_2, a_6, ...$ are in A.P. Hence common terms, when seen as terms of first

AP, are a_2, a_6, a_{10} and their common difference is 20.

$\therefore$ Sum of first 100 common terms

$$= \frac{100}{2}\left[2\times 21+(100-1)\times 20\right] = 10100.$$

7. **(d)** : As $n^{4/3} = (n^2)^{2/3} > (n^2 - 1)^{2/3}$
$= (n-1)^{2/3}(n+1)^{2/3}$

we have $x_n < \dfrac{n}{(n-1)^{4/3}+(n-1)^{2/3}(n+1)^{2/3}+(n+1)^{4/3}}$

$$= \frac{n\{(n+1)^{2/3}-(n-1)^{2/3}\}}{[(n+1)^{4/3}+(n+1)^{2/3}(n-1)^{2/3}+(n-1)^{4/3}][(n+1)^{2/3}-(n-1)^{2/3}]}$$

$$= \frac{1}{4}\{(n+1)^{2/3}-(n-1)^{2/3}\} \qquad ...(A)$$

(We've used $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ to simplify the above expression

using (A)

$$x_1 < \frac{1}{4}\left\{2^{2/3} - 0^{2/3}\right\}$$
$$x_2 < \frac{1}{4}\left\{3^{2/3} - 1^{2/3}\right\}$$
$$x_3 < \frac{1}{4}\left\{4^{2/3} - 2^{2/3}\right\}$$
$$x_4 < \frac{1}{4}\left\{5^{2/3} - 3^{2/3}\right\}$$

....

$$x_{999} < \frac{1}{4}\left\{1000^{2/3} - 998^{2/3}\right\}$$

Adding them

$$S = x_1 + x_2 + x_{999} < \frac{1}{4}\{1000^{2/3}+999^{2/3}-1^{2/3}-0^{2/3}\}$$
$$< \frac{1}{4}\{100+100-1\} < 50$$

Thus $S < 50$.

8. **(c)** : Observe that $1+\dfrac{x^4}{4}$ can be factored

$$1+\frac{x^4}{4} = 1+\left(\frac{x^2}{2}\right)^2 + 2\cdot 1\cdot\frac{x^2}{2} - x^2$$
$$= \left(1+\frac{x^2}{2}\right)^2 - x^2 = \left(1+x+\frac{x^2}{2}\right)\left(1-x+\frac{x^2}{2}\right)$$

Consider $1+2^{2^n-2} = 1+\dfrac{1}{4}\cdot 2^{2^n} = 1+\dfrac{1}{4}\cdot\left(2^{2^{n-2}}\right)^4$

$$= 1+\frac{1}{4}x^4 \text{ where } x = 2^{2^{n-2}},\ n\geq 3$$
$$= \left(1+x+\frac{x^2}{2}\right)\left(1-x+\frac{x^2}{2}\right)$$
$$= \left\{1+2^{2^{n-2}}+\frac{(2^{2^{n-2}})^2}{2}\right\}\left\{1-2^{2^{n-2}}+\frac{(2^{2^{n-2}})^2}{2}\right\}$$
$$= (1+2^{2^{n-2}}+2^{2^{n-1}-1})(1-2^{2^{n-2}}+2^{2^{n-1}}-1)$$

Hence $A = 1+2^{2^5-2} = (1+2^{2^3}+2^{2^4-1})(1-2^{2^3}+2^{2^4-1})$

Then A is composite. Similarly B is composite too.

9. **(d)** : $g(x+y)+g(x-y) = 2g(x)g(y)$

Set $x = y = 0 \Rightarrow g(0)+g(0) = 2g^2(0)$ $\quad [g^2(0) = (g(0))^2]$

$\Rightarrow g(0) = g^2(0)$ $\therefore$ $g(0) = 0$ or 1

Case 1: $g(0) = 0$, then set $x = a$, $y = 0$ we obtain

$g(a)+g(a) = 2g(a)g(0) \Rightarrow 2g(a) = 2g(a)g(0)$

$\Rightarrow g(a) = g(a)\times 0 = 0$

Then $g(a) = 0$ which is impossible $(g(a) = -1$ given)

$\therefore$ $g(0) = 1$

For $x = y = a$, we get $g(2a) = 1$

Replace x by $x+2a$ and y by $x-2a$ to get

$g(2x)+g(4a) = 2g(x+2a)g(x-2a)$ $\qquad$ ...(A)

Again $g(2x) = 2g^2(x)-1$ and $g(4a) = 2g^2(2a)-1$
$= 2\times 1 - 1 = 1$

Now (A) becomes $2g^2(x)-1+1 = 2g(x+2a)g(x-2a)$

$\Rightarrow g(x+2a)g(x-2a) = (g(x))^2$ $\qquad$ ...(B)

Similarly for x arbitrary and $y = 2a$, we get

$g(x+2a)+g(x-2a) = 2g(x)g(2a) = 2g(x)$

$(\because\ g(2a) = 1)$ $\qquad$ ...(C)

From (B) and (C), $g(x+2a) = g(x-2a) = g(x)$

Thus $g(x)$ has period $2a$. Hence choice (d) is false.

10. **(d)** : $\left(x^2+\dfrac{x}{2}+1\right)\left(x^2-\dfrac{x}{2}+1\right) = (x^2+1)^2 - \dfrac{x^2}{4}$

$$= x^4+2x^2+1-\frac{x^2}{4} = x^4+\frac{7}{4}x^2+1 \qquad ...(A)$$

Again $x^3+\dfrac{1}{x^3} = \left(x+\dfrac{1}{x}\right)\left(x^2+\dfrac{1}{x^2}\right)-\left(x+\dfrac{1}{x}\right)$

$$= \left(x+\frac{1}{x}+\frac{1}{2}\right)\left(x^2+\frac{1}{x^2}\right)-\left(x+\frac{1}{x}+\frac{1}{2}\right)$$
$$-\frac{1}{2}\left(x^2+\frac{1}{x^2}+\frac{7}{4}\right)+\frac{1}{2}+\frac{1}{2}\cdot\frac{7}{4}$$

$$\Rightarrow x^3+\frac{1}{x^3}-\frac{11}{8} = \left(x+\frac{1}{x}+\frac{1}{2}\right)\left(x^2+\frac{1}{x^2}\right)$$
$$-\left(x+\frac{1}{x}+\frac{1}{2}\right)-\frac{1}{2}\left(x^2+\frac{1}{x^2}+\frac{7}{4}\right)$$

$$\Rightarrow x^6-\frac{11}{8}x^3+1 = \left(x^2+\frac{x}{2}+1\right)(x^4+1)$$
$$-\left(x^2+\frac{x}{2}+1\right)x^2-\frac{1}{2}x\left(x^4+\frac{7}{4}x^2+1\right)$$

$$= \left(x^2+\frac{x}{2}+1\right)(x^4+1)-\left(x^2+\frac{x}{2}+1\right)x^2$$
$$-\frac{1}{2}x\left(x^2+\frac{x}{2}+1\right)\left(x^2-\frac{x}{2}+1\right) \quad \text{(from A)}$$

$$= \left(x^2+\frac{x}{2}+1\right)\left\{x^4+1-x^2-\frac{1}{2}x\left(x^2-\frac{x}{2}+1\right)\right\}$$

Thus $x^2+\frac{x}{2}+1$ divides $x^6-\frac{11}{8}x^3+1$.

11. (a) : Denote by a the first term of the A.P. and d the common difference.

Then $\quad x = a + (m-1)d \quad$ ....(i)

$\qquad\quad y = a + (n-1)d \quad$ ....(ii)

From (i) and (ii) on subtraction

$$y - x = (n-m)d \Rightarrow d = \frac{x-y}{m-n} \quad ...(A)$$

Sum of the first $(m+n)$ terms

$$= \frac{m+n}{2}\left[2a+(m+n-1)d\right] \quad ...(iii)$$

From (i) and (ii) on eliminating d

$$(n-1)x-(m-1)y = a(\overline{n-1} - \overline{m-1})$$

$$\Rightarrow a = \frac{(n-1)x-(m-1)y}{n-m} = \frac{(m-1)y-(n-1)x}{m-n} \quad ...(B)$$

(A) and (B) when plugged in (iii) yield

$$\text{Sum} = \frac{m+n}{2}\left[2\cdot\frac{(m-1)y-(n-1)x}{m-n}+(m+n-1)\frac{x-y}{m-n}\right]$$

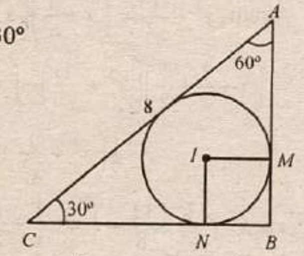
$$= \frac{m+n}{2}\left[2\cdot\frac{my-nx}{m-n}+2\frac{x-y}{m-n}+(m+n)\frac{x-y}{m-n}-\frac{x-y}{m-n}\right]$$

$$= \frac{m+n}{2}\left[\frac{2(my-nx)+(m+n)(x-y)}{(m-n)}+\frac{x-y}{m-n}\right]$$

$$= \frac{m+n}{2}\left[x+y+\frac{x-y}{m-n}\right] \qquad (\because m \neq n)$$

12. (b) : $AB = (AC)\sin 30°$

$$= 8\times\frac{1}{2} = 4 \text{ cm}$$



$BC = (AC)\cos 30°$

$$= 8\times\frac{\sqrt{3}}{2} = 4\sqrt{3} \text{ cm}$$

$$\text{area } (ABC) = \frac{1}{2}\times 4\times 4\sqrt{3} = \frac{16\sqrt{3}}{2} = 8\sqrt{3} \text{ cm}^2$$

$$r = \text{inradius} = \frac{\Delta}{s} = \frac{8\sqrt{3}}{\frac{4+4\sqrt{3}+8}{2}} = \frac{4}{(1+\sqrt{3})}$$

MI and NI are both inradii of the triangle

$$\therefore MI = NI = \frac{4}{1+\sqrt{3}}$$

$$BI^2 = MI^2 + NI^2 = 2MI^2 \ \therefore\ DI = \sqrt{2}\cdot MI = \frac{4\sqrt{2}}{1+\sqrt{3}}$$

$$= \frac{4\sqrt{2}(\sqrt{3}-1)}{2} = 2\sqrt{2}(\sqrt{3}-1) = 2(\sqrt{6}-\sqrt{2}) \text{ cm.}$$

13. (d) : We use a bit of number theory as also known as mean value theorem. Fermat's little theorem gives $a^5 - a \equiv 0 \pmod 5$, $b^5 - b = 0 \pmod 5$, $c^5 - c = 0 \pmod 5$, $d^5 - d = 0 \pmod 5$)

Hence $(a^5-a)+(b^5-b)-(c^5-c)-(d^5-d) \equiv 0 \pmod 5$

$\Rightarrow (a-c)+(b-d) \equiv 0 \pmod 5$ ....(A)

Claim : $a+b \neq c+d$

Assume $a+b = c+d$, then we may assume that $a > c > d > b$. Applying the mean value theorem to the function $f(t) = t^5$ on the interval $[c, a]$ and $[b, d]$, we have $t_1 \in (c, a)$ and $t_2 \in (b, d)$ such that

$5t_1^4(a-c) = 5t_2^4(d-b)$ but because $a-c = d-b$ we have $t_1 = t_2$. But this can't happen as they lie in different intervals.

Hence $a+b = c+d \Rightarrow (a-c)+(b-d) = 0$

Now $|a-c| + |b-d| \geq |(a-c)+(b-d)|$

$$\therefore |a-c|+|b-d| \geq 5 \ \ (\because (a-c)+(b-d) \equiv 0 \pmod 5)$$

14. (b) : Using complex numbers

Let $z = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$

Then $z^7 = \cos\pi + i\sin\pi = -1 \Rightarrow z^7 + 1 = 0$

From $S = \cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}$

$$= \frac{1}{2}\left(z+\frac{1}{z}\right)+\frac{1}{2}\left(z^3+\frac{1}{z^3}\right)+\frac{1}{2}\left(z^5+\frac{1}{z^5}\right)$$

$$= \frac{1}{2}\left(\frac{z^2+1}{z}+\frac{z^6+1}{z^3}+\frac{z^{10}+1}{z^5}\right)$$

$$= \frac{z^6+z^4+z^8+z^2+z^{10}+1}{2z^5}$$

$$= \frac{z^{10}+z^8+z^6+z^4+z^2+1}{2z^5} \quad ...(1)$$

Since $z^7 + 1 = 0$ we have $z^{10} = -z^3$ and $z^8 = -z$.

(1) gives $S = \dfrac{-z^3-z+z^6+z^4+z^2+1}{2z^5}$

$$= \frac{1}{2}\left(\frac{z^7+1}{z+1}+1\right) = \frac{1}{2}.$$

15. (c) : Suppose the equation $|x-58|+|x-65|+|x+58|+|x+65| = 123\lambda$ has at least one real solution x. Then

$123\lambda = |x-58|+|x-65|+|x+58|+|x+65|$

$\qquad = |x-58|+|x-65|+|-x-58|+|-x-65|$

$$(\because |x| = |-x|)$$

$\geq |(x-58)+(x-65)+(-x-58)+(-x-65)|$

$= |-2(58+65)| = 2\times 123$

$$(\because |a_1|+|a_2|+|a_3|+|a_4| \geq |a_1+a_2+a_3+a_4|)$$

Thus $\lambda \geq 2$

Conversely suppose that $\lambda \geq 2$, then we can show that the equation has at least one real solution.

Define $f(x) = |x - 58| + |x - 65| + |x + 58| + |x + 65|$

$f(0) = |-58| + |-65| + |58| + |65|$

$= 2(58 + 65) = 2 \times 123 \leq 123\lambda$

and $f(123\lambda) = |123\lambda - 58| + |123\lambda - 65| + |123\lambda + 58| + |123\lambda + 65|$

$= 4 \times 123\lambda > 123\lambda$ (all *Summands* are positive)

The existence of value x such that $f(x) = 123\lambda$ follows from the fact that f is continuous.

16. (a) : To start with note the transformation

$2(xy - uv) = 2xy - 2uv = [(x^2 + y^2) - (x - y)^2] - [(u^2 + v^2) - (u - v)^2]$

$= (x^2 + y^2 - u^2 - v^2) - [(x - y)^2 - (u - v)^2]$

$= (x^2+y^2-u^2-v^2)-(x-y+u-v)(x - y - u + v)$ ...(1)

Now $\dfrac{2(xy-uv)}{x-y+u-v} = \dfrac{x^2+y^2-u^2-v^2}{x-y+u-v} - (x-y-u+v)$ ...from (1)

$= \dfrac{x^2-y^2-u^2+v^2}{x+y+u+v} - \{(x+v)-(u+y)\}$ .... (from given hypothesis)

$= \dfrac{(x^2-y^2-u^2+v^2) - \{(x+v)^2-(u+y)^2\}}{x+y+u+v}$

$= \dfrac{(x^2+v^2)-(x+v)^2+(u+y)^2-(u^2+y^2)}{x+y+u+v}$

$= \dfrac{2(uy-vx)}{x+y+u+v} \Rightarrow \dfrac{xy-uv}{x-y+u+v} = \dfrac{uy-vx}{x+y+u+v}$.

17. (c) : $\{x\} + y + [z] = 3.1$ ....(i)

$x + [y] + \{z\} = 2.4$ ....(ii)

$[x] + \{y\} + z = 1.3$ ....(iii)

Adding all of them, we get

$2x + 2y + 2z = 6.8 \Rightarrow x + y + z = 3.4$ ....(A)

(We have used $\{x\} + [x] = x$, etc. to arrive at A)

Subtracting each of (i), (ii) and (iii) from (A)

$[x] + \{z\} = 0.3 \Rightarrow [x] = 0$ and $\{z\} = 0.3$

$\{y\} + [z] = 1.0 \Rightarrow [z] = 1$ and $\{y\} = 0$

$\{x\} + [y] = 2.1 \Rightarrow [y] = 2$ and $\{x\} = 0.1$

Then $x = [x] + \{x\} = 0 + 0.1 = 0.1$

$y = [y] + \{y\} = 2 + 0 = 2$

$z = [z] + \{z\} = 1 + 0.3 = 1.3$

Thus (0.1, 2, 1.3) is the only solution.

18. (b) : Let $S = x + y + z$. Then

$R = \dfrac{x}{S+x} + \dfrac{y}{S+y} + \dfrac{z}{S+z}$

Take the function $f : (0, \infty) \to (0, \infty)$ defined by

$f(t) = \dfrac{t}{S+t}$

We have $f'(t) = \dfrac{(S+t)1-t}{(S+t)^2} = \dfrac{S}{(S+t)^2}$

$f''(t) = -\dfrac{2S}{(S+t)^3} < 0 \ \forall \ t > 0$ so f is concave.

Now we can use a result on concave function

$\dfrac{f(x)+f(y)+f(z)}{3} \leq f\left(\dfrac{x+y+z}{3}\right)$ ...(A)

But $f\left(\dfrac{x+y+z}{3}\right) = f\left(\dfrac{S}{3}\right) = \dfrac{\frac{S}{3}}{S+\frac{S}{3}} = \dfrac{1/3}{4/3} = \dfrac{1}{4}$

So (A) gives

$\dfrac{f(x)+f(y)+f(z)}{3} \leq \dfrac{1}{4} \Rightarrow f(x)+f(y)+f(z) \leq \dfrac{3}{4}$

$\Rightarrow \dfrac{x}{2x+y+z} + \dfrac{y}{x+2y+z} + \dfrac{z}{x+y+2z} \leq \dfrac{3}{4}$ i.e. $R \leq \dfrac{3}{4}$.

19. (b) : Since $\sin\dfrac{3\pi}{8}$ and $\cos\dfrac{3\pi}{8}$ are not readily available, we try to see if the expression given turns out to be constant for all x. That's the idea.

$f_4(x) - f_6(x) = \dfrac{1}{4}(\sin^4 x + \cos^4 x) - \dfrac{1}{6}(\sin^6 x + \cos^6 x)$

$= \dfrac{1}{4}\{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x\} - \dfrac{1}{6}\{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)\}$

$= \dfrac{1}{4} - \dfrac{1}{2}\sin^2 x\cos^2 x - \dfrac{1}{6} + \dfrac{1}{2}\sin^2 x\cos^2 x$

20. (d) : We have $R = 8r \Rightarrow \dfrac{r}{R} = \dfrac{1}{8}$

$\Rightarrow 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \dfrac{1}{8}$

(where we've used $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$)

$\Rightarrow 2\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \dfrac{1}{10}$

$\Rightarrow \left(\cos\dfrac{A-B}{2} - \cos\dfrac{A+B}{2}\right)\sin\dfrac{C}{2} = \dfrac{1}{10}$

$\Rightarrow \left(\cos 60^\circ - \cos\dfrac{\pi - C}{2}\right)\sin\dfrac{C}{2} = \dfrac{1}{10}$ $(\because A - B = 120^\circ)$

$\Rightarrow \sin^2\dfrac{C}{2} - \dfrac{1}{2}\sin\dfrac{C}{2} + \dfrac{1}{16} = 0$

$\Rightarrow \left(\sin\dfrac{C}{2} - \dfrac{1}{4}\right)^2 = 0 \Rightarrow \sin\dfrac{C}{2} = \dfrac{1}{4}$ $\therefore \cos C = \dfrac{7}{8}$.

21. (d) : As $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= -\dfrac{1}{2} + \cos\alpha\sin\beta \Rightarrow \cos\alpha\sin\beta = \dfrac{1}{2} + \sin(\alpha+\beta)$ ...(i)

$\because -1 \leq \sin(\alpha+\beta) \leq 1$

we have $-\dfrac{1}{2} \leq \cos\alpha\sin\beta \leq \dfrac{3}{2}$ from (i)

Again $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$\Rightarrow \cos\alpha\sin\beta = \sin\alpha\cos\beta - \sin(\alpha-\beta) = -\frac{1}{2} - \sin(\alpha-\beta)$$

$\therefore -1 \le \sin(\alpha-\beta) \le 1$

we have $-\frac{3}{2} \le \cos\alpha\sin\beta \le \frac{1}{2}$

so, we get $-\frac{1}{2} \le \cos\alpha\sin\beta \le \frac{1}{2}$

But we've not yet established that $\cos\alpha\sin\beta$ infact attains all value is the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Consider $(\cos\alpha\sin\beta)^2 = (1-\sin^2\alpha)(1-\cos^2\beta)$
$= 1 - (\sin^2\alpha + \cos^2\beta) + \sin^2\alpha\cos^2\beta$
$= 1 - (\sin\alpha + \cos\beta)^2 + 2\sin\alpha\cos\beta + \frac{1}{4}$
$= \frac{5}{4} - (\sin\alpha+\cos\beta)^2 + 2\times -\frac{1}{2} = \frac{1}{4} - (\sin\alpha+\cos\beta)^2$

Let $x = \sin\alpha$, $y = \cos\beta$ then $-1 \le x \le 1, -1 \le y \le 1$ and $xy = -\frac{1}{2}$ given

The range of the sum $s = x + y = \sin\alpha + \cos\beta$ is to be obtained x and y are both roots of the equation

$$t^2 - st - \frac{1}{2} = 0 \qquad \text{...(A)}$$

Thus $(x, y) = \left(\frac{s+\sqrt{s^2+2}}{2}, \frac{s-\sqrt{s^2+2}}{2}\right)$

$$\frac{s-\sqrt{s^2+2}}{2} \le 1 \Rightarrow s \le \frac{1}{2}$$

other conditions can be similarly verified and ewe get that (D) has a solution (x, y) with $-1 \le x, y \le 1$ for all s satisfying $-\frac{1}{2} \le s \le \frac{1}{2}$. Also sine and cosine functions are onto from R to the interval $[-1, 1]$, the range of $s = \sin\alpha + \cos\beta$ is $[-1/2, 1/2]$ for $\sin\alpha\cos\beta = -1/2$. Thus the range of S^2 is $[0, 1/4]$. And that of $(\cos\alpha\sin\beta)^2$ is $[0, 2/4]$. It means that the range of $\cos\alpha\sin\beta$ is $[-1/2, 1/2]$.

22. (c) : $\sin^6x + 3\sin^2x\cos^2y + \cos^6y = 1$
$\Rightarrow \sin^6x + 3\sin^2x\cos^2y + \cos^6y - 1 = 0$
$\Rightarrow (\sin^2x)^3 + (\cos^2y)^3 + (-1)^3 - 3(\sin^2x)(\cos^2y)(-1) = 0 \qquad \text{...(A)}$
$(\sin^2x + \cos^2y - 1)[(\sin^2x - \cos^2y)^2 + (\cos^2y+1)^2 + (1+\sin^2x)^2] = 0$

The 2nd expression can't vanish for that could mean $\sin^2x + \cos^2y = -1$, impossible

Hence $\sin^2x + \cos^2y - 1 = 0 \Rightarrow \sin^2x = \sin^2y$

$$\Rightarrow x = y \quad (\because 0 \le x, y \le \pi/2)$$

When $x = y$, the converse also holds

Indeed, $\sin^6x + 3\sin^2x\cos^2x + \cos^6x = (\sin^2x + \cos^2x)^3 = 1$

Hence $\sin^6x + 3\sin^2x\cos^2y + \cos^6y = 1 \Leftrightarrow x = y$.
Note that choice (a) doesn't necessarily follow.

23. (d) : Observe that $121 - 341 + 220 = 0$. So let us find the value of $f(a, b, c)$, when $a + b + c = 0$
$a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$

$$\Rightarrow \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = 3 \qquad \text{...(i)}$$

Given, $f(a, b, c) = \frac{a^2}{2a^2+bc} + \frac{b^2}{2b^2+ca} + \frac{c^2}{2c^2+ab}$

$$= \frac{a^2/bc}{2\frac{a^2}{bc}+1} + \frac{b^2/ca}{2\frac{b^2}{ca}+1} + \frac{c^2/ab}{2\frac{c^2}{ab}+1}$$

$$= \frac{1}{2}\cdot\left[\frac{x}{1+x} + \frac{y}{1+y} + \frac{z}{1+z}\right] \qquad \text{...(ii)}$$

where $x = \frac{2a^2}{bc}$, $y = \frac{2b^2}{ca}$, $z = \frac{2c^2}{ab}$

Then (i) becomes $x + y + z = 6$ and

$$xyz = \left(\frac{2a^2}{bc}\right)\left(\frac{2b^2}{ca}\right)\left(\frac{2c^2}{ab}\right) = 8$$

Recase (ii) as $f = \frac{1}{2}\Sigma\frac{x}{1+x}$

$$= \frac{1}{2}\Sigma\left(1 - \frac{1}{1+x}\right) = \frac{1}{2}\left[3 - \Sigma\frac{1}{1+x}\right]$$

$$= \frac{1}{2}\left[3 - \frac{\Sigma(1+x)(1+y)}{(1+x)(1+y)(1+z)}\right]$$

$$= \frac{1}{2}\left[3 - \frac{3 + 2(x+y+z) + \Sigma xy}{1 + \Sigma x + \Sigma xy + xyz}\right]$$

$$= \frac{1}{2}\left[3 - \frac{3 + 2\times 6 + \Sigma xy}{1 + 6 + \Sigma xy + 8}\right] = \frac{1}{2}[3-1] = \frac{2}{2} = 1.$$

24. (d) : The proof uses a well-known result in an innovative way. Recall that $m^3 - m = m(m^2-1)$
$= m(m-1)(m+1)$ a product of three consecutive integers, is divisible by $3! = 6$.

Consider the number $N = n - (1001 + 1011 + 1000)$
$= (1001^3 + 1011^3 + 1000)^3 - (1001 + 1011 + 1000)$
$= \underbrace{(1001^3 - 1001)} + \underbrace{(1011^3 - 1011)} + \underbrace{(1000^3 - 1000)}$

Each of the three bracked terms on the right is divisible by 6. Hence N is divisible by 6. But $1001 + 1011 + 1000 = 3012$ is divisible by 6. Consequently the number n is divisible by 6.

25. (a) : Using $f(1) = 10, f(2) = 20, f(3) = 30$ to get a system of equations in a, b, c and d and then trying to find their values is not giving to bail us out. Instead we should use the given conditions to construct an auxiliary function $g(x)$ which becomes zero at $x = 1$, 2 and 3.
Let $g(x) = f(x) - 10x$ Then $g(1) = g(2) = g(3) = 0$

$g(x)$ is a fourth degree polynomial with coefficient of x^4 as 1

Hence $g(x) = (x-1)(x-2)(x-3)(x-\alpha)$ for some α

$\Rightarrow f(x) - 10x = (x-1)(x-2)(x-3)(x-\alpha)$

$\Rightarrow f(x) = (x-1)(x-2)(x-3)(x-\alpha) + 10x$

$$f(10) + f(-6) = 9 \times 8 \times 7 \times (10-\alpha) + 10 \times 10 + (-7)(-8)(-9)(-6-\alpha) + 10 \times (-6)$$

$$= 9 \times 8 \times 7 \times 10 - 9 \times 8 \times 7 \times \alpha + 100 + 7 \times 8 \times 9 \times 6 + 7 \times 8 \times 9 \times \alpha - 60$$

$$= 9 \times 8 \times 7(10+6) + 40 = 9 \times 8 \times 7 \times 16 + 40$$

$= 8104$ (Note that α's cancel out!).

26. **(b)** : Write 2934 in the base 5 representation

$2934 = 2500 + 375 + 50 + 5 + 4$

$= 4 \times 5^4 + 3 \times 5^3 + 2 \times 5^2 + 1 \times 5^2 + 4 \times 5^0$

$= (43214)^5$

Highest power of 5 in $|\underline{2934}$ is

$$\left[\frac{2934}{5}\right] + \left[\frac{2934}{25}\right] + \left[\frac{2934}{125}\right] + \left[\frac{2934}{625}\right] = 586 + 117 + 23 + 4 = 730$$

$$\text{Again } \frac{2934 - \text{sum of the digits in the base 5 representation of } 2934}{4}$$

$$= \frac{2934-(4+3+2+1+4)}{4} = \frac{2934-14}{4} = 730.$$

So choice (b) is correct.

(In general, the highest power of prime p in $n!$

$$\left. = \frac{n - \text{sum of digits in base } p \text{ representation of } n}{p-1} \right)$$

27. **(b)** : The equation is

$(x^2-3x+3)^2 + 3(x^2-3x+3) + 3 = x.$

At first this fourth degree equation looks intractable. But consider $g(x) = x^2 - 3x + 3$ and it is immediately see that the equation is equivalent to $g(g(x)) = x$

To solve $g(g(x)) = x$ we first solve $g(x) = x$. And then using the fact that solutions of $g(x) = x$ are also those of $g(g(x)) = x$, we solve the original equation

$x^2 - 3x + 3 = x \Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow (x-1)(x-3) = 0 \quad \therefore x = 1, 3$

Now $(x^2-3x+3)^2 - 3(x^2-3x+3) + 3 = x$ yields

$x^4 + 9x^2 + 9 - 6x^3 - 18x + 6x^2 - 3x^2 + 9x - 9 + 3 - x = 0$

$\Rightarrow x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$ ....(A)

We know that $(x-1)(x-3)$ is a factor of (A)

Hence by division we get the other factor and (A) reduces to $(x^2-4x+3)(x^2-2x+1) = 0$

$\Rightarrow (x-1)(x-3)(x-1)^2 = 0 \Rightarrow x = 1, 3$

Thus the equation has only two distinct real solutions.

28. **(d)** : Transform the given expression $n^2 - 19n + 89$ in two ways as below

$n^2 - 19n + 89 = n^2 - 18n + 81 - (n-8)$

$= (n-9)^2 - (n-8) < (n-9)^2$ ....(A)

(As $n > 11$ we have $n - 8 > 0$)

Again $n^2 - 19n + 89 = n^2 - 20n + 100 + n - 11$

$= (n-10)^2 + n - 11 > (n-10)^2$ ....(B)

(As $n > 11$, we have $n - 11 > 0$)

From (A) and (B)

$(n-10)^2 < n^2 - 19n + 89 < (n-9)^2$

Thus $n^2 - 19n + 89$ lies between two consecutive squares. Hence it's never a perfect square for any valur of $n > 11$.

29. **(c)** : *1st Solution*

$x = 1 + \sqrt{2} + \sqrt{3} \Rightarrow (x-1) = \sqrt{2} + \sqrt{3}$

Squaring $(x-1)^2 = 2 + 3 + 2\sqrt{6}$

$\Rightarrow x^2 - 2x + 1 = 5 + 2\sqrt{6} \Rightarrow x^2 - 2x - 4 = 2\sqrt{6}$

Squaring again

$(x^2 - 2x - 4)^2 = (2\sqrt{6})^2 = 24$

$\Rightarrow x^4 + 4x + 16 - 4x^3 - 8x^2 + 16x = 24$

$\Rightarrow x^4 - 4x^3 - 4x^2 + 16x - 8 = 0$

2nd Solution (Elegant)

To get the equation of the lowest degree with integral coefficients the other roots must be

$x_2 = 1 + \sqrt{2} - \sqrt{3},\ x_3 = 1 - \sqrt{2} + \sqrt{3},\ x_4 = 1 - \sqrt{2} - \sqrt{3}$

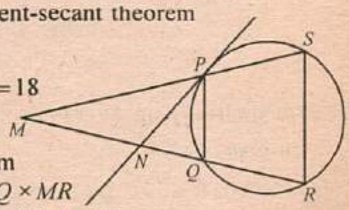
We have $x_1 + x_2 + x_3 + x_4 = 4$

$x_1x_2x_3x_4 = (1+\sqrt{2}-\sqrt{3})(1-\sqrt{2}+\sqrt{3})(1+\sqrt{2}+\sqrt{3}) = -8$

Similarly $\sum x_1x_2$ and $\sum x_1x_2x_3$ can be computed.

(But in multiple choice test there is no need to calculate all the coefficients. Once having obtained $\sum x_1$ and πx, we should use this information to eliminate the choices).

30. **(d)** : Using tangent-secant theorem



$NP^2 = NQ \times NR$

$\Rightarrow NR = \dfrac{NP^2}{NQ} = \dfrac{6^2}{2} = 18$

$\therefore QR = NR - NQ$

$= 18 - 2 = 16$ cm

Again $MP \times MS = MQ \times MR$

$\Rightarrow MP \times 40 = (MR - QR) \times MR$

$\Rightarrow MP = (20-16) \times \dfrac{20}{40} = 4 \times \dfrac{1}{2} = 2$ cm

Then $PS = MS - MP = 40 - 2 = 38$ cm

PQ = perimeter $-(PS + QR + SR)$

$= 80 - (38 + 16 + 6) = 20$ cm

Area of a cyclic quadrilateral is given by

$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$

where s = semi perimeter

$= \sqrt{(40-38)(40-16)(40-6)(40-20)}$

$= \sqrt{2 \times 24 \times 34 \times 20} = \sqrt{960 \times 34} = \sqrt{32 \times 30 \times 2 \times 17}$

$= \sqrt{64 \times 30 \times 17} = 8\sqrt{510}$ sq.cm.

31. The second equation $4xyz-(a^2x+b^2y+c^2z)=abc$ can be rewritten as $4-\left(\frac{a^2}{yz}+\frac{b^2}{zx}+\frac{c^2}{xy}\right)=\frac{abc}{xyz}$

$$\Rightarrow \frac{a^2}{yz}+\frac{b^2}{zx}+\frac{c^2}{xy}+\frac{abc}{xyz}=4 \qquad ...(i)$$

Set $x_1=\frac{a}{\sqrt{yz}},\ y_1=\frac{b}{\sqrt{zx}},\ z_1=\frac{c}{\sqrt{xy}}$, then (i) reduces to

$x_1^2+y_1^2+z_1^2+x_1y_1z_1=4$

where $0<x_1<2,\ 0<y_1<2,\ 0<z_1<2$

$$\Rightarrow\left(\frac{x_1}{2}\right)^2+\left(\frac{y_1}{2}\right)^2+\left(\frac{z_1}{2}\right)^2+2\left(\frac{x_1}{2}\right)\left(\frac{y_1}{2}\right)\left(\frac{z_1}{2}\right)=1$$

Recall that in a triangle

$\cos^2A+\cos^2B+\cos^2C+2\cos A\cos B\cos C=1$

So $\frac{x_1}{2}=\cos A,\ \frac{y_1}{2}=\cos B,\ \frac{z_1}{2}=\cos C$; A, B, C being angles of an acute triangle.

Now $\frac{a}{2\sqrt{yz}}=\cos A\Rightarrow a=2\sqrt{yz}\cos A$

Similarly $b=2\sqrt{zx}\cos B$ and $c=2\sqrt{xy}\cos C$

adding we have

$$a+b+c=2\sqrt{xy}\cos C+2\sqrt{yz}\cos B+2\sqrt{zx}\cos A$$

$$\Rightarrow x+y+z-2\sqrt{yz}\cos A-2\sqrt{zx}\cos B-2\sqrt{xy}\cos C=0$$

(using $a+b+c=x+y+z$)

$$\Rightarrow x(\sin^2B+\cos^2B)+y(\sin^2A+\cos^2A)-2\sqrt{yz}\cos A-2\sqrt{zx}\cos B+2\sqrt{xy}(\cos A\cos B-\sin A\sin B)=0$$

$$\Rightarrow(x\sin^2B-2\sqrt{xy}\sin A\sin B+y\sin^2A)+x\cos^2B+y\cos^2A+z+2\sqrt{xy}\cos A\cos B-2\sqrt{xz}\cos B-2\sqrt{yz}\cos A=0$$

$$\Rightarrow(\sqrt{x}\sin B-\sqrt{y}\sin A)^2+(\sqrt{x}\cos B+\sqrt{y}\cos A-\sqrt{z})^2=0$$

which gives

$$\sqrt{z}=\sqrt{x}\cos B+\sqrt{y}\cos A=\sqrt{x}\cdot\frac{b}{2\sqrt{zx}}+\sqrt{y}\cdot\frac{a}{2\sqrt{yz}}$$

$$\Rightarrow\sqrt{z}=\frac{b+a}{2\sqrt{z}}\quad\therefore z=\frac{a+b}{2}$$

By symmetry, $x=\frac{b+c}{2}$ and $y=\frac{c+a}{2}$.

32. Construct the auxiliary function

$g:[a,b]-R$ given by $g(x)=(x-a)(x-b)f(x)$

g is continuous on $[a,b]$, differentiable on (a,b)

Also $g(a)=0=g(b)$

Applying Rolle's theorem to g we have, $\exists$ one $c\in(a,b)$ such that $g'(c)=0$ ...(ii)

$$\Rightarrow(c-b)f(c)+(c-a)f(c)+(c-a)(c-b)f'(c)=0$$

Dividing by $(c-a)(c-b)$ gives $(\because c\neq a,\ c\neq b)$

$$\frac{f(c)}{c-b}+\frac{f(c)}{c-a}+f'(c)=0\Rightarrow f'(c)=\frac{f(c)}{b-c}+\frac{f(c)}{a-c}$$

$$\Rightarrow\frac{f'(c)}{f(c)}=\frac{1}{a-c}+\frac{1}{b-c}.$$

33. Note that $\frac{1}{\sqrt{k}}<\frac{2}{\sqrt{k}+\sqrt{k-1}}=2(\sqrt{k}-\sqrt{k-1})$

Again

$$\frac{1}{\sqrt{k}}=\frac{2}{\sqrt{k}+\sqrt{k}}>\frac{2}{\sqrt{k+1}+\sqrt{k}}=\frac{2(\sqrt{k+1}-\sqrt{k})}{(\sqrt{k+1}+\sqrt{k})(\sqrt{k+1}-\sqrt{k})}$$

$$=2(\sqrt{k+1}-\sqrt{k}) \qquad ...(iii)$$

Combining (i) and (ii) we get

$$=2(\sqrt{k+1}-\sqrt{k})<\frac{1}{\sqrt{k}}<2(\sqrt{k}-\sqrt{k-1}) \qquad ...(A)$$

Setting $k=2,3,\ldots$ in the inequality $\frac{1}{\sqrt{k}}<2(\sqrt{k}-\sqrt{k-1})$ and adding, we have

$$\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+....+\frac{1}{\sqrt{10,000}}<2\{(\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+....+(\sqrt{10,000}-\sqrt{9999})\}$$

$$=2\left(\sqrt{10,000}-1\right)=2(100-1)=2\times99=198$$

$\therefore S<198$ ....(B)

Again setting $k=2,3,\ldots$ in the inequality $\frac{1}{\sqrt{k}}>2(\sqrt{k+1}-\sqrt{k})$ and adding, we have

$$\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{10,000}}>2\{(\sqrt{3}-\sqrt{2})+(\sqrt{4}-\sqrt{3})+...+(\sqrt{10001}-\sqrt{10,000})\}$$

$$=2\{\sqrt{1001}-\sqrt{2}\}>197$$

Thus $S>197$ ...(C)

We have from (B) and (C) $197<S<198\ \therefore\ [S]=197$.

34. Inscribe the heptagon in a circle of radius R. The sides A_1A_2, A_1A_3 and A_1A_4 subtend arcs of measure $\frac{2\pi}{7},\frac{4\pi}{7},\frac{6\pi}{7}$ respectively.

Hence

$$A_1A_2=2R\sin\frac{\pi}{7},\ A_1A_3=2R\sin\frac{2\pi}{7},\ A_1A_4=2R\sin\frac{3\pi}{7}$$

The identity to be proved is equivalent to

$$\frac{1}{2R\sin\frac{\pi}{7}}=\frac{1}{2R\sin\frac{2\pi}{7}}+\frac{1}{2R\sin\frac{3\pi}{7}}$$

or $\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}=\sin\frac{\pi}{7}\sin\frac{3\pi}{7}+\sin\frac{\pi}{7}\sin\frac{2\pi}{7}$

or $-\cos\frac{5\pi}{7}+\cos\frac{\pi}{7}=-\cos\frac{4\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}$

$$\Rightarrow-\cos\frac{5\pi}{7}=-\cos\frac{4\pi}{7}+\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7} \qquad ...(1)$$

But (1) is evident, because

$$\frac{2\pi}{7}+\frac{5\pi}{7}=\pi=\frac{3\pi}{7}+\frac{4\pi}{7}$$

Hence $\cos\frac{2\pi}{7}=-\cos\frac{5\pi}{7}$ and $\cos\frac{3\pi}{7}=-\cos\frac{4\pi}{7}$, etc.

Thus we have proved that

$$\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}$$

35. Let $x=\cot\frac{A}{2}$, $y=\cot\frac{B}{2}$, $z=\cot\frac{C}{2}$

We have $x=\cot\frac{A}{2}=\frac{s-a}{r}$

Similarly $y=\cot\frac{B}{2}=\frac{s-b}{r}$ and $z=\cot\frac{C}{2}=\frac{s-c}{r}$

As $\frac{s}{r}=\frac{(s-a)+(s-b)+(s-c)}{r}=x+y+z$

The given relation

$$\left(\cot\frac{A}{2}\right)^2+\left(2\cot\frac{B}{2}\right)^2+\left(3\cot\frac{C}{2}\right)^2=\left(\frac{6s}{7r}\right)^2$$

is equivalent to $49(x^2+4y^2+9z^2=36(x+y+z)^2$

$\Rightarrow 13x^2+160y^2+405z^2-72(xy+yz+zx)=0$

$\Rightarrow (9x^2-72xy+144y^2)+(16y^2-72yz+81z^2)$
$+(324z^2-72zx+4x^2)=0$

$\Rightarrow (3x-12y^2)+(4y-9z)^2+(18z-2x)^2=0$

$\Rightarrow 3x=12y \Rightarrow x=4y$

Similarly $4y=5z \Rightarrow y=\frac{9}{4}z$ and $18z=2x \Rightarrow x=9z$

Thus $x:y:z=1:1/4:1/9$

Another way to arrive at the above is to employ Cauchy-Schwarz inequality

$(6^2+3^2+2^2)[(x^2+(2y^2)+(3z)^2]\ge(6x+3\cdot 2y+2\cdot 3z)^2$

Now $x:y:z::1:\frac{1}{4}:\frac{1}{9}\Rightarrow x:y:z::36:9:4$

$$\Rightarrow\frac{x}{36}=\frac{y}{9}=\frac{z}{4}\Rightarrow\frac{s-a}{36}=\frac{s-b}{9}=\frac{s-c}{4}$$

$$=\frac{2s-(b+c)}{9+4}=\frac{2s-(c+a)}{4+36}=\frac{2s-(a+b)}{36+9}$$

$$=\frac{a}{13}=\frac{b}{40}=\frac{c}{45}$$

Thus triangle ABC is similar to a triangle whose side lengths are 13, 40, 45.

36. We use induction to arrive at our result

For $n=1$, the polynomial $P_1(x)=1+x$ has the root -1

for $n=2$, $P_2(x)=1+\frac{x}{1!}+\frac{x(x+1)}{2!}=1+x+\frac{x^2+x}{2}$

$=\frac{x^2+3x+2}{2}=\frac{(x+1)(x+2)}{2}$ has roots -1, -2.

Claim : The polynomial $P_n(x)=0$ has roots $-1, -2, \ldots -n$

Proof : For $n=1$ the base case is verified

Suppose the assertion holds for n

the $P_n(x)=k(x+1)(x+2)\ldots(x+n)$

The number k being the coefficient of x^n

From $P_n(x)=1+\frac{x}{1!}+\frac{x(x+1)}{2!}+\ldots+\frac{x(x+1)\ldots(x+n-1)}{n!}$

we have $k=\frac{1}{n!}$ = coefficient of x^n

Hence $P_n(x)=\frac{1}{n!}(x+1)(x+2)\ldots(x+n)$

$$P_{n+1}(x)=P_n(x)+\frac{x(x+1)\ldots(x+n)}{(n+1)!}$$

$$=\frac{1}{n!}(x+1)(x+2)\ldots.(x+n)+\frac{x(x+1)\ldots(x+n)}{(n+1)!}$$

$$=\frac{1}{(n+1)!}(x+1)(x+2)\ldots.(x+n)(x+n+1)$$

Hence the roots of $P_{n+1}(x)=0$ are $-1,-2,\ldots-n,-(n+1)$

Thus by principle of mathematical induction we have established that the roots of

$$P_n(x)=1+\frac{x}{1!}+\frac{x(x+1)}{2!}+\ldots+\frac{x(x+1)(x+n-1)}{n!}$$

are $-1, -2, \ldots., -n$.

37. For minimum value $P=\cos x\sin y\cos z$

$=\frac{1}{2}\cos x(2\sin y\cos z)$

$=\frac{1}{2}\cos x[\sin(y+z)+\sin(y-z)]\ge\frac{1}{2}\cos x\sin(y+z)$

$=\frac{1}{2}\cos^2 x\quad\left(\because \frac{\pi}{2}\ge 0\ge z \text{ we have } \sin(y-z)\ge 0\right)$

Observe that $x=\frac{\pi}{2}-(y+z)\le\frac{\pi}{2}-2\times\frac{\pi}{12}=\frac{\pi}{3}$

Thus the minimum value of P is

$$\frac{1}{2}\cos^2\frac{\pi}{3}=\frac{1}{2}\times\left(\frac{1}{2}\right)^2=\frac{1}{8}$$

attained when $x=\frac{\pi}{3}$, $y=z=\frac{\pi}{12}$

For maximum value

$P=\cos x\sin y\cos z=\frac{1}{2}\cos z(2\cos x\sin y)$

$$=\frac{1}{2}\cos z[\sin(x+y)-\sin(x-y)]\le\frac{1}{2}\cos^2 z$$

$$\left(\because \frac{\pi}{2}\ge x\ge y\ \therefore\ \sin(x-y)\ge 0\right)$$

Thus $P \le \frac{1}{4}(1+\cos 2z) = \frac{1}{4}\left(1+\cos\frac{\pi}{6}\right) = \frac{2+\sqrt{3}}{8}$

This maximum value is attained at $x = y = \frac{5\pi}{24}$ and $z = \frac{\pi}{12}$.

38. Write $N = 99899 = 9x^4 + 9x^3 + 8x^2 + 9x + 9$, where $x = 10$. We will attempt to factorize the polynomial to see whether its prime or composite.

The coefficients 9, 9, 8, 9, 9 remind us of reciprocal equation. To factorize, pair up the terms equidistant from beginning and end.

$$\begin{aligned}
9x^4 + 9x^3 + 8x^2 + 9x + 9 &= 9(x^4 + 1) + 8x^2 + 9x(x^2 + 1)\\
&= 9\{(x^2 + 1)^2 - 2x^2\} + 8x^2 + 9x(x^2 + 1)\\
&= 9(x^2 + 1)^2 + 9x(x^2 + 1) - 10x^2\\
&= 9(x^2 + 1)^2 + 15x(x^2 + 1) - 6x(x^2 + 1) - 10x^2\\
&= \{3(x^2 + 1) - 2x\}\{3(x^2 + 1) + 5x\}\\
&= (3x^2 - 2x + 3)(3x^2 + 5x + 3)
\end{aligned}$$

For $3x^2 - 2x + 3$, $D = 4 - 4 \times 3 \times 3 < 0$

and $3x^2 + 5x + 3$, $D = 25 - 4 \times 3 \times 3 < 0$

Thus $(3x^2 - 2x + 3)$ and $(3x^2 + 5x + 3)$ can't be further factored in linear polynomials.

Thus $N = (3x^2 - 2x + 3)(3x^2 + 5x + 3)$

on putting $x = 10$ we have

$$\begin{aligned}
N = 99899 &= (300 - 20 + 3)(300 + 50 + 3)\\
&= (283)(353) = 283 \times 353
\end{aligned}$$

showing that N is composite.

39. As $|ax + by + cz| + |bx + cy + az| + |cx + ay + bz| = |x| + |y| + |z|$ holds

For all real numbers x, y, z we will try to uncover as much information as we can by setting strategic values for x, y, z.

Take $x = y = z = 1$ in the given equation to get

$|a + b + c| = 1$ ....(A)

For $x = 1, y = z = 0$, we get

$|a| + |b| + |c| = 1$ ...(B)

(A) and (B) give $|a + b + c| = |a| + |b| + |c|$,

we immediately realize that a, b, c all have the same sign. Again let $x = 1, y = -1, z = 0$, it follows that

$|a - b| + |b - c| + |c - a| = 2$ ...(C)

Now $|a - b| \le |a| + |b|$, with equality if and only if a and b have opposite sign or if one of them is zero.

Writing the three inequalities

$$\left.\begin{aligned}|a-b| &\le |a| + |b|\\ |b-c| &\le |b| + |c|\\ |c-a| &\le |c| + |a|\end{aligned}\right\} *$$

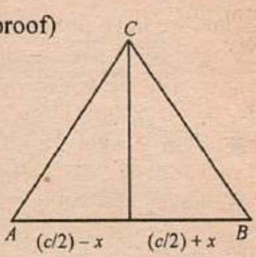
and adding them, we get

$|a - b| + |b - c| + |c - a| \le 2(|a| + |b| + |c|) = 2$ ...(D)

From (C) and (D) we find that the septem(*) becomes all equalities and thus in each pair (a, b), (b, c), (c, a) one of the number must be 0.

Hence the required triplets (a, b, c) are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(-1, 0, 0)$, $(0, -1, 0)$, $(0, 0, -1)$.

40. *1st solution* (Geometric proof)



An equilateral triangle with side c has altitude $\frac{c}{2}\sqrt{3}$.

Any triangle with side c will have an altitude perpendicular to c of length $\frac{c}{2}\sqrt{3} + y$. The side c is split into two parts $\frac{c}{2} - x$ and $\frac{c}{2} + x$. x, y are deviations from an equilateral triangle.

$$a^2 + b^2 + c^2 - 4\sqrt{3}A = \left(\frac{c}{2} - x\right)^2 + \left(\frac{c}{2} + x\right)^2 + 2\left(y + \frac{c}{2}\sqrt{3}\right)^2 + c^2 - 2\sqrt{3}c\left(y + \frac{c}{2}\sqrt{3}\right)$$

$$= \left(\frac{c^2}{4} - cx + x^2\right) + \left(\frac{c^2}{4} + cx + x^2\right) + 2\left(y^2 + cy\sqrt{3} + \frac{c^2}{4}\cdot 3\right) + c^2 - 2\sqrt{3}cy - 3c^2$$

$$= 2x^2 + 2y^2 \ge 0$$

we have equality iff $x = y = 0$, i.e. for the equilateral triangle.

2nd Solution (Proof by contradiction)

Suppose to the contrary that $4\sqrt{3}A > a^2 + b^2 + c^2$

$$\Rightarrow 4\sqrt{3}A \cdot \frac{1}{2}bc\sin\alpha > a^2 + b^2 + c^2$$

$$\Rightarrow 2bc\sin\alpha > \frac{1}{\sqrt{3}}(a^2 + b^2 + c^2) \quad ...(1)$$

Using cosine law, $2bc\cos\alpha = b^2 + c^2 - a^2$ ...(2)

Squaring and adding the two relations (1) and (2)

$$4b^2c^2 > \frac{(a^2 + b^2 + c^2)^2}{3} + (b^2 + c^2 - a^2)^2$$

$$\begin{aligned}
&\Rightarrow 12b^2c^2 > (a^2 + b^2 + c^2)^2 + 3(b^2 + c^2 - a^2)^2\\
&\Rightarrow 12b^2c^2 > a^4 + b^4 + c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2 + 3\{b^4 + c^4 + a^2 + 2b^2c^2 - 2a^2b^2 - 2a^2c^2\}\\
&\Rightarrow 0 > 4a^4 + 4b^4 + 4c^4 - 4a^2b^2 - 4b^2c^2 - 4c^2a^2\\
&\Rightarrow 0 > a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2\\
&\Rightarrow 0 > (a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2\\
&\Rightarrow (a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 < 0
\end{aligned}$$

Sum of squares is negative. We get a contradiction. ■

Section-III
(Subjective questions) (Q.No. 44 to 48)

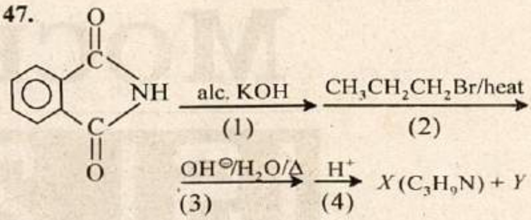
44. The equilibrium constants for amino acids are given in terms of succesive ionisation constants of the protonated form, for example, equilibrium constants for glycine (NH_2CH_2COOH) are $K_{a1} = 5 \times 10^{-3}$ M and $K_{a2} = 2 \times 10^{-10}$ M. What will be the pH at the isoelectric point for this amino acid and pH of 0.02 M protonated glycine in pure water respectively? (Express pH value after multiplying these by 100 and use three significant digits to express your answer)
[take log 2 = 0.30].

1st 2007

45. A sample consisting of chocolate-brown powder of PbO_2 is allowed to react with excess of KI and iodine liberated is reacted with N_2H_4 in another container. The volume of gas liberated from this second container at STP was measured as 1.12 litre. Find out volume of decimolar NaOH required to dissolve given amount PbO_2 completely. (Assume all reactions are 100% complete).

46. 2665 mg of an octahedral complex $CrCl_3 \cdot 6H_2O$ needs 3.4 g of $AgNO_3$ for complete precipitation of its all the free chloride ions. If the given complex is subjected to dehydration then what is the weight (in mg) of anhydrous complex obtained?
[Cr = 52, Cl = 35.5, Ag = 108, N = 14, O = 16, H = 1]

47.



Calculate molecular weight of Y.

48. A water insoluble organic mixture contained following compounds
(1) = Benzoic acid (2) = Salicylaldehyde
(3) = p-Hydroxybenzaldehyde
(4) = α-Naphthylamine (5) = Naphthalene
The following sequence of reagents are used to separate this mixture

$\boxed{1+2+3+4+5}$ mixture
(Step-X) aq. HCl → [Insoluble] ; [Soluble]*
[Insoluble] (Step-Y) aq. $NaHCO_3$ → [Insoluble] ; [Soluble]*
[Insoluble] (Step-Z) aq. NaOH → [Insoluble]* ; [Soluble]
[Soluble] (Step-W) Steam distillation → [Distilled]* ; [Left behind]

Fill up the serial number of starred compound obtained in the steps X, Y, Z and W respectively.

X	Y	Z	W

Answers to the problems of
IIT-JEE 2007 Practice Paper
on page no. 76 of March 2007 issue.

1. (b) 2. (c) 3. (b) 4. (c) 5. (b)
6. (d) 7. (b) 8. (a) 9. (c) 10. (a)
11. (b) 12. (c) 13. (b) 14. (d) 15. (b)
16. (a) 17. (a) 18. (d) 19. (b,d) 20. (a)
21. (a, b) 22. (a,b,c,d) 23. (a,b,c)
24. (a,b,c,d)
25. (a,b,c) 26. (a,b,c)
27. (a,b) 28. None of these

ANSWER

1. (c) 2. (a) 3. (a) 4. (d) 5. (c)
6. (b) 7. (c) 8. (d) 9. (a)
10. (a) P, W (b) R, W (c) Q, V (d) P, T
11. (a) Q, U (b) S, W (c) R, V (d) S, V
12. 10 13. 75 14. 9 15. 110 16. 5
17. (b) 18. (c) 19. (d) 20. (b) 21. (a)
22. (d) 23. (b) 24. (a) 25. (b)
26. (a) J, R (b) K, P (c) L, S (d) M, Q
27. (a) K, R (b) J, Q (c) L, S (d) M, P
28. 30 29. 7 30. 8 31. 1 32. 11
33. (a) 34. (d) 35. (b) 36. (a) 37. (c)
38. (b) 39. (c) 40. (b) 41. (a)
42. (A) L, S (B) K, R (C) M, Q (D) J, P
43. (A) L, S (B) M, P (C) K, Q (D) J, R
44. 600200 45. 2000 46. 2485 47. 166 48. 4152

(For Paper I refer Chemistry Today)

MOCK TEST
FOR
ISI 2007

Exam on 6th May 2007

By **Alok kumar**, B.Tech, IIT Kanpur

The Indian Statistical Institute (ISI), Kolkata, is considered as one of the foremost centres in the world for training and research in statistics and the related sciences. The B.Stat (Hons) degreé program, the flagship programme of the institute, offers comprehensive instruction in the theory, method and application of statistics, in addition to several areas of Mathematics and some basic areas of computer science.

Each candidate applying for admission to this programme has to take a selection test comprising Objective type and Short-answer type questions in mathematics at the Higher Secondary level (10 + 2 year's programme). The selection tests consists of

(1) A multiple choice type test having about 30 questions, and

(2) A short-answer type test having about 10 questions.

Questions will be set on the following and related topics.

Algebra : Sets, operations on sets, prime numbers, factorization of integers and divisibility, rational and irrational numbers, permutations and combinations, binomial theorem, logarithms, theory of quadratic equations, polynomial and remainder theorem, arithmetic and geometric progressions, inequalities involving A.M., G.M., and H.M., complex numbers.

Geometry : Plane geometry of class X level. Geometry of 2 dimensions with cartesian and polar co-ordinates. Concept of a locus, equation of a line, angle between two lines, distance from a point to a line. Areas of a triangle, equations of a circle, parabola, ellipse and hyperbola and equations of their tangents and normals, mensuration.

Trigonometry : Measures of angles, trigonometric and inverse trigonometric functions, trigonometric identities including addition formulae, solutions of trigonometric equations. Properties of triangles, heights and distances.

Calculus : Functions, one-one functions, onto functions, limits and continuity, derivatives and methods of differentiation, slope and curve, tangents and normals, maxima and minima, use of calculus in sketching graph of functions, methods of integration, definite and indefinite integrals, evaluation of area using integrals.

Logical Reasoning : Consistency of statements.

In response to growing demand from students preparing for the ISI, we bring to you the first Mock ISI paper, which closely simulates the real exam. There is more to follow in the coming months.

MULTIPLE CHOICE TEST

1. Among the following pairs of quadratic equations which has the same nature of their roots?

(a) $5x^2 + 11x + 17$ and $33x^2 + 21x + 5$

(b) $21x^2 + 25x + 13$ and $59x^2 + 67x + 21$

(c) $31x^2 + 14x + 7$ and $52x^2 + 76x + 31$

(d) all of the above.

2. Given that $N = 2^n(2^{n+1} - 1)$ and $2^{n+1} - 1$ is a prime number. Which of the following is true?

(a) Sum of divisors of N is $2N$

(b) Sum of reciprocals of the divisors of N is 1

(c) Sum of the reciprocals of the divisors of N is 4

(d) Sum of divisors of N is $4N$.

3. A fair coin is tossed 10,000 times. The probability p of obtaining at least 3 heads in a row satisfies

(a) $1/4 \le p < 1/2$ (b) $3/4 \le p < 1$

(c) $0 \le p < 1/4$ (d) $1/2 \le p < 3/4$.

4. $a_1, a_2, ..., a_{2007}$ are distinct positive real numbers. Let $S = \frac{a_1}{a_2} + \frac{a_2}{a_3} + ... + \frac{a_{2006}}{a_{2007}} + \frac{a_{2007}}{a_1}$. Then S satisfies

(a) $S > 2007$ (b) $S < 2007$
(c) $S < 1$ (d) $S > 1$.

5. Each number from 1, 2, 3, ... 100 (decimal system) is written in base 6 and their product is also written in base 6. The number of zeroes at the end of this product is
(a) 97 (b) 48 (c) 24 (d) 16.

6. Let $S = \{1, 2, ..., 100\}$. The number of unordered pairs (A, B) of subsets of S such that A and B have no elements in common, where A or B or both may be empty, is
(a) 3^{100} (b) $\frac{3^{100}-1}{2}$ (c) $\frac{3^{100}+1}{2}$ (d) $\frac{3^{100}}{2}$

7. The largest positive integer that cannot be written in a form $5a + 7b$, where a and b are positive integers is
(a) 25 (b) 26 (c) 33 (d) 35.

8. For how many positive integers n is $n^2 + 96$ a perfect square?
(a) exactly three (b) four
(c) none (d) infinite.

9. Three straight lines are drawn through a point P, lying in the interior of a triangle ABC, parallel to its sides. The areas of the resulting three triangles are 1, 2 and 4 cm. The area of the triangle ABC is (in sq. cm)
(a) 21 (b) $3+\sqrt{2}$
(c) $11+6\sqrt{2}$ (d) 14.

10. The altitude CN is drawn from the vertex C of the right angled triangle ABC. The radii of the circles inscribed into the triangles ACN and BCN are respectively 16 cm and 63 cm. The radius of the circle inscribed in the triangle ABC is (in cm)
(a) 79/2 (b) 65 (c) 47 (d) 56.

11. Triangle PQR and $P'Q'R'$ have side lengths as x, y, z and x', y', z' respectively, satisfying the relation

$$\sqrt{(x+y+z)(x'+y'+z')} = \sqrt{xx'} + \sqrt{yy'} + \sqrt{zz'}.$$

Then of the following statements only one is true. Which one is it?
(a) triangle PQR and $P'Q'R'$ are similar
(b) triangle PQR and $P'Q'R'$ are congruent
(c) exactly one of the two triangles is right angled
(d) exactly one of the two triangles is isosceles.

12. The number of solutions of the following system of equations

$$\{x\} + y + [z] = 2.3 \quad ; \quad x + [y] + \{z\} = 4.5$$
$$[x] + \{y\} + z = 6.2 \text{ is}$$

(a) none (b) exactly one
(c) exactly two (d) infinite.

13. Consider a sequence of numbers generated by the recursive relation $v_{n+3} = v_{n+2} \cdot v_n$, $n \geq 1$. Let $v_1 = 1 = v_2$ and $v_3 = -1$. Then v_{446} is
(a) 1 (b) -1
(c) 1 or -1
(d) none of the foregoing numbers.

14. If $\frac{x^2+y^2-z^2-w^2}{x-y+z-w} = \frac{x^2-y^2-z^2+w^2}{x+y+z+w}$, then of the following equalities which one is correct?

(a) $\frac{xy-zw}{x+y-z-w} = \frac{2(yz-xw)}{x-y+z-w}$

(b) $\frac{yz-xw}{x+y-z-w} = \frac{z(xy-zw)}{x-y+z-w}$

(c) $\frac{2(xy-zw)}{x-y+z-w} = \frac{yz-xw}{x+y+z+w}$

(d) $\frac{xy-zw}{x-y+z-w} = \frac{yz-xw}{x+y+z+w}$.

15. Let $N = 2^{1224} - 1$, $\alpha = 2^{153} + 2^{77} + 1$ and $\beta = 2^{408} - 2^{204} + 1$. Then which of the following statements is correct?
(a) α divides N but β doesn't
(b) β divides N but α doesn't
(c) α and β both divide N
(d) neither α nor β divides N.

16. The sum to the series
$(2^2-1)(6^2-1)+(4^2-1)(8^2-1)+...+(100^2-1)(104^2-1)$ is

(a) $\frac{1}{100}[101\cdot103\cdot105\cdot107\cdot109 + 1\cdot3\cdot5\cdot7\cdot9]$

(b) $\frac{1}{10}[99\cdot101\cdot103\cdot105\cdot107 + 1\cdot3\cdot5\cdot7]$

(c) $\frac{1}{10}[99\cdot101\cdot103\cdot105\cdot107 - 1\cdot3\cdot5\cdot7]$

(d) $\frac{1}{100}[101\cdot103\cdot105\cdot107\cdot109 - 1\cdot3\cdot5\cdot7\cdot9]$

17. What is the remainder when the repunit $\underbrace{111 \ldots 11}_{670\text{ 1's}}$ is divided by 1221? (A repunit is a number made up entirely of 1's)
(a) 1111 (b) 1213 (c) 11 (d) 121.

18. In a certain city all licence plates are of five characters that is, a sequence of three digits followed by a sequence of two letters. Each licence-plate uses only the digits 1, 2, 3, 4 and 5 and only the first thirteen letters of the English alphabet, i.e. A to M. Also, no licence-plate can have simultaneously the letter A and the digit 3. The maximum possible number of licence-plates that can be issued is

(a) 19600 (b) $4^3 \times 12^3$

(c) $4^3 \times 13^2 + 5^3 \times 12^2$ (d) none of the above.

19. In the inequality $\frac{a}{1+bc}+\frac{b}{1+ca}+\frac{c}{1+ab} \geq k$. What is the least number k that can be substituted so that the inequality holds for all non-negative real numbers a, b, c satisfying $a + b + c = 1$

(a) 9/10 (b) 10/9

(c) 1 (d) the least k doesn't exist.

20. The value of α and β for which the equation

$$\sqrt{x+\alpha\sqrt{x}+\beta}+\sqrt{x} = 4$$

has infinitely many real solution are

(a) $\alpha = -8, \beta = -16$ (b) $\alpha = -8, \beta = 16$

(c) $\alpha = 8, \beta = -16$ (d) $\alpha = 8, \beta = 16$.

21. There is an internal tangency between two circles at a point P. A circle passing through the centre O of the smaller circle meets the larger circle at A and D and smaller circles at B and C. Also let $AB:BC:CD::2:4:1$. The sum of the squares of radii equals 10 cm^2, what is the diameter of the smaller circle?

(a) 6 cm (b) 4 cm (c) 2 cm (d) 3 cm

22. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the area of a quadrilateral is $4\sqrt{3}$, the sum of remaining two sides is (in cm)

(a) 5 (b) 4 (c) 10 (d) 6.

23. The sum of real roots of the equation

$x^2 - 2^{2006}x + |x - 2^{2005}| + 2 \cdot (2^{4009} - 1) = 0$ is

(a) $2^{2005} + 2^{2004}$ (b) 2^{2006}

(c) 2^{2007} (d) $2^{2006} + 2^{2007}$.

24. Triangle ABC has one of its angle $\angle ABC = 45°$. A point D is taken on the side BC such that $CD = 2DB$. If $\angle DAB = 15°$, then the measure of angle ACB is

(a) 45° (b) 60°

(c) 75° (d) cannot be determined.

25. How many ordered solution (x, y) does the system of equations $3x(x + y - 2) = 2y$, $y(x + y - 1) = 9x$ have

(a) none (b) infinite

(c) exactly six (d) exactly three.

26. The number of isosceles triangles with integer sides if no side exceeds 1994 is

(a) 994009 (b) 1988018

(c) 949009 (d) 2982027.

27. The number of integer solutions of $a + b + c = 24$ subject to the condition $1 \leq a \leq 5$, $12 \leq b \leq 18$, $-1 \leq c \leq 12$ is

(a) 34 (b) 35 (c) 53 (d) 43.

28. The number of geometric progressions of three terms comprising unequal natural numbers less than or equal to 100 is

(a) 106 (b) 53 (c) 52 (d) 104.

29. How many times is the digit 0 written when listing all numbers from 1 to 3333?

(a) 936 (b) 365 (c) 963 (d) 356.

30. The set of points (x, y) in the plane satisfying $|x| + |x - y| = 4$ enclose

(a) a rhombus (b) a square

(c) a trapezium (d) a parallelogram.

SHORT ANSWER TYPE TEST

31. Suppose that $-1 \leq ax^2 + bx + c \leq 1 \ \forall \ x \in [-1, 1]$ where a, b, c are real numbers. Show that

$$-4 \leq 2ax + b \leq 4 \ \forall \ x \in [-1, 1].$$

32. If α and β are positive numbers, show that the equation $\frac{1}{x}+\frac{1}{x-\alpha}+\frac{1}{x+\beta} = 0$ has real roots, one between $\alpha/3$ and $2\alpha/3$, and one between $-2\beta/3$ and $-\beta/3$.

33. Let $f(x)$ be a function satisfying

$$f\left(x+\frac{5}{6}\right)+f(x) = f\left(x+\frac{1}{2}\right)+f\left(x+\frac{1}{3}\right) \ \forall \ x \in R$$

then show that $f(x + 2) + f(x) = 2f(x + 1)$.

34. Evaluate the product $\dfrac{\left(1^4+\frac{1}{4}\right)\left(3^4+\frac{1}{4}\right)\cdots\left(49^4+\frac{1}{4}\right)}{\left(2^4+\frac{1}{4}\right)\left(4^4+\frac{1}{4}\right)\cdots\left(50^4+\frac{1}{4}\right)}$.

35. Find all three digit numbers that when divided by 11 are equal to the sum of the squares of their digits.

36. Find all integral values of λ for which the quadratic expressions $(x+\lambda)(x+1991)+1$ can be factored as $(x+\alpha)(x+\beta)$, when α and β are integers.

37. Let f be a function defined on are positive $\{(i, j): i$ and j are positive integers$\}$ satisfying

(i) $f(i, i+1)=\frac{1}{4}\ \forall\ i$

(ii) $f(i,j)=f(i,k)+f(k,j)-2f(i,k)f(k,j)$

$\forall\ k$ such that $i<k<j$

Evaluate $\lim\limits_{n\to\infty} f(1, n)$

38. Let f be a function from $\{1, 2, 3\}$ into $\{1, 2, 3, 4, 5\}$. find the number of functions satisfying

$$f(i) \le f(j)\ \forall\ i<j.$$

39. Let $x=by+cz+du$, $y=cz+ax+du$, $z=ax+by+du$ and $u=ax+by+cz$. Prove that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}+\frac{d}{1+d}=1.$$

40. Solve for real x, the equation

$$\sqrt{1-x}=2x^2-1+2x\sqrt{1-x^2}.$$

SOLUTIONS

1. **(d)** : It is at once apparent that nobody in his right mind is going to calculate the discriminant of each equation. Observe the pattern of the equations

$5x^2+11x+17$ and $33x^2+21x+5$

$33x^2+21x+5=(5+11+17)x^2+(11+2\cdot 5)x+5$

Every pair of equations is of the forms

ax^2+bx+c and $(a+b+c)x^2+(b+2a)x+a$

Discriminant of second equation

$=(b+2a)^2-4a(a+b+c)$

$=b^2+4ab+4a^2-4a^2-4ab-4ac$

$=b^2-4ac=$ Discriminant of first equation

We have shown that both equations of any pair exhibit the same nature of roots.

2. **(a)** : *1st Solution* : $N=2^n(2^{n+1}-1)$

As $2^{n+1}-1$ is a prime number, call it p then $N=2^np$

Divisors of N are $1, 2, 2^2, ..., 2^n, p, 2p, 2^2p, 2^3p ... 2^np$

sum of divisors of N, $\sigma_N=$

$1+2+2^2+...+2^n+p+2p+2^2p+...+2^np$

$=(1+2+...+2^n)+p(1+2+...+2n)$

$=(1+p)(1+2+...+2^n)=(1+p)\cdot\dfrac{2^{n+1}-1}{2-1}$

$(1+p)(2^{n+1}-1)=(1+2^{n+1}-1)(2^{n+1}-1)$

$=2^{n+1}(2^{n+1}-1)=2\cdot 2^n(2^{n+1}-1)=2N$

Sum of reciprocal of divisors of N

$$\sigma_N=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^n}\right)+\frac{1}{p}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{2^2}+...\frac{1}{2^n}\right)$$

$$=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^n}\right)\left(1+\frac{1}{p}\right)$$

$$=\frac{1-(1/2)^{n+1}}{1-(1/2)}\cdot\frac{p+1}{p}=\frac{2^{n+1}-1}{2^{n+1}\times(1/2)}\cdot\frac{p+1}{p}$$

$$=\frac{2^{n+1}-1}{2^n}\cdot\frac{2^{n+1}}{2^{n+1}-1}=2.$$

Second Solution : The first result could also be had by the formula for the sum of divisors σ_n of a positive integers n given by

$$\sigma_n=\left(\frac{p_1^{k+1}-1}{p_1-1}\right)\cdot\left(\frac{p_2^{k+1}-1}{p_2-1}\right)\left(\frac{p_3^{k+1}-1}{p_3-1}\right)....\left(\frac{p_r^{k+1}-1}{p_r-1}\right)$$

where $n=p_1^{k_1}, p_2^{k_2}, ... p_r^{k_r}$ is the prime factorization of n for our problem

$$\sigma_N=\frac{2^{n+1}-1}{2-1}\frac{p^2-1}{p-1}=(2^{n+1}-1)(p+1)$$

$$=(2^{n+1}-1)2^{n+1}=2\cdot 2^n(2^{n+1}-1)=2N.$$

3. **(c)** : Let the probability of getting a head in a single trial be P_H and that of getting a tail be P_T

For a fair coin $P_H=P_T=1/2$

Suppose we get tails in first r trials and then get in a row 3 heads. All cases are covered by varying r from $r=0$ to $r=9997$.

The desired probability $p=\sum\limits_{r=0}^{9997} P_T^r P_H^3$

$$=P_H^3\sum_{r=0}^{9997}P_T^r=\left(\frac{1}{2}\right)^3\sum_{r=0}^{9997}\left(\frac{1}{2}\right)^r$$

$$=\frac{1}{8}\cdot\left\{1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+....+\left(\frac{1}{2}\right)^{9997}\right\}$$

$$=\frac{1}{8}\cdot\frac{1-\frac{1}{2^{9998}}}{1-\frac{1}{2}}=\frac{1}{8}\cdot\left(1-\frac{1}{2^{9998}}\right)\times 2$$

$$=\frac{1}{4}\left(1-\frac{1}{2^{99998}}\right)<\frac{1}{4}$$

Thus $p<1/4$ we also have $p\ge 0$. Thus $0\le p<1/4$.

4. **(a)** : Let $u_n=\dfrac{a_n}{a_{n+1}}$, $n=1, 2, 3, ... 2006$

Then $\frac{a_{2007}}{a_1} = \frac{1}{u_1 u_2 \dots u_{2006}}$

Applying AM-GM inequality between

$u_1, u_2, \dots u_{2006}$ and $\frac{1}{u_1 u_2 \dots u_{2006}}$

We have $\dfrac{u_1 + u_2 + \dots + u_{2006} + \frac{1}{u_1 \cdot u_2 \dots u_{2006}}}{2007} \geq$

$$\left\{u_1 \cdot u_2 \dots u_{2006} \cdot \frac{1}{u_1 \cdot u_2 \cdot \dots u_{2006}}\right\}^{1/2007}$$

$$\Rightarrow u_1 + u_2 + \dots + u_{2006} + \frac{1}{u_1 \cdot u_2 \cdot \dots u_{2006}} \geq 2007$$

$$\Rightarrow \frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{2006}}{a_{2007}} + \frac{a_{2007}}{a_1} \geq 2007 \Rightarrow S \geq 2007$$

For equality to hold all terms must be equal to 1.

i.e. $\frac{a_1}{a_2} = \frac{a_2}{a_3} = \dots = 1$

But that means $a_1 = a_2 = \dots = a_{2007}$. A contradiction, because we are given that the numbers are all distinct. This same result could also be obtained by straightaway multiplying all the numbers involved..

5. **(b)** : The problem amounts to finding out the highest power of 6 in $\lfloor 100$

Power of 3 in $\lfloor 100 = \left[\frac{100}{3}\right] + \left[\frac{100}{9}\right] + \left[\frac{100}{27}\right] + \left[\frac{100}{81}\right]$

$= 33 + 11 + 3 + 1 = 48$

Power of 2 in $\lfloor 100 > 48$

So power of 6 in $\lfloor 100 =$

min(power of 2 in $\lfloor 100$, power of 3 in $\lfloor 100$) = 48

Thus there are 48 zeroes in the end when the product is written in base 6.

6. **(c)** : In this problem, it is easier to calculate first the ordered pairs, (A, B).

To each element x of s assign the numbers 1, 2 or 0 according as x is in A, or in B or in neither of them. Each way of disposing of the numbers of S can be associated with a 100-digit string made up of 1, 2 or 0. For instance, 0121012... 1 means element '1' is in neither of A and B, element '2' is in A, element '3' is in B, i.e., first number tells where first element is, second number tells where 2nd element is, etc.

Any selection of A and B corresponds to the number of 100 digit number which can be made using digits 0, 1 or 2 with leading zeros allowed.

There are obviously 3^{100} such number.

But we have to count the number unordered pairs. Every unordered pair is counted twice in the number 3^{100}, except for the case when both A and B are empty. For example (ϕ, ϕ) gives one selection.

Hence number of unordered pairs $= \frac{3^{100} - 1}{2} + 1$

$$= \frac{3^{100} + 1}{2}.$$

7. **(d)** : The number 26 can be written as

$$26 = 5 \times 1 + 7 \times 3$$

Also note that $33 = 5 \times 1 + 7 \times 4$

and $34 = 5 \times 4 + 7 \times 2$

Hence the number required is greater than 34

Every number greater than 35 can be written as $5a + 7b$ for suitable values of a and b.

For example 103 can be written as

$$103 = 7 \times 9 + 5 \times 8$$

But $35 = 5m + 7n$, representation in positive integers is not possible. $5m$ is 5, 10, 15, 20, 15 – none of which is a multiple of 7. For $m = 7$, n becomes zero. But n has to be a positive integer. Thus it follows that 35 can't represented in the form $5m + 7n$.

8. **(b)** : Let $n^2 + 96 = k^2, k \in N$

$\Rightarrow \quad k^2 - n^2 = 96 \Rightarrow (k - n)(k + n) = 96$

$(k - n) + (k + n) = 26$ = even. It follows that $(k - n)$ and $(k + n)$ are either both odd or both even. But if both are odd, product can be even. Thus both must be even.

Now $k + n > k - n$ and 96 can be factored in the following way as a product of even numbers]

$96 = 2 \times 48 = 4 \times 24 = 6 \times 16 = 8 \times 12$

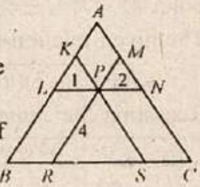
We have $k - n = 2, k + n = 48 \Rightarrow (k, n) = (25, 23)$

$k - n = 4, k + n = 24 \Rightarrow (k, n) = (14, 10)$

$k - n = 6, k + n = 16 \Rightarrow (k, n) = (11, 5)$

$k - n = 8, k + n = 12 \Rightarrow (k, n) = (10, 2)$

Thus there are four positive integers n such that $n^2 + 96$ is a perfect square.



9. **(c)** : Observe that the triangle LPK, PNM and RSP are similar to ΔABC. Let A be the area of triangle ABC

$$\frac{1}{A} = \frac{LP^2}{BC^2}, \ \frac{2}{A} = \frac{PN^2}{BC^2}, \ \frac{4}{A} = \frac{RS^2}{BC^2}$$

Now $\frac{1}{\sqrt{A}} = \frac{LP}{BC}, \ \frac{\sqrt{2}}{\sqrt{A}} = \frac{PN}{BC}, \frac{\sqrt{4}}{\sqrt{A}} = \frac{RS}{BC}$

Adding them all

$$\frac{1+\sqrt{2}+2}{\sqrt{A}} = \frac{LP+PN+RS}{BC}$$

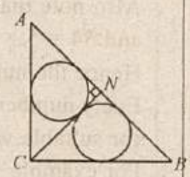
$$= \frac{BR+SC+RS}{BC} = \frac{BC}{BC} = 1.$$

(Note that $LP \parallel BR$ and $LP = BR$, $PN \parallel SC$ and $PN = SC$)

$$\Rightarrow \frac{3+\sqrt{2}}{\sqrt{A}} = 1 \Rightarrow \sqrt{A} = 3+\sqrt{2}$$

$$\Rightarrow A = (3+\sqrt{2})^2 = 11+6\sqrt{2}.$$

10. **(b) :** Denote by r, the radius of circle inscribed in triangle ACB.



From similarity of triangle ACN and ABC

$$\frac{r}{16} = \frac{c}{b} \Rightarrow br = 16c \quad ...(A)$$

Again from similarity of triangles CBN and ABC, we have $\frac{r}{63} = \frac{c}{a} \Rightarrow ar = 63c \quad ...(B)$

Squaring and adding we obtain

$$r^2(a^2+b^2) = (16^2+63^2)c^2$$

$$\Rightarrow \quad r^2c^2 = 65^2c^2 \quad \therefore \; r = 65 \text{ cm}.$$

11. **(a) :** Recall the identity for $a, b, c, x, y, z \in R$

$$(a^2+b^2+c^2)(x^2+y^2+z^2) = (ax+by+cz)^2 + (ay-bx)^2 + (az-cx)^2 + (bz-cy)^2$$

From the above equality it follows that

$$(a^2+b^2+c^2)(x^2+y^2+z^2) \ge (ax+by+cz)^2$$

Because square of a real number is non negative

Equality hold iff $ay - bx = az - cx = bz - cy = 0$

$$ay - bx = 0 \Rightarrow \frac{a}{x} = \frac{b}{y}$$

Similarly $bz - cy = 0 \Rightarrow \frac{b}{y} = \frac{c}{z}$

Hence $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$ is the condition for the equality to hold. ...(A)

The given relation is

$$\sqrt{(x+y+z)(x'+y'+z')} = \sqrt{xx'+yy'+zz'}$$

Recasting the above as

$$\sqrt{(X^2+Y^2+Z^2)(X'^2+Y'^2+Z'^2)} = XX'+YY'+ZZ'$$

where $X^2 = x$, $Y^2 = y$, $Z^2 = z$ etc.

on squaring

$$(X^2+Y^2+Z^2)(X'^2+Y'^2+Z'^2) = (XX'+YY'+ZZ')^2$$

Thus for (B) to hold

$$\frac{X}{X'} = \frac{Y}{Y'} = \frac{Z}{Z'} \quad \text{(using (A))}$$

$$\frac{X^2}{X'^2} = \frac{Y^2}{Y'^2} = \frac{Z^2}{Z'^2} \Rightarrow \frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'}$$

establishing that triangles PQR and $P'Q'R$ are similar.

12. **(b):**

$$\{x\} + y + [z] = 2 \cdot 3 \quad ...(1)$$

$$x + [y] + \{z\} = 4 \cdot 5 \quad ...(2)$$

$$[x] + \{y\} + z = 6 \cdot 2 \quad ...(3)$$

Adding all of them we obtain,

$$x + [x] + \{x\} + y + [y] + \{y\} + z + [z] + \{z\} = 2.3 + 4.5 + 6.2$$

$$\Rightarrow \; 2x + 2y + 2z = 13 \Rightarrow x + y + z = 6.5 \quad ...(4)$$

From (1) and (4) on subtraction

$$x - \{x\} + z - [z] = 6.5 - 2.3 \Rightarrow [x] + \{z\} = 4.2$$

(Recall that $x = [x] + \{x\}$, etc.)

implying that $[x] = 4$, $\{z\} = 0.2$

Similarly from (4) and (2) on subtraction

$\{y\} + [z] = 2 \Rightarrow \{y\} = 0$, $[z] = 2$

And from (4) and (3) on subtraction

$\{x\} + [y] = 0.3 \Rightarrow \{x\} = 0.3$ and $[y] = 0$

Now $x = [x] + \{x\} = 4 + 0.3 = 4.3$

$y = [y] + \{y\} = 0 + 0 = 0$

$z = [z] + \{z\} = 2 + 0.2 = 2.2$

thus (4.3, 0, 2.2) is the only solution.

13. **(b) :** $v_{n+3} = v_{n+2} \cdot v_n$, $n \ge 1$

Let's compute a few terms of the series. The idea is to see the sequence repeats itself.

$v_4 = v_3 \cdot v_1 = -1$, $v_5 = v_4 \cdot v_2 = -1$

$v_6 = v_5 \cdot v_3 = (-1)(-1) = 1$

$v_7 = v_6 \cdot v_4 = (1)(-1) = -1$

$v_8 = v_7 \cdot v_5 = (-1)(-1) = 1 = v_1$

$v_9 = v_8 \cdot v_6 = (1)(1) = 1 = v_2$

$v_{10} = v_9 \cdot v_7 = (1)(-1) = v_3$

From v_8 it's like a fresh beginning. The sequence repeats itself after every 7 terms. Accordingly the period is 7.

The sequence goes like 1, 1, –1, –1, –1, 1, –1, v_1, v_2, ...

Now $446 = 7 \times 63 + 5$ so $v_{446} = v_5 = -1$.

14. **(d) :** Observe that

$$2(xy - zw) = 2xy - 2zw$$

$$= \{(x^2+y^2) - (x-y)^2\} - \{(z^2+w^2) - (z-w)^2\}$$

$$= (x^2+y^2-z^2-w^2) - (x-y+z-w)(x-y-z+w) \quad ...(A)$$

Again

$$\frac{2(xy-zw)}{x-y+z-w} = \frac{x^2+y^2-z^2-w^2}{x-y+z-w} - (x-y-z-w)$$

$$= \frac{x^2-y^2-z^2+w^2}{x+y+z+w} - \{(x+w)-(y+z)\}$$

Contd. on page no. 68

Contd. from page no. 26

(From the hypothesis of the problem)

$$= \frac{x^2 - y^2 - z^2 + w^2 - \{(x+w)^2 - (y+z)^2\}}{x+y+z+w}$$

$$= \frac{(x^2+w^2) - (x+w)^2 - \{(y^2+z^2) - (y+z)^2\}}{x+y+z+w}$$

$$= \frac{-2xw + 2yz}{x+y+z+w} = \frac{2(yz - xw)}{x+y+z+w}$$

Thus $\dfrac{2(xy - zw)}{x - y + z - w} = \dfrac{2(yz - xw)}{x+y+z+w}$

$$\Rightarrow \frac{xy - zw}{x - y + z - w} = \frac{yz - xw}{x+y+z+w}.$$

15. (c) : The solution rests on being able to factorize the number N, using elementary identities.

$N = 2^{1224} - 1 = (2^{153})^8 - 1$

Now $x^8 - 1 = (x^4 - 1)(x^4 + 1) = (x^2 - 1)(x^2 + 1)(x^4 + 1)$
$= (x - 1)(x + 1)(x^2 + 1)(x^4 + 1)$

$(2^{153})^8 - 1 = (2^{153} - 1)(2^{153} + 1)[(2^{153})^2 + 1][(2^{153})^4 + 1]$

Consider $(2^{153})^2 + 1 = (2^{153})^2 + (1)^2 + 2 \cdot 2^{153} \cdot 1 - 2^{154}$
$= (2^{153} + 1)^2 - (2^{77})^2$
$= (2^{153} + 2^{77} + 1)(2^{153} - 2^{77} + 1)$
$= \alpha(2^{153} - 2^{77} + 1)$

Also $(2^{153})^4 + 1 = 2^{153 \times 4} + 1 = 2^{612} + 1 = (2^{204})^3 + 1$
$= (2^{204} + 1)(2^{408} - 2^{204} + 1) = (2^{204} + 1)\beta$

Thus α and β both divide N.

16. (b) : We will express the series as a telescopic sum

$t_n = ((2n)^2 - 1)[(2n + 4)^2 - 1)]$
$= (2n - 1)(2n + 1)(2n + 3)(2n + 5)$

Now $t_1 = 1 \cdot 3 \cdot 5 \cdot 7$, $t_2 = 3 \cdot 5 \cdot 7 \cdot 9$, etc

Note that $1\cdot3\cdot5\cdot7 = \frac{1}{10}[1\cdot3\cdot5\cdot7\cdot9 - (-1)1\cdot3\cdot5\cdot7]$

$$3\cdot5\cdot7\cdot9 = \frac{1}{10}[3\cdot5\cdot7\cdot9\cdot11 - 1\cdot3\cdot5\cdot7\cdot9]$$

$$5\cdot7\cdot9\cdot11 = \frac{1}{10}[5\cdot7\cdot9\cdot11\cdot13 - 3\cdot5\cdot7\cdot9\cdot11]$$

....

$$99\cdot101\cdot103\cdot105 = \frac{1}{10}[99\cdot101\cdot103\cdot105\cdot107 - 97\cdot99\cdot101\cdot103\cdot105]$$

The sum telescopes yielding

$1 \cdot 3 \cdot 5 \cdot 7 + 3 \cdot 5 \cdot 7 \cdot 9 + ... + 99 \cdot 101 \cdot 103 \cdot 105$

$$= \frac{1}{10}[99\cdot101\cdot103\cdot105\cdot107 + 1\cdot3\cdot5\cdot7]$$

17. (a) : The trick lies in being able to find the factors of the given repunit

$111111 = 33 \times 37 \times 91 = 1221 \times 91$

Now $\underbrace{1111...1}_{670\text{ 1s}} = \underbrace{111\cdot111...10000}_{666\text{ 1s}} + 1111$

As 666 is a multiple of 6, the number

$\underbrace{1111...10000}_{666\text{ 1's}}$ is divisible by 1221

Hence the remainder is 1111.

18. (a) : Denote by boxes the letters and digits to be filled up. $\underbrace{\square\square\square}_{\text{3 digits}}\underbrace{\square\square}_{\text{2 letters}}$

Number of licence plates that don't use the digit '3'
$= 4^3 = 13^2$

Because each of the place of digits can be filled in 4 ways, but each letter place can be filled in 13 ways.

Number of licence-plates that don't use the letter 'A'
$= 5^3 \times 12^2$

Because each of the places of digits can be filled in 5 ways, but each letter place can be filled in 12 ways

But these two categories – 'Not 3' and 'Not A' have some overlap – namely, those licence - plates that used neither '3' nor 'A'.

Number of such licence plates $= 4^3 \times 12^2$, because now digit 3 is not available and also the letter A is not available.

Thus the desired number of licence-plates
$= 4^3 \times 13^2 + 5^3 \times 12^2 - 4^3 \times 12^2 = 19600.$

19. (d) : Assume none of a, b, c is zero. That is a, b, c are all positive.

By AM-GM inequality

$$\frac{a+b+c}{3} \ge \sqrt[3]{abc} \Rightarrow \frac{1}{3} \ge \sqrt[3]{abc} \Rightarrow 1 \ge 27abc$$

$$\Rightarrow 3abc \le \frac{1}{9} \Rightarrow 1 + 3abc \le 1 + \frac{1}{9}$$

$$\Rightarrow 1 + 3abc \le \frac{1}{9} \Rightarrow 1 + 3abc \ge \frac{9}{10}$$

Now by weighted mean inequality

$$\frac{a\cdot\frac{1}{1+bc} + b\cdot\frac{1}{1+ca} + c\cdot\frac{1}{1+ab}}{a+b+c} \ge \frac{a+b+c}{a\cdot(1+bc) + b(1+ca) + c(1+ab)}$$

(In notation $M_1(x) \ge M_{-1}(x)$)

$$\Rightarrow \frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \ge \frac{1}{(a+b+c) + 3abc}$$

$$\Rightarrow \frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \ge \frac{1}{1+3abc} \ge \frac{9}{10}$$

$$\therefore \frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \ge \frac{9}{10}$$

Note that 9/10 is the greatest number that can be

$y = x$ at (1, 1) and also touches the x-axis is

(a) $2-\sqrt{2}$ (b) $2+\sqrt{2}$ (c) $\sqrt{2}-1$ (d) $1+\sqrt{2}$.

30. $A \equiv (-4, 0)$, $B \equiv (4, 0)$. M and N are the variable points of y-axis such that M lies below N, $MN = 4$. Line joining AM and BN intersect at 'P'. Locus of 'P' is

(a) $2xy - 16 - x^2 = 0$ (b) $2xy + 16 - x^2 = 0$

(c) $2xy + 16 + x^2 = 0$ (d) $2xy - 16 + x^2 = 0$.

ANSWERS

1. (c)	2. (c)	3. (d)	4. (c)	5. (c)
6. (b)	7. (c)	8. (b)	9. (b)	10. (c)
11. (b)	12. (c)	13. (c)	14. (b)	15. (b)
16. (a)	17. (d)	18. (a)	19. (b)	20. (c)
21. (d)	22. (b)	23. (b)	24. (d)	25. (b)
26. (d)	27. (a)	28. (b)	29. (a)	30. (d)

Patwatoli, village of IITians
48 Dalits have made it to institute on merit

IN the land of "quota politics", Patwatoli village of Bihar truly stands out to be an exception.

Unlike the champions of Mandalisation in polity, students here do not suffer from any "quota syndrome", which is often perceived as the only ladder for upward mobility for the Dalits and backwards.

Known as the IIT village of Bihar, success here is redefined as "where there is a will, there is a way". This Dalit village in Gaya district has 48 students who have made it to the IIT on merit, and not quota now applicable in premier institutions, in the past one decade. Four of them made it last year alone.

The village has perpetual power crisis but IIT aspirants study together and are coached by their successful seniors.

Santosh Kumar, who owns a small shop in Gaya, told that the story began in 1992 when the first student from village had cleared the exams and there has been no looking back since. Santosh said, "The aspiring poor students here help their fathers in the shops to earn their livelihood on one hand, and prepare for the IIT entrance on the other".

The fathers of the aspiring students are equally happy hoping their sons move out of the village. Soon after the first few students made it to IIT till 1993, the successful seniors then set up an organisation called "Nav Prayas".

This organisation helps aspirants from the village prepare for the exams. Social activist Ajay Kumar, who hails from Gaya, admits that the Nav Prayas is doing good social work. "Seniors are the guides and they instill confidence among new aspirants the need to compete". Officials in the district administration too admire the teracity of the Dalit students of Patwatoli. As Patwatoli emerges as a model for others to follow by producing 48 IIT students, it perhaps also shows how commitment combined with the concerted action can overcome toughest of odds, without quota.

10 SUBJECTIVE PROBLEMS

PROBABILITY

1. The digits 1, 2, 3... 9 are arranged at random to write a 9-digit number. What's the chance that the difference of the digits at equal distances from both the ends is always one?

2. Two bishops' one black and the other white are placed at random on a chess board. What is the chance that they kill each other ?

3. Six couples are arranged at random for a photograph. What's the chance that the two members of each couple are always together ?

4. 3 identical green balls, 2 identical red balls and 5 identical blue balls are arranged in a row at random. What's the chance that the first green ball comes before first red ball?

5. Two four digit number X and Y are written down at random. What's the chance that Y can be subtracted from X without borrowing ?

6. A bag has two cubical dice one fair and the other unfair for which the chance of getting k dots on the die is proportional to k itself. One of the dice is selected at random from the bag and rolled. If it turns up an even number what's the chance that the selected die is fair ?

7. There are two urns in which the first urn has six white and four black balls where as the second has three white and two black balls respectively. Three balls are drawn from the first urn and the balls of predominant color are dropped into the second urn. Later a ball is drawn from the second urn. What's the chance that it is white ?

8. The numbers X, Y are chosen at random from the set of natural numbers $\{1, 2, 3, ..., N\}$ $N > 2$ with replacement. What is the chance that $X^2 - Y^2$ is divisible by 3 ?

9. X and Y are two independent binomial variates such that $X(5, 1/2)$ and $Y(7, 1/2)$ then what's the chance that $P(X+Y) = 3$?

10. A fair coin is tossed 6 times. The random variable X is the difference between the number of heads and that of tails in the random experiment. What's the mean and variance of X ?

SOLUTION

1. The total number of ways of writing down the number = 9!

The 9 digits can be split into pairs (1, 2), (2, 3), (3, 4)....... (8, 9) such that the difference of the digits is 1.

The total number of ways of filling up of first place is 9 and its pair should be placed in the last place. The second place from the beginning can be filled in 7 ways. The second place from the last can be filled with its pair in only one way. The third place from the beginning can be filled in 5 ways and the third place from the last can be filled in 1 way and the fourth place from the beginning can be filled in 3 ways and the fourth from the last can be filled in 1 way with its pair and finally the middle place can be filled in 1 way.

The total number of favourable ways $= 9 \times 7 \times 5 \times 3 \times 1$

Required chance $= \dfrac{9\times7\times5\times3\times1}{9!} = \dfrac{1}{8\times6\times4\times2} = \dfrac{1}{384}$

2. First the total number of ways of putting the two bishops on the chess board successively $= {}^2C_1 \times 64 \times 63 = 2 \times 64 \times 63$

One of the two bishops can be selected in 2 ways.

Case 1 : When the first bishop occupies any of 28 squares in the external border the number of ways of putting the second bishop such that it is killed is 7 in each case and the number of ways is $28 \times 7 = 196$

Case 2 : When the first bishop occupies any one of the 20 squares in the next inner large square then the number of ways of putting the second bishop is $20 \times 9 = 180$

Case 3 : When the first bishop occupies any one of the 12 squares in the next inner boundary then the number of ways of putting the second bishop is $12 \times 11 = 132$

Case 4 : When the first bishop occupies any one of the 4 squares in the middle then the number of ways of putting the second bishop is $4 \times 13 = 52$

The total number or favourable ways of placing the

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second bishop such that it is killed

$= 196 + 180 + 132 + 52 = 560$

The total number of favourable ways is

$2(196 + 180 + 132 + 52) = 2(560)$

Required chance $= \dfrac{2 \times 560}{2 \times 64 \times 63} = \dfrac{5}{36}$

3. First six couples can be arranged at random in a row in a total number of 12! ways. If the two members of each couple are together then each couple should be taken as one entity and the two members in each entity can be arranged in 2! ways and the six entities can be arranged in 6! ways $\therefore$ the total number of favorable ways is $6!\, 2^6$

$\therefore$ Required probability $= \dfrac{6!\, 2^6}{12!}$

4. The total no. of ways of arranging the ten balls $= \dfrac{10!}{2! \cdot 3! \cdot 5!} = 2520$

Let E be the event that first green ball comes before first red ball. Let G, B, R denote green, blue, red balls.

Position of the first green ball	No. of ways of arranging the remaining balls
G	$\dfrac{9!}{2! \cdot 2! \cdot 5!}$
BG	$\dfrac{8!}{2! \cdot 2! \cdot 4!}$
BBG	$\dfrac{7!}{2! \cdot 2! \cdot 3!}$
BBBG	$\dfrac{6!}{2! \cdot 2! \cdot 2!}$
BBBBG	$\dfrac{5!}{2! \cdot 2!}$
BBBBBG	$\dfrac{4!}{2! \cdot 2!}$

Total number of favourable out comes =

$\dfrac{9!}{2! \cdot 2! \cdot 5!} + \dfrac{8!}{2! \cdot 2! \cdot 4!} + \dfrac{7!}{2! \cdot 2! \cdot 3!} + \dfrac{6!}{2! \cdot 2! \cdot 2!} + \dfrac{5!}{2! \cdot 2!} + \dfrac{4!}{2! \cdot 2!}$

$= 756 + 840 + 210 + 90 + 30 + 6 = 1512$

$\therefore$ Required chance $= \dfrac{1512}{2520}$

5. Total numbers of ways of writing X and Y is $(9 \times 10^3)^2$

If Y is to be subtracted from X without borrowing then the digit in every position of X should not be less than the digit in the corresponding place of Y. The following cases arise for the digits in the units place of X and Y

Digit in Units place of Y	Digit in Units place of X	Number of ways
0	0, 1, ...9	10
1	1, 2, ...9	9
2	2, 3, ...9	8
3	3, 4, ...9	7
4	4, 5, ...9	6
5	5, 6, ...9	5
6	6, 7, 8, 9	4
7	7, 8, 9	3
8	8, 9	2
9	9	1

Number of favourable ways for the unit's digit of the two numbers such that Y can be subtracted from X without borrowing $= 10 + 9 + 8 + ...+ 1 = 55$

Similarly number of ways of filling up the digit in tens and hundreds places is each 55, for both the numbers, but the unit's place can be filled for both the numbers in 45 ways as the first position should not be occupied by 0.

Total number of favourable ways is $55^3 \times 45$

Required probability $= \dfrac{55 \times 55 \times 55 \times 45}{81 \times 1000 \times 1000} = \dfrac{5(0.55)^3}{9}$

6. Let E be the event of getting an even number and A, B be the events of selecting the fair and unfair dice respectively.

Probability of selecting the fair die $A = P(A) = 1/2$

Probability of getting an even number on the fair die $= P(E/A) = 3/6 = 1/2$

Probability of selecting the die $B = P(B) = 1/2$

On the unfair die assume the chances of getting 1, 2, 3... 6 is k, $2k$, $3k$, $4k$, $5k$, $6k$

Now $k + 2k + 3k + 4k + 5k + 6k = 1 \Rightarrow 21k = 1 \Rightarrow k = 1/21$

Probability of getting an even number on the unfair die

$= P(E/B) = 2k + 4k + 6k = 12k = \dfrac{12}{21}$

By Baye's theorem

Required chance $= P(B/E) = \dfrac{P(B)\,P(E/A)}{P(A)\,P(E/A) + P(B)\,P(E/B)}$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{12}{21}} = \frac{21}{45} = \frac{7}{15}$$

7. Let E be the event of drawing a white ball from the second urn. Let A_1, A_2, A_3, A_4 be the events of drawing 3 White; 2 White, 1 Black; 1 White, 2 Black; 3 Black balls respectively.

$$P(A_1)=\frac{^6C_3}{^{10}C_3}=\frac{1}{6};\ P(A_2)=\frac{^6C_2\times{}^4C_1}{^{10}C_3}=\frac{1}{2};$$

$$P(A_3)=\frac{^6C_1\times{}^4C_2}{^{10}C_3}=\frac{3}{10};\ \ P(A_4)=\frac{^4C_3}{^{10}C_3}=\frac{1}{30}$$

Also $P(E/A_1)=\frac{6}{8}; P(E/A_2)=\frac{5}{7};$

$P(E/A_3)=\frac{3}{7}; P(E/A_4)=\frac{3}{8}$

$\therefore$ Required probability is $\sum_{i=1}^{4}P(A_i)\,P(E/A_i)$

$$=\frac{1}{6}\times\frac{6}{8}+\frac{1}{2}\times\frac{5}{7}+\frac{3}{10}\times\frac{3}{7}+\frac{1}{30}\times\frac{3}{8}=\frac{1}{8}+\frac{5}{14}+\frac{9}{70}+\frac{1}{80}=\frac{349}{560}$$

8. The total number of ways of selecting both X and $Y = N\times N = N^2$

Every number is in the form $3k$; $3k-1$; $3k+1$

If both numbers are of the form $3k$; say $X = 3l$, $Y = 3m$ then $X^2 - Y^2 = 9l^2 - 9m^2 = 9(l^2 - m^2)$ which is divisible by 3.

If both numbers are not multiple of 3 say $X = 3l - 1$, $Y = 3m - 1$ then $X^2 - Y^2 = (3l-1)^2 - (3m-1)^2$ $= 9l^2 - 6l - 9m^2 + 6m = 3(3l^2 - 2l - 3m^2 + 2m)$ and this is divisible by 3.

Number of multiples of 3 from 1 to $N=\left[\frac{N}{3}\right]$

Number of numbers which are not multiples of $3=N-\left[\frac{N}{3}\right]$

Number of ways of selecting two multiples of $3=\left[\frac{N}{3}\right]^2$

Number of ways of selecting two numbers which are not multiples of $3=\left(N-\left[\frac{N}{3}\right]\right)^2$

Total no. of favourable ways $=\left[\frac{N}{3}\right]^2+\left(N-\left[\frac{N}{3}\right]\right)^2$

Required probability $=\dfrac{\left[\frac{N}{3}\right]^2-\left(N-\left[\frac{N}{3}\right]\right)^2}{N^2}$

9. Range of $X = \{1, 2, 3, 4, 5\}$

Range of $Y = \{1, 2, 3, 4, 5, 6, 7\}$

If $X + Y = 3$ then $(X, Y) \in \{(0, 3), (1, 2), (2, 1), (3, 0)\}$

$P(X+Y)=3 \Rightarrow P(X=0\cap Y=3)\cup P(X=1\cap Y=2)\cup P(X=2\cap Y=1)\cup\ P(X=3\cap Y=0)$

i.e. $P(X+Y)=3\Rightarrow P(X=0)\,P(Y=3)+P(X=2)\,P(Y=1)+P(X=1)\,P(Y=2)+P(X=3)\,P(Y=0)$

Required chance

$$=\frac{^5C_3}{2^5}\cdot\frac{^7C_0}{2^7}+\frac{^5C_2}{2^5}\cdot\frac{^7C_1}{2^7}+\frac{^5C_1}{2^5}\cdot\frac{^7C_2}{2^7}+\frac{^5C_0}{2^5}\cdot\frac{^7C_3}{2^7}$$

$$=\frac{10+70+105+35}{4096}=\frac{220}{4096}=\frac{55}{1024}$$

10. Given $X = |\text{No. of Heads} - \text{No. of Tails}|$

Range of $X = \{0, 2, 4, 6\}$

Probability of getting 3 heads and 3 tails

$=P(X=0)=\frac{^6C_3}{2^6}=\frac{20}{64};$

Probability of getting 4 heads and 2 tail or 2 head and 4 tails $=P(X=2)=\frac{2\cdot{}^6C_2}{2^6}=\frac{30}{64};$

Probability of getting 5 heads and 1 tail or 1 head and 5 tails $=P(X=4)=\frac{2\cdot{}^6C_5}{2^6}=\frac{12}{64};$

Probability of getting 6 heads and 0 tails or 0 heads and 6 tails $=P(X=6)=\frac{2\cdot{}^6C_6}{2^6}=\frac{2}{64}$

X	0	2	4	6
$P(X=x_r)$	20/64	30/64	12/64	2/64

Mean of X, μ

$$=\sum_{r=0}^{6}x_r\,P(X=x_r)=\frac{60}{64}+\frac{48}{64}+\frac{12}{64}=\frac{120}{64}=1.875$$

Variance of $X=\sum_{r=0}^{6}x_r^2P(X=x_r)-\mu^2$

$$=\frac{120}{64}+\frac{192}{64}+\frac{72}{64}=\frac{384}{64}=6-(1.875)^2=2.4844$$

MOCK TEST

FOR

ISI 2008

Exam on 7th May 2008

By **Alok Kumar**, B.Tech, IIT Kanpur

MULTIPLE CHOICE TYPE TEST

1. For how many values of p are the numbers p, $\frac{p-1}{4}$ and $\frac{p+1}{2}$ are all prime?

(a) exactly one (b) atleast two
(c) infinitely many (d) no value.

2. Let $A = \{a_1, a_2, ..., a_7\}$ a set of seven elements and $B = \{b_1, b_2, b_3\}$ is a set of three elements. The number of function f from A to B such that

(i) f is onto, and
(ii) there are exactly three elements x in A such that $f(x) = b_1$ is

(a) 558 (b) 560 (c) 490 (d) 1680.

3. Sum $1\,\underline{|1} + 2\,\underline{|2} + 3\,\underline{|3} + ... + 2008\,\underline{|2008}$ equals

(a) $\underline{|2008} - 1$ (b) $\underline{|2009} - 1$
(c) $2\underline{|2008} - 1$ (d) $2\underline{|2009} - 1$

4. The minimum possible value of $|z|^2 + |z-3|^2 + |z-6i|^2$ is

(a) 30 (b) 45 (c) 15 (d) 20.

5. The number of integer (positive, negative or zero) solutions of $xy - 6(x + y) = 0$ with $x \le y$ is

(a) 10 (b) 9 (c) 12 (d) 6.

6. The number of integers $n > 1$, such that n, $n + 2$, $n + 4$ are all prime numbers is

(a) zero (b) one
(c) infinite
(d) more than one, but finite.

7. The remainder $R(x)$, when polynomial x^{100} is divided by $x^2 - 3x + 2$ is

(a) $(2^{100} - 1)\,x - 2(2^{99} - 1)$
(b) $2^{100}x - 2(2^{100} - 1)$
(c) $(2^{100} - 1)\,x + 2(2^{99} - 1)$
(d) $(2^{100} + 1)\,x - 2(2^{99} + 1)$.

8. The expression

$\sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x}$ simplifies to

(a) $\cos 2x$ (b) $\cos x$ (c) $\sin 2x$ (d) $\sin x$.

9. $(1 + \tan 1°)(1 + \tan 2°) \dots (1 + \tan 45°) = 2^n$, the value of n is

(a) 22 (b) 23 (c) 24 (d) 21.

Directions for questions 10 to 12 : Answer the following questions on the basis of the passage given.

On a rainy day n people go to a party in a hotel. Each of them leaves his umbrella at property counter. Denote by D_n the number of ways in which the umbrellas are handed back to them after the party in such a manner that no person receives his own umbrella.

10. D_n satisfies

(a) $D_n = (n-1)(D_{n-1} + D_{n-2})$
(b) $D_n = (n-1)D_{n-1} + (n-2)D_{n-2}$
(c) $nD_n = (n-1)(D_{n-1} + D_{n-2})$
(d) $nD_n = (n-1)D_{n-1} + (n-2)D_{n-2}$

11. Let $n = 6$, how many ways are there in which exactly one perspon gets back his own umbrella?

(a) 135 (b) 264 (c) 275 (d) 305.

12. Let $D = 6$, how many ways are there in which exactly two persons get back their own umbrellas?

(a) 135 (b) 264 (c) 275 (d) 305.

13. The sum

$$\frac{3}{1!+2!+3!} + \frac{4}{2!+3!+4!} + ... + \frac{2008}{2006!+2007!+2008!}$$

equals

(a) $\frac{1}{2} - \frac{1}{2006!}$ (b) $\frac{1}{2} - \frac{1}{2008!}$
(c) $\frac{1}{2006!} - \frac{1}{2008!}$ (d) $\frac{1}{2007!} - \frac{1}{2008!}$

14. The line $x + y = \alpha$, $\alpha > 0$ meets the axis of x and y at A and B respectively. A triangle AMN is inscribed in the ΔOAB, O being the origin of coordinates with right angle at N, M and N lie on OB and AB respectively. If the area of the triangle AMN is three eighths of the area of ΔOAB, then $\frac{AN}{BN}$ is

(a) 3 (b) 1/3 (c) 2/3 (d) 3/2.

15. Let $f(x,y)=\sqrt{x^2+y^2}+\sqrt{(x-1)^2+y^2}$
$$+\sqrt{x+(y-1)^2}+\sqrt{(x-3)^2+(y-4)^2},$$
where x and y range over all real numbers. The minimum value of $f(x, y)$ is

(a) $2+\sqrt{5}$ (b) $5+\sqrt{2}$
(c) $5-\sqrt{2}$ (d) none of these.

Directions for questions 16-18 : Answer the following questions on the basis of the passage given.

Let polynomials $A(x)$, $B(x)$, $C(x)$, if they exist, satisfy for all x.

$$|A(x)|-|B(x)|+|C(x)|=\begin{cases}-1 & x<-1\\ 3x+2 & -1\le x\le 0\\ -2x+2 & x>0\end{cases}$$

16. $A(x)$ is

(a) $\frac{3x-3}{2}$ (b) $\frac{-5x}{2}$ (c) $x-\frac{1}{2}$ (d) $\frac{3x+3}{2}$

17. $B(x)$ is

(a) $\frac{-5x}{2}$ (b) $\frac{5x}{2}$ (c) $x-\frac{1}{2}$ (d) $\frac{3x-3}{2}$

18. $C(x)$ is

(a) $-x+\frac{1}{2}$ (b) $\frac{-5x}{2}$ (c) $\frac{3x-3}{2}$ (d) $x-\frac{1}{2}$

19. Orthocentre of triangle with vertices (0, 0), (3, 4) and (3, 0) is

(a) $\left(3, \frac{5}{4}\right)$ (b) (3, 12) (c) (3, 4) (d) (3, 0).

20. If $x^2+x+1=0$, then $\sum_{r=1}^{2005}\left(x^r+\frac{1}{x^r}\right)^3=$

(a) 4007 (b) 4006
(c) 4005 (d) none of these.

21. The value of
$$\binom{2000}{2}+\binom{2000}{5}+\binom{2000}{8}+....+\binom{2000}{2000}$$ is

(a) $\frac{2^{2000}+1}{3}$ (b) $\frac{2^{2000}-1}{3}$

(c) $\frac{2^{2001}-1}{3}$ (d) $\frac{2^{2001}+1}{3}$

22. Let ABC be an acute angled triangle with area A.

Then $\sqrt{a^2b^2-4A^2}+\sqrt{b^2c^2-4A^2}+\sqrt{c^2a^2-4A^2}$ simplifies to

(a) $\frac{a^2+b^2+c^2}{4}$ (b) $\frac{ab+bc+ca}{4}$

(c) $\frac{a^2+b^2+c^2}{2}$ (d) $\frac{ab+bc+ca}{2}$

23. The incentre of the triangle with vertices $(1, \sqrt{3})$, (0, 0) and (2, 0) is

(a) $\left(1, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$

(c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(1, \frac{1}{\sqrt{3}}\right)$

24. The area of the triangle whose vertices are (a, a), $(a+1, a+1)$, $(a+2, a)$ is

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 1/2 (d) 1.

25. Let $I=\int_0^1\frac{e^t}{1+t}dt$ then $I=\int_{a-1}^{a}\frac{e^{-t}}{t-a-1}dt$ in terms of I equals

(a) $-Ie^{-a}$ (b) Ie^{-a} (c) Ie^{a} (d) $-Ie^{a}$.

26. Let P be a point in the first quadrant lying on the ellipse $\frac{x^2}{8}+\frac{y^2}{18}=1$. Let AB be the tangent at P to the ellipse meeting the x-axis at A and y-axis at B. If O is the origin, the minimum possible area of triangle OAB is

(a) 12 (b) 9π (c) 6π (d) 4.

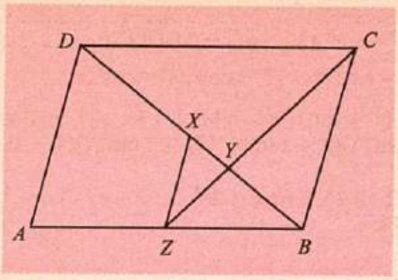
27. The equation of the circle circumscribing the triangle formed by the points (0, 0), (1, 0) and (0, 1) is

(a) $x^2+y^2+x+y=0$ (b) $x^2+y^2+x-y+2=0$
(c) $x^2+y^2-x-y=0$ (d) $x^2+y^2-x+y+2=0$.

28. Suppose that a, b, c are three distinct real numbers. The expression
$$\frac{(x-a)(x-b)}{(c-a)(c-b)}+\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}=1$$
takes the value zero for

(a) no real x
(b) exactly two distinct real x
(c) exactly three distinct real x
(d) more than three real x.

29. In the picture, $ABCD$ is a parallelogram. AD is parallel to ZX and $\frac{AZ}{ZB}$ equals 2/3. Then $\frac{XY}{BD}$ equals



(a) 1/4 (b) 3/5 (c) 9/40 (d) 9/25.

30. The least positive integer k such that $\binom{2n}{n}^{1/n} < k$ for all positive integers n is

(a) $k=7$ (b) $k=4$ (c) $k=3$ (d) $k=5$.

SHORT ANSWER TYPE TEST

1. Let $f(x)=\int_0^1 |t-x|\,t\,dt$ for all real x. Sketch the graph of $f(x)$. What is the minimum value of $f(x)$?

2. Find all (x, y) such that $\sin x + \sin y = \sin(x+y)$ and $|x| + |y| = 1$.

3. Suppose f is a real valued differentiable function defined on $[1, \infty)$ with $f(1) = 1$. Suppose, moreover, that f satisfies $f'(x)=\dfrac{1}{x^2+f^2(x)}$. Show that $f(x) \le 1 + \pi/4$ for every $x \ge 1$

(By the symbol $f^2(x)$ we mean $\{f(x)\}^2$)

4. Let $a_1, a_2, \ldots, a_n$ be n numbers such that each a_2 is either 1 or -1. If $a_1a_2a_3a_4 + a_2a_3a_4a_5 \ldots\ldots + a_na_1a_2a_3 = 0$ then prove that 4 divides n.

5. Let $a_0 = 0 < a_1 < a_2 \ldots. < a_n$ be real numbers. Suppose $p(t)$ is a real valued polynomial of degree n such that

$$\int_{a_j}^{a_{j+1}} p(t)dt = 0 \text{ for all } 0 \le j \le n-1,$$

show that, for $0 \le j \le n-1$, the polynomial $p(t)$ has exactly one root in the interval (a_j, a_{j+1}).

6. Suppose that the three equations

$ax^2 - 2bx + c = 0$; $bx^2 - 2cx + a = 0$

$cx^2 - 2ax + b = 0$

all have only positive roots. Such that a, b, c are all equal.

7. Let a, b, c, d satisfy

$$\frac{a^2}{2^2-1^2}+\frac{b^2}{2^2-3^2}+\frac{c^2}{2^2-5^2}+\frac{d^2}{2^2-7^2}=1$$

$$\frac{a^2}{4^2-1^2}+\frac{b^2}{4^2-3^2}+\frac{c^2}{4^2-5^2}+\frac{d^2}{4^2-7^2}=1$$

$$\frac{a^2}{6^2-1^2}+\frac{b^2}{6^2-3^2}+\frac{c^2}{6^2-5^2}+\frac{d^2}{6^2-7^2}=1$$

$$\frac{a^2}{8^2-1^2}+\frac{b^2}{8^2-3^2}+\frac{c^2}{8^2-5^2}+\frac{d^2}{8^2-7^2}=1$$

Find the value of $a^2 + b^2 + c^2 + d^2$.

8. Let P be a point inside the triangle ABC such that $\angle APB = \angle BPC = \angle CPA$. Prove that

$$PA+PB+PC=\sqrt{\frac{a^2+b^2+c^2}{2}+2\sqrt{3}\,\Delta}$$

where a, b, c, Δ are the sides and area of the triangle ABC.

9. Find the angle at the vertex of an isosceles triangle of given area such that the radius of the incircle is maximum.

10. In how many ways can one fill a 4×4 matrix with ±1 so that the product of the entries in each row and each column equal to -1?

SOLUTIONS TO MULTIPLE CHOICE TYPE TEST

1. (a) : Let $\dfrac{p-1}{4}=k$, some positive integer k, then $p = 4k + 1$

We then have $\dfrac{p+1}{2}=\dfrac{4k+2}{2}=2k+1$

Then the given numbers become $4k + 1$, k, $2k + 1$

Observe that $4k + 1 = 3k + k + 1$

$$2k + 1 = 3k - (k - 1)$$

The remainder when $2k + 1$, k, $4k + 1$ are divided by 3 are the same as the remainder when $k - 1$, k, $k + 1$ are divided by 3. But these are three consecutive integers, so one of them is a multiple of 3.

Now the only multiple of 3 that is prime is 3 itself. Only for $k = 3$ all three $4k + 1$, k, $2k + 1$ are prime *viz.* 13, 3, 7.

2. (c) : The three elements in A which form the pre-image set of b_1 can be chosen in ${}^7C_3=\dfrac{7\cdot6\cdot5}{1\cdot2\cdot3}=35$ ways

Now the remaining 4 elements in A can be associated with the remaining two elements in B such that the function is onto is $2^4 - 2 = 14$.

Thus the number of function satisfying the given condition, by product rule, is $35 \times 14 = 490$.

3. **(b)** : Let's telescope the series. Note that

$$k\underline{|k} = [(k+1)-1]\underline{|k} = \underline{|k+1} - \underline{|k}$$

Thus, $1\cdot\underline{|1} = \underline{|2} - \underline{|1}$

$2\cdot\underline{|2} = \underline{|3} - \underline{|2}$

....

$2008\cdot\underline{|2008} = \underline{|2009} - \underline{|2008}$

Adding we get

$$1\underline{|1} + 2\underline{|2} + ... + 2008\underline{|2008} = \underline{|2009} - 1$$

4. **(a)** : Look at the problem in the geometric setting. Let $A \equiv 0$, $B \equiv 3$ and $C \equiv 6i$ and $P \equiv z$. Note that A, B, C determine a triangle. Then the problem reduces to finding the minimum possible value of $PA^2 + PB^2 + PC^2$. Geometrically, we know that the minimum occurs at centroid of the triangle ABC, which is at

$$z = \frac{0+3+6i}{3} = 1+2i$$

The minimum possible value is

$$|1+2i|^2 + |1+2i-3|^2 + |1+2i-6i|^2$$
$$= (1^2+2^2) + (2^2+2^2) + (1^2+4^2)$$
$$= 1+4+4+4+1+16 = 30.$$

5. **(a)** : $xy - 6(x+y) = 0$

$\Rightarrow\ xy - 6(x+y) + 36 = 36$

$\Rightarrow\ (x-6)(y-6) = 36 \quad \Rightarrow \quad uv = 36$

where $u = (x-6)$, $v = y-6$. $x \le y$ means $u \le v$

Once (u, v) is a solution, so is $(-v, -u)$.

The solution in positive integers are

$(1, 36), (2, 18), (3, 12), (4, 9), (6, 6)$

Also $(-36, -1), (-18, -2), (-12, -3), (-9, -4), (-6, -6)$ are the solutions.

We have in all 10 solutions.

6. **(b)** : n, $n+2$, $n+4$ when divided by 3 leave the same remainder as modulo 3.

When 0, 2, 4 are divided by 3, so the remainders are congruent to 0, 2, 1 in that order. But these are consecutive integers. So one of them is necessarily divisible by 3. Now the only number that is prime and divisible by 3 is 3 itself.

For $n = 3$ only the numbers n, $n+2$, $n+4$ are prime.

7. **(a)** : By division algorithm,

$$x^{100} = (x-1)(x-2)g(x) + ax + b$$

$$\left.\begin{aligned} \text{At } x = 1 &\Rightarrow 1 = a + b \\ x = 2 &\Rightarrow 2^{100} = 2a + b \end{aligned}\right\} \quad \begin{aligned} a &= 2^{100} - 1 \\ b &= 2 - 2^{100} \end{aligned}$$

Thus, $R(x) = ax + b = (2^{100} - 1)x + (2 - 2^{100})$

$= (2^{100} - 1)x + 2(1 - 2^{99})$.

8. **(a)** : $\sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x}$

$= \sqrt{\sin^4 x + 4(1-\sin^2 x)} - \sqrt{\cos^4 x + 4(1-\cos^2 x)}$

$= \sqrt{(2-\sin^2 x)^2} - \sqrt{(2-\cos^2 x)^2}$

$= (2 - \sin^2 x) - (2 - \cos^2 x) = \cos 2x.$

9. **(b)** : $(1+\tan 1°)(1+\tan 2°) + ... (1+\tan 45°)$

$= \{(1+\tan 1°)(1+\tan 44°)\}\{(1+\tan 2°)(1+\tan 43°)\}$ .... $(1+\tan 45°)$

$= 2\cdot 2\cdot 2 ...$ 23 terms $= 2^{23}$

$\therefore\ n = 23.$

10. **(a)**

11. **(b)**

12. **(c)** : Let the persons $A_1, A_2, ... A_n$ and their umbrellas be $a_1, a_2, ... a_n$. There are $(n-1)$ possible choices for A_1, to receive a wrong umbrella. The umbrella a_1 be disposed off in two ways. Give umbrella a_1 to A_2 or do not give umbrella a_1 to A_2. So we have a recurrence relation.

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

Recall that we learnt

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^n\frac{1}{n!}\right]$$

The number of ways that exactly one of six persons picks up the umbrella $= {}^6C_1 \times D_5 = 6 \times 44 = 264$.

The number of ways that exactly two of six persons picked up their own umbrellas $= {}^6C_2 \times D_4 = 15 \times 9 = 135$.

13. **(b)** : We employ the method of difference to telescope the sum.

$$t_k = \frac{k+2}{k! + (k+1)! + (k+2)!} = \frac{k+2}{k!(1+k+1+(k+1)(k+2)}$$

$$= \frac{k+2}{k!(k+2)^2} = \frac{(k+2)}{k!(k+2)^2} = \frac{1}{k!(k+2)} = \frac{k+1}{(k+2)!}$$

$$= \frac{(k+2)-1}{(k+2)!} = \frac{1}{(k+1)!} - \frac{1}{(k+2)!}$$

$$t_1 = \frac{1}{2!} - \frac{1}{3!}$$

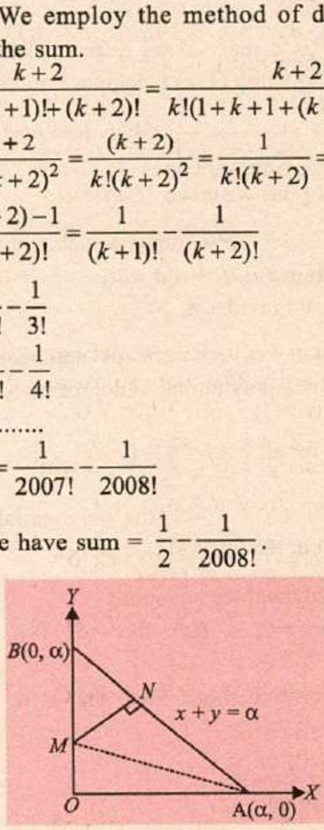
$$t_2 = \frac{1}{3!} - \frac{1}{4!}$$

....

$$t_{2006} = \frac{1}{2007!} - \frac{1}{2008!}$$

Adding we have sum $= \dfrac{1}{2} - \dfrac{1}{2008!}$.

14. **(a)** :



Let $\dfrac{AN}{BN} = k$

$$N=\left(\frac{\alpha}{k+1},\ \frac{\alpha k}{k+1}\right)$$

$MN \perp AB$

$$\therefore\ \text{Slope of } MN = -\frac{1}{\text{slope of } AB} = -\left(\frac{1}{-1}\right)=1$$

Equation of MN

$$y-\frac{\alpha k}{k+1}=(1)\left(x-\frac{\alpha}{k+1}\right)$$

$$x-y=\frac{\alpha(1-k)}{(1+k)}.$$

Thus the co-ordinate of M is $\left(0,\ \alpha\frac{k-1}{k+1}\right)$

Area of $AMN = \frac{3}{8}\times(\text{area of } OAB)$

$$=\frac{1}{2}(AN)(MN)=\frac{3}{8}\times\frac{1}{2}(\alpha)(\alpha)$$

$$=\frac{1}{2}\left|\frac{\alpha k\sqrt{2}}{1+k}\right|\left|\frac{\alpha\sqrt{2}}{1+k}\right|=\frac{3}{16}\alpha^2 \Rightarrow \frac{k}{(1+k)^2}=\frac{3}{16}$$

$\Rightarrow\ k = 3,\ 1/3.$

But if $k = 1/3$, then M lies outside the segment OB and thus the sought for value of $k = 3$.

$$\frac{AN}{BN}=3$$

15. (b) : The expression

$|z| + |z - 1| + |z - i| + |z - (3 + 4i)|$

$= |z| + |z - (3 + 4i)| + |z - 1| + |z - i|$

The least distance is $(5+\sqrt{2})$.

16. (d) **17. (b)**

18. (a) : As $x = -1$, and $x = 0$, are two critical points of the absolute function, we can assume that

$f(x) = \alpha|x + 1| + \beta|x| + \gamma|x| + \delta$

$= (\gamma - \alpha - \beta)x + \gamma - \alpha,\ x < -1$

$= (\alpha - \beta + \gamma)x + \alpha + \delta,\ -1 \le x \le 0$

$= (\alpha + \beta + \gamma)x + \alpha + \delta,\ x > 0$

which gives $\alpha=\frac{3}{2},\ \beta=\frac{-5}{2},\gamma=-1,\ \delta=\frac{1}{2}$

Then $A(x)=\frac{3x+3}{2},\ B(x)=\frac{5x}{2},\ C(x)=-x+\frac{1}{2}.$

19. (d) : Let $A = (0, 0)$, $B = (3, 4)$, $C = (3, 0)$

$AB = 5$, $BC = 4$, $AC = 3$

Orthocentre will be at $(3, 0)$.

20. (a) : $x = \omega, \omega^2$

$$\left(x+\frac{1}{x}\right)^3+\left(x^2+\frac{1}{x^2}\right)^3+\left(x^3+\frac{1}{x^3}\right)^3$$

$$= (-1)^3 + (-1)^3 + 2^3 = 6$$

The sum of any three consecutive term is 6.

Sum up to 2005th term $= 668 \times 6 - 1 = 4007$.

21. (b) : Let $f(x) = (1 + x)^{2000} = \sum_{r=0}^{2000}\binom{2000}{r}x^r$

Set ω, 1, ω^2 in turn and form

$f(1) + \omega f(\omega) + \omega^2 f(\omega)$

$$=3\left[\binom{2000}{2}+\binom{2000}{5}+\dots+\binom{2000}{2000}\right]$$

$$= 2^{2000} + \omega(1 + \omega)^{2000} + \omega^2(1 + \omega^2)^{2000} = 3S$$

$$\therefore\ S=\frac{1}{3}(2^{2000}-1).$$

22. (c) : $2A = ab\sin C = bc\sin A = ca\sin B$

The expression equals

$$\sum\sqrt{a^2b^2-a^2b^2\sin^2 C}=\sum ab\cos C$$

$$=\frac{1}{2}\sum(ab\cos C+ac\cos B)$$

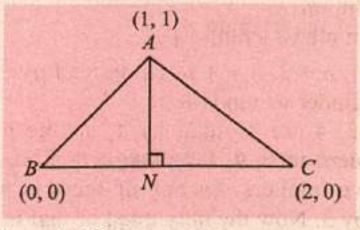
$$=\frac{1}{2}\sum a(b\cos C+c\cos B)$$

$$=\frac{1}{2}\sum a\cdot a=\frac{a^2+b^2+c^2}{2}$$

23. (d) : The triangle formed by the vertices $(1, \sqrt{3})$, $(0, 0)$ and $(2, 0)$ is an equilateral triangle. So its incentre and centroid will be same.

24. (d) : Shifting the origin to (a, a), the co-ordinates of the vertices become $(0, 0)$, $(1, 1)$, $(2, 0)$.

The triangle is as under



The equation of BC is $x = 2$

The perpendicular distance from A on BC is $AN = 1$

Area of $ABC = \frac{1}{2}\times BC\times AN=\frac{1}{2}\times 2\times 1=1$

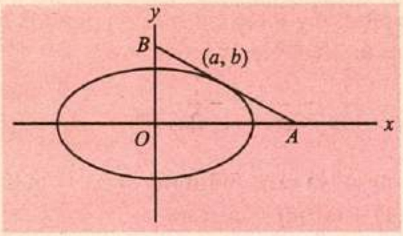
25. (a) : Let $J=\int_{a-1}^{a}\frac{e^{-t}}{t-a-1}dt$

Set $t - a = -u$, so that $dt = -du$ and limits become 1 and 0

$$J=\int_1^0\frac{e^{u-a}}{-u-1}(-du)=\int_1^0\frac{e^u e^{-a}du}{u+1}=e^{-a}\int_1^0\frac{e^u du}{u+1}=-e^{-a}\int_0^1\frac{e^u du}{1+u}$$

$$= -e^{-a}\cdot I = -Ie^{-a}.$$

26. (a) :



$$\frac{x^2}{8}+\frac{y^2}{18}=1$$

Let the tangent at $P(a, b)$ meet the x-axis at A and y-axis at B.

The equation of the tangent to the ellipse at (a, b) is

$$\frac{ax}{8}+\frac{by}{18}=1 \text{ i.e. } \frac{x}{8/a}+\frac{y}{18/b}=1$$

So that $A\equiv\left(\frac{8}{a}, 0\right)$ and $B\equiv\left(0, \frac{18}{b}\right)$

Area of triangle $OAB = \frac{1}{2}\cdot\frac{8}{a}\cdot\frac{18}{b}=\frac{72}{ab}$

As (a, b) lies on the ellipse, we have

$$\frac{a^2}{8}+\frac{b^2}{18}=1$$

From AM-GM inequality, $\dfrac{\frac{a^2}{8}+\frac{b^2}{18}}{2}\geq\sqrt{\dfrac{a^2b^2}{8\cdot18}}$

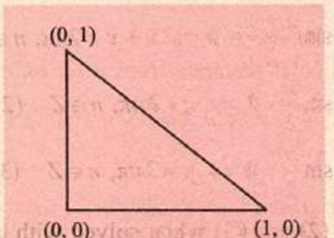
$$\Rightarrow \frac{1}{2}\geq\frac{ab}{12} \Rightarrow 6\geq ab \Rightarrow ab\leq 6 \Rightarrow \frac{1}{ab}\geq\frac{1}{6}$$

The area of the triangle $= \dfrac{72}{ab}$

As $\dfrac{1}{ab}\geq\dfrac{1}{6}$

The minimum area of the triangle $= 72\cdot\frac{1}{6}=12.$

27. (c) :



As the triangle is right angled, the circle circumscribing the triangle is the circle with its extremities of diameter being (0, 1) and (1, 0).

The equation of the circle is

$$(x-0)(x-1)+(y-1)(y-0)=0$$

$$\Rightarrow x^2+y^2-x-y=0.$$

28. (c) : Let

$$f(x)=\frac{(x-a)(x-b)}{(c-a)(c-b)}+\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}-1$$

Then $f(x)$ is a quadratic in x.

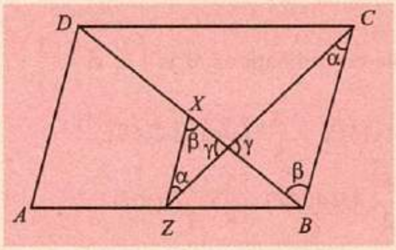
Now $f(a) = 0 + 1 + 0 - 1 = 0$

$f(b) = 0 + 0 + 1 - 1 = 0$

$f(c) = 1 + 0 + 0 - 1 = 0$

Then $f(x)$, a polynomial of degree 2, vanishes for 3 distinct real values of x. Thus it must be an identity.

29. (c) :



$$\frac{AZ}{ZB}=\frac{1}{3} \Rightarrow \frac{BZ}{BA}=\frac{3}{5} \text{ So that } \frac{BX}{BD}=\frac{3}{5} \quad \text{... (i)}$$

Triangle XYZ and BYC are similar, then

$$\frac{XY}{YB}=\frac{ZX}{BC}=\frac{3}{5}$$

As $\dfrac{XY}{YB}=\dfrac{3}{5}$, we have $\dfrac{XY}{BX}=\dfrac{3}{8}$ ... (ii)

From (i) and (ii), $\dfrac{XY}{BD}=\dfrac{3/8}{5/3}=\dfrac{9}{40}$.

30. (b) : We have

$$\binom{2n}{0}<\binom{2n}{0}+\binom{2n}{4}+\ldots+\binom{2n}{2n}=(1+1)^{2n}=4^n$$

For $n = 5$, we have

$\binom{10}{5}=252>3^5$. Thus, $k = 4$.

SOLUTIONS TO SHORT ANSWER TYPE QUESTIONS

1. We first need to find the explicit form of the function. Note that the integration is with respect to t, so that the limits on t are from 0 to 1. Now depending upon whether x lies in $(-\infty, 0]$, $(0, 1)$ or $[1, \infty)$ we have different expressions for the function.

$\boldsymbol{x \leq 0}$

$$f(x)=\int_0^1|t-x|\,tdt = \int_0^1(t-x)tdt = \int_0^1 t^2dt - x\int_0^1 tdt$$

$$=\left[\frac{t^3}{3}\right]_0^1-x\left[\frac{t^2}{2}\right]_0^1=\frac{1}{3}-\frac{x}{2}=-\frac{x}{2}+\frac{1}{3}.$$

$\boldsymbol{0 < x < 1}$

Now both x and t lie in the same interval and so to decide the sign of $t - x$, we need to break (0, 1) further into (0, x] and [x, 1).

$$f(x) = \int_0^x |t-x|\,t\,dt + \int_x^1 |t-x|\,t\,dt$$

$$= \int_0^x (x-t)t\,dt + \int_x^1 (t-x)t\,dt$$

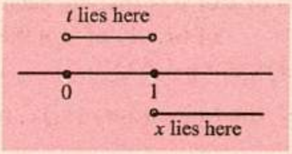
$$= x\int_0^x t\,dt - \int_0^x t^2 dt + \int_x^1 t^2 dt - x\int_x^1 t\,dt$$

$$= x\left[\frac{t^2}{2}\right]_0^x - \left[\frac{t^3}{3}\right]_0^x + \left[\frac{t^3}{3}\right]_x^1 - x\left[\frac{t^2}{2}\right]_x^1$$

$$= \frac{x^3}{2} - \frac{x^3}{3} + \frac{1}{3}(1-x^3) - \frac{x}{2}(1-x^2)$$

$$= \frac{x^3}{6} + \frac{1}{3} - \frac{x^3}{3} - \frac{x}{2} + \frac{x^3}{2} = \frac{x^3}{3} - \frac{x}{2} + \frac{1}{3}.$$

$x \geq 1$



$$f(x) = \int_0^1 |t-x|\,t\,dt = \int_0^1 (x-t)t\,dt$$

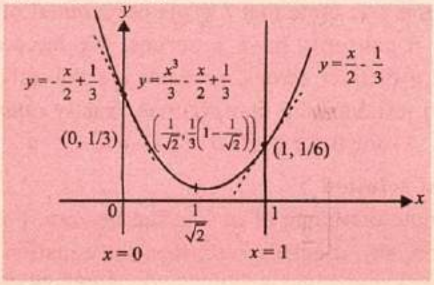
$$= x\int_0^1 t\,dt - \int_0^1 t^2 dt = x\left[\frac{t^2}{2}\right]_0^1 - \left[\frac{t^3}{3}\right]_0^1 = \frac{x}{2} - \frac{1}{3}$$

Thus we summarise,

$$f(x) = \begin{cases} -\dfrac{x}{2} + \dfrac{1}{3}, & x \leq 0 \\ \dfrac{x^3}{3} - \dfrac{x}{2} + \dfrac{1}{3}, & 0 < x < 1 \\ \dfrac{x}{2} - \dfrac{1}{3}, & x \geq 1 \end{cases}$$

The function is continuous at $x = 0$ and $x = 1$, which can be easily verified by little calculation.

The graph of $y = f(x)$ can be drawn as under



The minimum value of $f(x)$ occurs in (0, 1).

As $y = \dfrac{x^3}{3} - \dfrac{x}{2} + \dfrac{1}{3}$, $0 < x < 1$

$$\frac{dy}{dx} = \frac{3x^2}{3} - \frac{1}{2} = x^2 - \frac{1}{2}$$

dy/dx vanishes at $x^2 = 1/2$ *i.e.* $x = \dfrac{1}{\sqrt{2}}$ in (0, 1)

$$\frac{d^2y}{dx^2} = 2x$$

And thus $\dfrac{d^2y}{dx^2}$ is positive at $x = \dfrac{1}{\sqrt{2}}$, so $y_{\min}$ occurs at $x = \dfrac{1}{\sqrt{2}}$.

$$y_{\min} = \left[\frac{x^3}{3} - \frac{x}{2} + \frac{1}{3}\right]_{\text{at } x = \frac{1}{\sqrt{2}}}$$

$$= \left[\frac{x}{6}(2x^2 - 3) + \frac{1}{3}\right]_{\text{at } x = \frac{1}{\sqrt{2}}}$$

$$= \frac{1}{6\sqrt{2}}\left[2 \cdot \frac{1}{2} - 3\right] + \frac{1}{3} = -\frac{2}{6\sqrt{2}} + \frac{1}{3}$$

$$= \frac{-1}{3\sqrt{2}} + \frac{1}{3} = \frac{1}{3}\left(1 - \frac{1}{\sqrt{2}}\right).$$

It can be verified that $\dfrac{1}{3}\left(1 - \dfrac{1}{\sqrt{2}}\right) < \dfrac{1}{6}$.

2. $\sin x + \sin y = \sin(x + y)$

Applying transformation formulae we get

$$2\sin\frac{x+y}{2}\cos\frac{x-y}{2} = 2\sin\frac{x+y}{2}\cos\frac{x+y}{2}$$

$$\Rightarrow \quad 2\sin\frac{x+y}{2}\left\{\cos\frac{x-y}{2} - \cos\frac{x+y}{2}\right\} = 0$$

$$\Rightarrow \quad 2\sin\frac{x+y}{2}\sin\frac{x}{2}\sin\frac{y}{2} = 0$$

Thus either $\sin\dfrac{x+y}{2} = 0 \Rightarrow x + y = 2n\pi,\ n \in Z$ (1)

or $\sin\dfrac{x}{2} = 0 \Rightarrow x = 2n\pi,\ n \in Z$ (2)

or $\sin\dfrac{y}{2} = 0 \Rightarrow y = 2n\pi,\ n \in Z$ (3)

Each of (1), (2) and (3) when solved with $|x| + |y| = 1$ produce the solution to the system. Let's take them one by one.

(I) $\left.\begin{array}{l} x + y = 2n\pi,\ n \in z \\ |x| + |y| = 1 \end{array}\right\}$

Using triangle inequality, $|x + y| \leq |x| + |y|$

Thus $|2n\pi| \le 1 \Rightarrow |n| \le \frac{1}{2\pi}$

Thus the only integral value that n can take is 0.

So our system reduces to $\left.\begin{array}{l} x+y=0 \\ |x|+|y|=1 \end{array}\right\}$

which has two solutions, $\left(\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right)$

(II) $\left.\begin{array}{l} x=2n\pi, n\in z \\ |x|+|y|=1 \end{array}\right\}$

Plugging the first equation in second, we have

$|2n\pi| + |y| = 1 \qquad \Rightarrow |y| = 1 - |2n\pi|$

As $|y| \ge 0 \qquad \Rightarrow 1 - |2n\pi| \ge 0$

$\Rightarrow |2n\pi| \le 1 \qquad \Rightarrow |n| \le \frac{1}{2\pi}$

The only integral value that n can take is 0. Thus there are two solutions in this case (0, 1), (0, –1).

(III) $\left.\begin{array}{l} y=2n\pi, n\in z \\ |x|+|y|=1 \end{array}\right\}$

Arguing as in case (II) we have two statement in this case (1, 0), (–1, 0).

Thus all together we have 6 solutions, viz.

$$\left(\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right), (0, 1), (0, -1), (1, 0), (-1, 0)$$

3. $f'(x) = \frac{1}{x^2 + f^2(x)}$

We observe that $f'(x) > 0$ on $[1, \infty)$

Thus f is increasing on $[1, \infty)$

As $f(1) = 1$ we have $f(x) \ge 1 \ \forall x \in [1, \infty)$

Now $f'(x) = \frac{1}{x^2 + f^2(x)} \le \frac{1}{x^2+1}$

As $f'(x)$ and $\frac{1}{x^2+1}$ are both positive

We have $\int_1^x f'(t)dt \le \int_1^x \frac{1}{1+t^2}dt$

$\Rightarrow [f(t)]_1^x \le \left[\tan^{-1} t\right]_1^x$

$\Rightarrow f(x) - f(1) \le \tan^{-1}x - \tan^{-1}1$

$\Rightarrow f(x) - 1 \le \tan^{-1}x - (\pi/4)$

$\Rightarrow f(x) \le 1 + \tan^{-1}x - \frac{\pi}{4}$

As $x \in [1, \infty)$, $\tan^{-1}x \le \pi/2$ giving

$f(x) \le 1 + \frac{\pi}{2} - \frac{\pi}{4}$

That is $f(x) \le 1 + \frac{\pi}{4} \ \forall x \ge 1$

4. Let $\alpha_1 = a_1a_2a_3a_4$

$\alpha_2 = a_2a_3a_4a_5$

....

$\alpha_n = a_na_1a_2a_3$

Since each a_i is either 1 or –1, so is each α_i, that is, each α_i is 1 or –1

The given condition,

$a_1a_2a_3a_4 + a_2a_3a_4a_5 + ... + a_na_1a_2a_3 = 0$

turns into $\alpha_1 + \alpha_2 + ... + \alpha_n = 0$

Let k of α_i's be 1, so $(n-k)$ of α_i's will be –1. The above relation gives

$k \cdot 1 + (n-k)\cdot(-1) \Rightarrow k - (n-k) = 0$

$\therefore \ n = 2k.$

Now, $\prod_{i=1}^n \alpha_i =$ product of all α_i's

= (product of k of α_i's that are each 1)·(product of $(n-k)$ of α_i's that are each –1)

$= (1)^k(-1)^{n-k} = (-1)^{n-k}.$

Again, $\prod_{i=1}^n \alpha_i = (a_1a_2a_3a_4)(a_2a_3a_4a_5).....(a_na_1a_2a_3)$

$= a_1^2a_2^2a_3^2 \ .... \ a_n^2$

$= 1 \cdot 1 \ ..... \ 1 = (1)^n = 1.$

Equating the two expressions for $\prod_{i=1}^n \alpha_i$ we have

$(-1)^{n-k} = 1$

$\Rightarrow n - k = 2l$ for some integer l

But $n = 2k$, which gives $2k - k = 2l$. $\therefore k = 2l$.

Thus $n = 2k = 4l$, which shows that 4 divides n.

5. Define $f(t) = \int_{a_j}^t p(x)dx$, so that $f'(t) = p(t)$

Observe that $f(a_j) = 0$ are $f(a_{j+1}) = 0$ (by hypothesis of the problem).

$f(t)$ has a zero in the interval (a_j, a_{j+1}) for $0 \le j \le n-1$.

From Rolle's theorem applied to this polynomial, $f'(t)$ has at least one zero in the interval (a_j, a_{j+1}) for $0 \le j \le n-1$. Note that $f'(t)$ is polynomial of degree n. So it can only have n zeroes. We have already found n distinct zeroes, so these are the only zeroes of $p(t)$, establishing that $p(t)$ has exactly one zero in (a_j, a_{j+1}) for $0 \le j \le n-1$.

6. Ist solution

First note that none of a, b, c can be zero. For if any of them, say, a equals zero, then the equation

Contd. on page no.67

Mathematics Olympia for IIT-JEE 2008

COMPREHENSIONS

A. A ray of light coming along the line $y = 1$ from positive direction of x-axis and strikes a concave mirror whose intersection with x-y plane is a parabola $y^2 = 4x$, then

1. The slope of reflected ray

(a) $-3/4$ (b) $-4/3$ (c) $-1/2$ (d) $-1/3$.

2. Length of the reflected ray (contained within the parabola) if reflected ray makes an angle θ with positive direction of its axis

(a) $4\sec^2\theta$ (b) $4\operatorname{cosec}^2\theta$

(c) $2\tan^2\theta$ (d) $2\sec^2\theta$

3. If tangents are drawn to the parabola at the ends of reflected ray (contained within the parabola), then angle between the tangents is

(a) 45° (b) 60° (c) 30° (d) 90°.

B. Let θ_1, θ_2 and θ_3 are acute angles and are the roots of the equation E_1, E_2 and E_3 respectively where

$E_1 : (2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$E_2 : 3\cos^2 x - 10\cos x + 3 = 0$ and

$E_3 : 1 - \sin 2x = \cos x - \sin x$

1. The value of $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$ will be

(a) $\frac{3\sqrt{3}+7}{6}$ (b) $\frac{3\sqrt{3}-2}{6}$

(c) $\frac{3\sqrt{6}+2\sqrt{2}+6}{6\sqrt{2}}$ (d) None of these.

2. The value of $\sin\theta_1 + \sin\theta_2 + \sin\theta_3$ will be

(a) $\frac{14+3\sqrt{2}}{6\sqrt{2}}$ (b) $\frac{3+4\sqrt{2}}{6}$

(c) $\frac{\sqrt{2}+1}{2}$ (d) 6/5.

3. The value of $\sin(\theta_1 - \theta_2)$ is equal to

(a) 1 (b) 0

(c) $\frac{1-2\sqrt{6}}{6}$ (d) $\frac{\sqrt{3}+2\sqrt{2}}{6}$

C. Consider a real valued function $f(x)$, we define a limit as follows

$$<f(x)> = \lim_{h\to 0}\frac{f^2(x+h)-f^2(x)}{h} \quad \text{where } f^2(x) = (f(x))^2$$

1. If $u = f(x)$, $v = g(x)$, then the value of $<u \cdot v>$ is

(a) $<u>v + <v>u$ (b) $u^2<v> + v^2<u>$

(c) $<u> + <v>$ (d) $uv<u + v>$.

2. If $u = f(x)$, $v = g(x)$ then value of $<u/v>$ is

(a) $\frac{u^2<v>-v^2<u>}{v^4}$ (b) $\frac{u<v>-v<u>}{v^2}$

(c) $\frac{v^2<u>-u^2<v>}{v^4}$ (d) $\frac{v<u>-u<v>}{v^2}$

3. The value of $<\tan x>$ will be

(a) $\sec^2 x$ (b) $2\sec^2 x$

(c) $\tan x \cdot \sec^2 x$ (d) $2\tan x \cdot \sec^2 x$

D. Consider an ellipse having foci $A(\vec{r}_1)$ and $B(\vec{r}_2)$ in the cartesian plane. If eccentricity and area of the ellipse be 1/2 and 4 units respectively.

1. The equation of ellipse is

(a) $|\vec{r}-\vec{r}_1|+|\vec{r}-\vec{r}_2| = |\vec{r}_1-\vec{r}_2|$

(b) $|\vec{r}-\vec{r}_1|+|\vec{r}-\vec{r}_2| = 2$

(c) $|\vec{r}-\vec{r}_1|+|\vec{r}-\vec{r}_2| = 2|\vec{r}_1-\vec{r}_2|$

(d) None of these.

2. If origin is an interior point of the ellipse then

(a) $2|\vec{r}_1-\vec{r}_2| > |\vec{r}_1|+|\vec{r}_2|$

(b) $2|\vec{r}_1-\vec{r}_2| < |\vec{r}_1|+|\vec{r}_2|$

(c) $|\vec{r}_1-\vec{r}_2| > |\vec{r}_1|+|\vec{r}_2|$

(d) $|\vec{r}_1-\vec{r}_2| < |\vec{r}_1|+|\vec{r}_2|$.

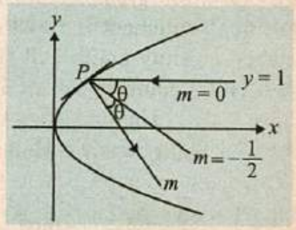
3. If P is any arbitrary point on the ellipse then maximum area of ΔPAB is

(a) $4/\pi$ (b) $2/\pi$ (c) $1/\pi$ (d) $\pi/4$.

SOLUTIONS

A.

1. (b) : Let reflected ray strikes at P (1/4, 1) slope of tangent at P



$$= \left(\frac{dy}{dx}\right)_{(1/4,\,1)} = 2$$

by : **Er. Akhlak Ahmad**, Chtrahar Building, III[rd] Floor, Hazratganj, Lucknow

$\Rightarrow$ Slope of normal at $P = -(1/2)$

Applying concept of reflection

$$\frac{-\frac{1}{2}-m}{1+\left(-\frac{1}{2}\right)(m)}=\frac{0+\frac{1}{2}}{1+0\left(-\frac{1}{2}\right)}$$

$\Rightarrow$ $m = -4/3$

2. **(b) :** Since reflected ray must pass through focus

$\Rightarrow$ Length of reflected ray = length of focal chord $= 4a\,\text{cosec}^2\theta = 4\text{cosec}^2\theta$ ($\because$ $a = 1$)

3. **(d) :** Since, tangents drawn at the end of focal chord intersect on the directrix and hence, perpendicular $\Rightarrow$ angle $= 90°$

B.

$E_1 : (2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\Rightarrow$ $(1 + \cos x)(2\sin x - \cos x - 1 + \cos x) = 0$

$\Rightarrow$ $(1 + \cos x)(2\sin x - 1) = 0$

$\Rightarrow$ $\cos x = -1$ (rejected as acute angle)

or $\sin x = 1/2 \Rightarrow \sin\theta_1 = 1/2 \quad \Rightarrow \cos\theta_1 = \frac{\sqrt{3}}{2}$

Next

$E_2 : 3\cos^2 x - 10\cos x + 3 = 0$

factorizing we get

$\cos x = 1/3$ or $\cos x = 3$ (rejected)

$\Rightarrow$ $\cos\theta_2 = 1/3 \quad \Rightarrow \sin\theta_2 = \frac{2\sqrt{2}}{3}$

Next

$E_3 : 1 - \sin 2x = \cos x - \sin x$

$\Rightarrow$ $\sin^2 x + \cos^2 x - 2\sin x \cdot \cos x = \cos x - \sin x$

$\Rightarrow$ $(\cos x - \sin x)(\cos x - \sin x - 1) = 0$

$\Rightarrow$ $\sin x = \cos x$ or $\cos x - \sin x = 1$

$\Rightarrow$ $\sin\theta_3 = \cos\theta_3 = \frac{1}{\sqrt{2}}$ or $\cos\theta_3 = 1$, $\sin\theta_3 = 0$

(rejected as acute angle)

So,

1. **(c) :** $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$

$$= \frac{\sqrt{3}}{2}+\frac{1}{3}+\frac{1}{\sqrt{2}}=\frac{3\sqrt{6}+2\sqrt{2}+6}{6\sqrt{2}}$$

2. **(a) :** $\sin\theta_1 + \sin\theta_2 + \sin\theta_3$

$$=\frac{1}{2}+\frac{2\sqrt{2}}{3}+\frac{1}{\sqrt{2}}=\frac{14+3\sqrt{2}}{6\sqrt{2}}$$

3. **(c) :** $\sin(\theta_1 - \theta_2) = \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2$

$$=\frac{1}{2}\times\frac{1}{3}-\frac{\sqrt{3}}{2}\times\frac{2\sqrt{2}}{3},\quad \sin(\theta_1-\theta_2)=\frac{1-2\sqrt{6}}{6}$$

C.

1. **(b) :** $<u \cdot v> = <f(x)\cdot g(x)>$

$$=\lim_{h\to 0}\frac{f^2(x+h)\,g^2(x+h)-f^2(x)g^2(x)}{h}$$

$$=\lim_{h\to 0} f^2(x+h)\left[\frac{g^2(x+h)-g^2(x)}{h}\right] + g^2(x)\left[\frac{f^2(x+h)-f^2(x)}{h}\right]$$

$= \quad f^2(x) < g(x) > + g^2(x) < f(x) >$

$= \quad u^2 < v > + v^2 < u >$

2. **(c) :** $<\frac{u}{v}> = <\frac{f(x)}{g(x)}>$

$$=\lim_{h\to 0}\frac{\left[\frac{f(x+h)}{g(x+h)}\right]^2-\left[\frac{f(x)}{g(x)}\right]^2}{h}$$

$$=\lim_{h\to 0}\frac{f^2(x+h)g^2(x)-g^2(x+h)f^2(x)}{h\,g^2(x+h)g^2(x)}$$

$$=\lim_{h\to 0}\frac{g^2(x)\frac{[f^2(x+h)-f^2(x)]}{h}-f^2(x)\frac{[g^2(x+h)-g^2(x)]}{h}}{g^2(x+h)g^2(x)}$$

$$=\frac{g^2(x)< f(x) > - f^2(x) < g(x) >}{g^4(x)}$$

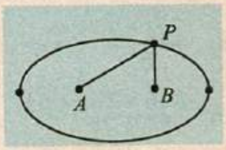
$$<\frac{u}{v}> = \frac{v^2 < u > - u^2 < v >}{v^4}$$

3. **(d) :** $< \tan x > = \lim_{h\to 0}\frac{\tan^2(x+h)-\tan^2 x}{h}$

$$=\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}\cdot\lim_{h\to 0}(\tan(x+h)+\tan x)$$

$= \sec^2 x \cdot (2\tan x), \quad < \tan x > \; = 2\tan x \cdot \sec^2 x$

D.



1. **(c) :** $PA + PB = 2a$

Let position vector of $P(\vec{r})$

$2ae = AB = |\vec{r}_1 - \vec{r}_2|$

$\Rightarrow |\vec{r}-\vec{r}_1|+|\vec{r}-\vec{r}_2| = \frac{|\vec{r}_1-\vec{r}_2|}{e} \qquad (\because e = 1/2)$

$= 2|\vec{r}_1 - \vec{r}_2|$

2. **(a) :** If origin is an interior point of the ellipse then

$S_1 < 0, \; |0-\vec{r}_1|+|0-\vec{r}_2| < 2|\vec{r}_1-\vec{r}_2|$

$\Rightarrow \; |\vec{r}_1|+|\vec{r}_2| < 2|\vec{r}_1-\vec{r}_2| \;\Rightarrow\; 2|\vec{r}_1-\vec{r}_2| > |\vec{r}_1|+|\vec{r}_2|$

3. **(b) :** Area of $\Delta PAB = \frac{1}{2}(AB)\,h$

max area $= \frac{1}{2}(2ae)\,h_{\max} = \frac{1}{2}(2ae)(b) \quad (\because h_{\max} = b)$

max area $= abe \quad (\because \pi ab = 4)$

$= \frac{4}{\pi}\cdot\frac{1}{2} = \frac{2}{\pi}$

Contd. from page no.26

$$bx^2 - 2cx + a = 0 \quad \Rightarrow \; bx^2 - 2cx = 0$$

will have a root $x = 0$, which is not positive. Thus, $a, b, c \neq 0$.

Further since the equation have only positive roots, a, b, c are either all positive or all negative.

Let a, b, c be all positive. If they are all negative, then multiply each equation with (-1) and then $-a$, $-b$, $-c$ will be all positive.

For the roots to exist, $D \geq 0$, which gives

$$b^2 \geq ac \; ; \; c^2 \geq ab \; ; \; a^2 \geq bc$$

We have to prove that a, b, c are all equal. Assume to the contrary, that any two of them, say a and b are not equal. Take for definiteness, $a > b$

Now $a > b = 1 \Rightarrow ba > b^2$ (multiplying by b, which is positive)

$\Rightarrow \; ba > b^2 \geq ac$ (from the condition given)

$\Rightarrow \; ba > ac \qquad \Rightarrow \; b > c$

Again, multiplying with c,

$cb > c^2 \geq ab$ (from the condition given)

$\Rightarrow \; cb > ab \qquad \Rightarrow \; c > a$

Thus we have arrived at $a > b > c > a$, which is absurd.

Thus, a, b, c are equal.

2nd solution

Having established that a, b, c are all positive or are negative and thus taking a, b, c to be positive, without any loss of generality, we assume the roots of

$ax^2 - 2bx + c = 0$ to be α_1, β_1

$bx^2 - 2cx + c = 0$ to be α_2, β_2

$cx^2 - 2ax + b = 0$ to be α_3, β_3

We have $\alpha_1 + \beta_1 = \dfrac{2b}{a} \qquad \alpha_1\beta_1 = \dfrac{c}{a}$

$$\alpha_2 + \beta_2 = \frac{2c}{b} \qquad \alpha_2\beta_2 = \frac{a}{b}$$

$$\alpha_3 + \beta_3 = \frac{2a}{c} \qquad \alpha_3\beta_3 = \frac{b}{c}$$

On multiplying, $\left.\begin{aligned}(\alpha_1+\beta_1)(\alpha_2+\beta_2)(\alpha_3+\beta_3) = 8\\ (\alpha_1\beta_1)(\alpha_2\beta_2)(\alpha_3\beta_3) = 1\end{aligned}\right\} \ldots \text{(i)}$

From AM-GM inequality,

$$\frac{\alpha_1+\beta_1}{2} \geq \sqrt{\alpha_1\beta_1} \; ; \frac{\alpha_2+\beta_2}{2} \geq \sqrt{\alpha_2\beta_2}; \; \frac{\alpha_3+\beta_3}{2} \geq \sqrt{\alpha_3\beta_3}$$

Multiplying them all,

$$(\alpha_1 + \beta_1)(\alpha_2 + \beta_2)(\alpha_3 + \beta_3) \geq 8(\alpha_1\beta_1)(\alpha_2\beta_2)(\alpha_3\beta_3) \qquad \ldots \text{(ii)}$$

From (i) and (ii),

$$(\alpha_1 + \beta_1)(\alpha_2 + \beta_2)(\alpha_3 + \beta_3) = (\alpha_1\beta_1)(\alpha_2\beta_2)(\alpha_3\beta_3)$$

Thus the equality holds in the inequality. That means $\alpha_1 + \beta_1 = \alpha_1\beta_1$, $\alpha_2 + \beta_2 = \alpha_2\beta_2$, $\alpha_3 + \beta_3 = \alpha_3\beta_3$ which gives

$$\frac{b}{a} = \frac{c}{a} \Rightarrow b = c$$

Similarly, $c = a$ and $a = b$.

Thus $a = b = c$, *i.e.* a, b, c are all equal.

7. Consider the equation,

$$\frac{a^2}{x-1^2} + \frac{b^2}{x-3^2} + \frac{c^2}{x-5^2} + \frac{d^2}{x-7^2} = 1$$

When cross-multiple is a fourth degree equation, whose roots are $x = 2^2, 4^2, 6^2, 8^2$.

$$\begin{aligned} a^2(x-9)(x-25)(x-49) + b^2(x-1)(x-25)(x-45) \\ + c^2(x-1)(x-9)(x-49) \\ + d^2(x-1)(x-9)(x-25) \\ = (x-1)(x-9)(x-25)(x-49) \end{aligned}$$

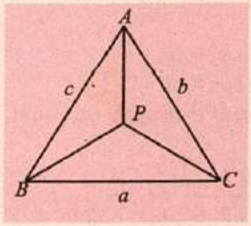
Now,

$$\begin{aligned} a^2(x-9)(x-25)(x-45) + b^2(x-1)(x-25)(x-49) \\ + c^2(x-1)(x-9)(x-49) + d^2(x-1)(x-9)(x-25) \\ - (x-1)(x-9)(x-25)(x-49) \\ \equiv (x-4)(x-16)(x-36)(x-64) \end{aligned}$$

Comparing the coefficient of x^3 and simplifying, we get

$$a^2 + b^2 + c^2 + d^2 = 36.$$

8.



Since $\angle APB = \angle BPC = \angle CPA$

$\Rightarrow$ each of these angle is equal to $2\pi/3$

In triangle ABC we have

$$PA^2 + PC^2 - 2PA \cdot PC \cos\frac{2\pi}{3} = b^2$$

$\Rightarrow \; b^2 = PA^2 + PC^2 + PA \cdot PC \qquad \ldots \text{(i)}$

Similarly, $a^2 = PC^2 + PB^2 + PB \cdot PC \qquad \ldots \text{(ii)}$

$c^2 = PA^2 + PB^2 + PA \cdot PB \qquad \ldots \text{(iii)}$

Adding (i), (ii) and (iii)

$$\begin{aligned} a^2 + b^2 + c^2 &= 2(PA^2 + PB^2 + PC^2) + PA \cdot PB \\ &\quad + PB \cdot PC + PC \cdot PA \\ &= 2(PA + PB + PC)^2 - 4\Sigma\, PA \cdot PB + \Sigma PA \cdot PB \end{aligned}$$

$$\Rightarrow \; a^2 + b^2 + c^2 = 2(PA + PB + PC)^2 - 3\Sigma PA \cdot PB \qquad \ldots \text{(iv)}$$

$$\Delta = \Delta_{ABC} = \Delta_{APC} + \Delta_{BPC} + \Delta_{BPA}$$

$$= \frac{1}{2}\sin\frac{2\pi}{3}(PA\cdot PC + PB\cdot PC + PA\cdot PB)$$

$$= \frac{\sqrt{3}}{4}(\sum PA\cdot PB)$$

$$\sum PA\cdot PB) = \frac{4\Delta}{\sqrt{3}}$$

Putting the value of $PA \cdot PB$ in (iv)

$$a^2 + b^2 + c^2 = 2(PA + PB + PC)^2 - 3\cdot\frac{4\Delta}{\sqrt{3}}$$

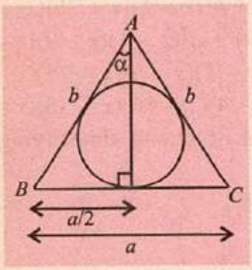
$$= 2(PA+PB+PC)^2 - 4\sqrt{3}\Delta$$

$$\Rightarrow \quad \frac{a^2+b^2+c^2}{2} = (PA+PB+PC)^2 - 2\sqrt{3}\Delta$$

$$\Rightarrow \quad (PA+PB+PC)^2 = \frac{a^2+b^2+c^2}{2} + 2\Delta\sqrt{3}$$

$$PA+PB+PC = \sqrt{\frac{a^2+b^2+c^2}{2} + 2\sqrt{3}\Delta}$$

9.



Let 2α be the angle of the vertex.
Area of the triangle = Δ (constant)
Let r = radius of incircle, b = length of equal sides and a = length of base.

Now, $r = \frac{\Delta}{s}$, where $s = \frac{a+b+c}{2}$ or $s = \frac{a}{2} + b$ $\quad (\because b = c)$

Now, $\frac{(a/2)}{b} = \sin\alpha \Rightarrow a = 2b\sin\alpha$

$$r = \frac{\Delta}{b(1+\sin\alpha)}$$

Again, $\Delta = \frac{1}{2}b^2\sin 2\alpha \Rightarrow b^2 = \frac{2\Delta}{\sin 2\alpha}$

Therefore,

$$z = r^2 = \frac{\Delta^2}{\frac{2\Delta}{\sin 2\alpha}(1+\sin\alpha)^2} = \frac{\Delta}{2}\cdot\frac{\sin 2\alpha}{(1+\sin\alpha)^2},\ 0 < 2\alpha < \pi$$

For maximum r (that means for maximum z)

$$\frac{dz}{d\alpha} = \frac{\Delta}{2}\left\{\frac{2\cos 2\alpha(1+\sin\alpha) - 2\sin 2\alpha\cdot\cos\alpha}{(1+\sin\alpha)^3}\right\} = 0$$

$$2\cos 2\alpha(1 + \sin\alpha) - 2\sin 2\alpha\cdot\cos\alpha = 0$$

$\Rightarrow \quad (1 - 2\sin^2\alpha)(1 + \sin\alpha) - 2\sin\alpha\cos^2\alpha = 0$

$\Rightarrow \quad 1 - 2\sin^2\alpha + \sin\alpha - 2\sin^3\alpha - 2\sin\alpha + 2\sin^3\alpha = 0$

$\Rightarrow \quad 2\sin^2\alpha + \sin\alpha - 1 = 0$ or $\sin\alpha = 1/2, -1$

Neglecting $\sin\alpha = -1$

$\sin\alpha = 1/2 \Rightarrow \alpha = 30°$.

$\angle A = 2\alpha = 60°$ (angle of vertex)

$\angle B = \angle C$ (given)

$\angle A + 2\angle B = 180°$

$2\angle B = 120° \quad \Rightarrow \angle B = 60°$

$\Rightarrow$ ΔABC is an equilateral triangle.

10. Let's fill the 3×3 matrix $\{a_{ij}\}$, $i, j \in \{1, 2, 3\}$ by element 1 or -1 in all possible ways. The number of ways to fill is $2^9 = 512$.
We claim that this is number of ways to fill 4×4 matrix $\{a_{ij}\}$; $i, j \in \{1, 2, 3, 4\}$. Once the 3×3 matrix elements are decided, the remaining elements are automatically determined. The remaining elements in each row and each column is determined by the condition that the product of all entries in each row and each column is -1. Our task is to show that the element a_{44} also gets determined.

Let $P = (a_{11}a_{12}a_{13})(a_{21}a_{22}a_{23})(a_{31}a_{32}a_{33})$

Now $P(a_{41}a_{42}a_{43}) = (a_{11}a_{12}a_{13})(a_{21}a_{22}a_{23})(a_{31}a_{32}a_{33})(a_{41}a_{42}a_{43})$

$= (a_{11}a_{21}a_{31}a_{41})(a_{12}a_{22}a_{32}a_{42})(a_{13}a_{23}a_{33}(a_{43})$

$= (-1)(-1)(-1)(-1) = 1 \quad$ ... (i)

Similarly, $P(a_{14}a_{24}a_{34}) = 1 \quad$ ... (ii)

From (i) and (ii), $a_{41}a_{42}a_{43} = a_{14}a_{24}a_{34}$

Thus chain of a_{44} is consistent with the overall arrangement.

Maths CliniC

If you have any difficult/unsolved problem or you are unable to understand one, then write to us. Our team of experts will diagnose your problems. The diagnosed problems will be published in the subsequent issues.

1. If z is any complex number such that $2|z|^2 + z^2 - 5 + i\sqrt{3} = 0$ then what will be the values of z? *[Gourav, Patna]*

Soln.: Let $z = x + iy$ be any complex number then

$\bar{z} = x - iy$

$|z|^2 = z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$ ...(i)

given equation is

$2|z|^2 + z^2 - 5 + i\sqrt{3} = 0$

$2(x^2 + y^2) + (x + iy)^2 = 5 - i\sqrt{3}$ (Using (i))

$\Rightarrow 2x^2 + 2y^2 + x^2 - y^2 + 2xy\,i = 5 - i\sqrt{3}$

$\Rightarrow (3x^2 + y^2) + 2xy\,i = 5 - i\sqrt{3}$

comparing Re and Im parts

$3x^2 + y^2 = 5,$ $\quad 2xy = -\sqrt{3}$

Using (ii) | squaring both sides

$3x^2 + \dfrac{3}{4x^2} = 5$ | $4x^2y^2 = 3$ or $y^2 = \dfrac{3}{4x^2}$ ...(ii)

$\dfrac{12x^4 + 3}{4x^2} = 5$

$12x^4 + 3 = 20x^2$ or $12x^4 - 20x^2 + 3 = 0$

or $12x^4 - 18x^2 - 2x^2 + 3 = 0$

or $(6x^2 - 1)(2x^2 - 3) = 0$

either $6x^2 - 1 = 0$ or $2x^2 - 3 = 0$

$6x^2 = 1$ $\qquad x^2 = \dfrac{3}{2}$

or $x^2 = \dfrac{1}{6}$ $\qquad x = \pm\sqrt{\dfrac{3}{2}}$

or $x = \pm\sqrt{\dfrac{1}{6}}$

Putting $x^2 = \dfrac{1}{6}$ in (ii) $y^2 = \dfrac{3\times 6}{4\times 1} = \dfrac{9}{2} \Rightarrow y = \pm\dfrac{3}{\sqrt{2}}$

Putting $x^2 = \dfrac{3}{2}$ in (ii) $y^2 = \dfrac{3\times 2}{4\times 3} = \dfrac{1}{2} \Rightarrow y = \pm\dfrac{1}{\sqrt{2}}$

Complex numbers are

$$x + iy = \frac{1}{\sqrt{6}} - i\frac{3}{\sqrt{2}},\quad \frac{-1}{\sqrt{6}} + i\frac{3}{\sqrt{2}},\quad \sqrt{\frac{3}{2}} - i\frac{1}{\sqrt{2}},\quad -\sqrt{\frac{3}{2}} + i\frac{1}{\sqrt{2}}$$

2. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ *[Venkatesh, A.P.]*

$$\text{L.H.S} = |z_1 + z_2|^2 + |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| + |z_1|^2 + |z_2|^2 - 2|z_1||z_2| = 2|z_1|^2 + 2|z_2|^2 = \text{R.H.S}$$

So $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$ ...(i)

Taking $\left(|a + \sqrt{a^2 - b^2}| + |a - \sqrt{a^2 - b^2}|\right)^2$

$$|a + \sqrt{a^2 - b^2}|^2 + |a - \sqrt{a^2 - b^2}|^2 + 2\,|(a + \sqrt{a^2 - b^2})||a - \sqrt{a^2 - b^2}|$$

$(\because |z_1||z_2| = |z_1 z_2|)$

$= 2|a|^2 + 2|\sqrt{a^2 - b^2}|^2 + 2|a^2 - a^2 + b^2|$ Using (i)

$= 2|a|^2 + 2|b|^2 + 2|a^2 - b^2|$

Taking

$(|a + b| + |a - b|)^2 = |a + b|^2 + |a - b|^2 + 2|a^2 - b^2|$ ...(ii)

$= 2|a|^2 + 2|b|^2 + 2|a^2 - b^2|$ (Using (i))

Hence from (i) and (ii), we conclude that

$\left(|a + \sqrt{a^2 - b^2}| + |a - \sqrt{a^2 - b^2}|\right)^2 = (|a + b|) + |a - b|)^2$

$\Rightarrow |a + \sqrt{a^2 - b^2}| + |a - \sqrt{a^2 - b^2}| = |a + b| + |a - b|$

3. Evaluate $\displaystyle\int \frac{\sqrt{\sin x}}{\cos x}\,dx$ *[Ravi, Chennai]*

$$= \int \frac{\sqrt{\sin x}}{\sqrt{1 - \sin^2 x}}\,dx = \int \frac{\sqrt{\sin x}}{\sqrt{(1 + \sin x)(1 - \sin x)}}\,dx$$

Put $\sin x = t$

differentiate w.r.t x

$\Rightarrow \cos x\,dx = dt$

or $dx = \dfrac{1}{\sqrt{1 - t^2}}\,dt$

$$= \int \frac{\sqrt{t}\qquad dt}{\sqrt{(\sqrt{1+t})(\sqrt{1-t})(\sqrt{1+t})(\sqrt{1-t})}}$$

$$= \int \frac{\sqrt{t}\;dt}{(1+t)(1-t)}$$

$$= \int \frac{\sqrt{t}\;dt}{(1 + (\sqrt{t})^2)(1 - (\sqrt{t})^2)}$$

Taking $\sqrt{t} = y$ ...(i)

or $t = y^2$

So $$\frac{\sqrt{t}}{(1+t)(1-t)} = \frac{y}{(1+y^2)(1-y^2)} = \frac{Ay + B}{1 + y^2} + \frac{Cy + D}{1 - y^2} \quad ...(ii)$$

where A, B, C, D are constant (By using partial fraction)

or $y = (Ay + B)(1 - y^2) + (Cy + D)(1 + y^2)$

By comparing coeff of y^3, y^2, y and constant term we get $A = \dfrac{1}{2}, B = 0, C = \dfrac{1}{2}, D = 0$

Substituting these values in (ii)

$$\frac{1}{2}\int \frac{y}{1+y^2}\,dy + \frac{1}{2}\int \frac{y}{1-y^2}\,dy = \frac{1}{2\times 2}\int \frac{2y}{1+y^2}\,dy - \frac{1}{2\times 2}\int \frac{-2y}{1-y^2}\,dy$$

$$= \frac{1}{4}\log(1 + y^2) - \frac{1}{4}\log(1 - y^2) + c$$

$$= \frac{1}{4}\log\left(\frac{1+y^2}{1-y^2}\right) - \frac{1}{4}\log\left(\frac{1+t}{1-t}\right) + c = \frac{1}{4}\log\left(\frac{1+\sin x}{1-\sin x}\right) + c$$

(Using (i))

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. If a, b, c, d are positive then

$$\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx}$$

(a) $e^{d/b}$ (b) $e^{c/a}$
(c) $e^{(c+d)/(a+b)}$ (d) e

2. $\lim_{x\to 1}\frac{\sqrt{1+\cos 2(x-1)}}{x-1} =$

(a) exists and it equals $\sqrt{2}$
(b) exists and it equals $-\sqrt{2}$
(c) does not exist because $x-1\to 0$
(d) does not exist because L.H.L. is not equal to R.H.L

3. The function $\frac{\log(1+ax)-\log(1-bx)}{x}$ is not defined at $x = 0$. The value which should be assigned to f at $x = 0$ so that it is continuous at $x = 0$ is

(a) $a-b$ (b) $1+b$
(c) $\log a + \log b$ (d) none of these

4. If a line OP through the origin O makes angle α, $45°$ and $60°$ with x, y and z axis respectively then the direction cosines of OP are

(a) $\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}$ (b) $\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}$
(c) $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$ (d) none of these

5. If O is the origin and the line OP of length r makes an angle α with x-axis and lies in the xz plane, then the co-ordinates of P are

(a) $(r\cos\alpha, 0, r\sin\alpha)$ (b) $(0, 0, r\sin\alpha)$
(c) $(0, 0, r\cos\alpha)$ (d) $(r\cos\alpha, 0, 0)$

6. Evaluate $\lim_{x\to\pi/2}\frac{\cos x}{(1-\sin x)^{2/3}}$

7. Evaluate $\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x$

8. If f and g are differentiable at $a\in R$ such that $f(a) = g(a) = 0$ and $g'(a) \neq 0$, then show that $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

9. Show that the function g, defined by $g(x) = \sin\alpha + \cos\alpha - 1$, $\alpha = \sin^{-1}\sqrt{\{x\}}$, $\{\cdot\}$ denotes fractional part function, is an even function.

10. Evaluate $\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{4x+1}$

By : **Prof. Shyam Bhushan**, Director, Narayana Institute, Jamshedpur. Mobile: 09334870021

SOLUTION

1. **(a)** : $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx}$

$= \lim_{x\to\infty}\left[\left(1+\frac{1}{a+bx}\right)^{a+bx}\right]^{\frac{c+dx}{a+bx}}$

$e^{\lim_{x\to\infty}\frac{(c/x)+d}{(a/x)+b}}$

2. **(c)** : Put $x-1=t$

$\lim_{x\to 1}\frac{\sqrt{1+\cos 2(x-1)}}{x-1} = \lim_{t\to 0}\frac{\sqrt{2}\,|\cos t|}{t}$

$= \lim_{t\to 0}\frac{\sqrt{2}\cos t}{t}$ does not exist because $t\to 0$, $(x-1)\to 0$

3. **(d)** : For continuity at $x=0$

$\lim_{x\to 0} f(x) = f(0)$

$\lim_{x\to 0}\left(\frac{a.\log(1+ax)}{ax}+\frac{b.\log(1-bx)}{-bx}\right) = f(0)$

$\Rightarrow a+b=f(0)$

4. **(c)** : Direction cosines of OP are $\cos\alpha$, $\cos 45°$, $\cos 60°$ and $\cos^2\alpha+\cos^2 45°+\cos^2 60° = 1$

$\Rightarrow \cos^2\alpha+\left(\frac{1}{2}\right)+\left(\frac{1}{4}\right)=1\Rightarrow\cos^2\alpha=\frac{1}{4}$

$\Rightarrow \cos\alpha = 1/2.$

5. **(a)** : Let the coordinate of P be (x, y, z). Since OP lies in xz plane and makes an angle α with the x-axis, it makes angle $(\pi/2)-\alpha$ with z-axis and $\pi/2$ with y-axis. so, $x=r\cos\alpha$, $y=r\cos\frac{\pi}{2}$, $z=r\cos\left(\frac{\pi}{2}-\alpha\right)$ are the required co-ordinates and therefore are $(r\cos\alpha, 0, r\sin\alpha)$

6. $\lim_{x\to\pi/2}\frac{\cos x}{(1-\sin x)^{2/3}} = \lim_{t\to 0}\frac{\sin t}{(1-\cos t)^{2/3}}$

$= \lim_{t\to 0}\frac{2\sin(t/2)\cos(t/2)}{(2\sin^2(t/2))^{2/3}} = \lim_{t\to 0}\frac{1}{2^{1/3}}\frac{\cos(t/2)}{(\sin(t/2))^{1/3}}$

Limit value is ∞ as $t\to 0^+$ and limit value is $-\infty$ as $t\to 0^-$

7. $\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^x = e^{\lim_{x\to\infty} x\left(\sin\frac{1}{x}+\cos\frac{1}{x}-1\right)}$

$= e^{\lim_{x\to\infty}\frac{\sin(1/2x)}{1/2x}\left(\cos\frac{1}{2x}-\sin\frac{1}{2x}\right)} = e$

8. $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)}$,

as $f(a)=g(a)=0$ and $g'(a)\neq 0$

9. $g(x)=\sin(\sin^{-1}\sqrt{\{x\}})+\cos(\sin^{-1}\sqrt{\{x\}})-1$

$=\sqrt{\{x\}}+\cos(\cos^{-1}\sqrt{1-\{x\}})-1$

$=\sqrt{\{x\}}+\sqrt{1-\{x\}})-1$

If $x\in I$, then $\{x\}=0\Rightarrow g(x)=0\Rightarrow g(x)=g(-x)$

If $x\notin I$, then $\{-x\}=1-\{x\}\Rightarrow g(-x)$

$=\sqrt{1-\{x\}}+\sqrt{\{x\}}-1=g(x)$

Hence g is an even function.

10. $\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{4x+1} = \lim_{x\to-\infty}-\frac{\sqrt{2+\frac{1}{x^2}}}{4+\frac{1}{x}}$, as $x<0$

$=-\frac{1}{2\sqrt{2}}$

PRACTICE PAPER

IIT-JEE 2008

Exam on 13th April 2008

By **Alok Kumar**, B.Tech, IIT Kanpur

SECTION-I

(only one correct option)

1. Which of the following inequalities is true?

(a) $\sin 10 > 0, \cos 8 > 0$ (b) $\sin 10 < 0, \cos 8 > 0$

(c) $\sin 10 < 0, \cos 8 < 0$ (d) $\sin 10 > 0, \cos 8 < 0$

2. The expression $\dfrac{1}{\log(1/5)^{1/2}}+\dfrac{1}{\log(1/3)^{1/2}}$ lies in the interval

(a) $(2, 3)$ (b) $(3, 4)$

(c) $(-3, -2)$ (d) $(-4, -3)$

3. Let $\theta \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$ which of the following statements is true?

(a) $(\cos\theta)^{\cos\theta} < (\sin\theta)^{\cos\theta} < (\cos\theta)^{\sin\theta}$

(b) $(\cos\theta)^{\sin\theta} < (\cos\theta)^{\cos\theta} < (\sin\theta)^{\cos\theta}$

(c) $(\sin\theta)^{\cos\theta} < (\cos\theta)^{\cos\theta} < (\cos\theta)^{\sin\theta}$

(d) $(\cos\theta)^{\cos\theta} < (\cos\theta)^{\sin\theta} < (\sin\theta)^{\cos\theta}$

4. The area of quadrilateral whose vertices are 1, i, ω, ω^2 where ω ($\neq 1$) is a complex cube root of unity, is

(a) $\dfrac{3+2\sqrt{3}}{4}$ (b) $\dfrac{3+2\sqrt{3}}{2}$

(c) $\dfrac{\sqrt{3}+2}{4}$ (d) $\dfrac{\sqrt{3}+2}{2}$

5. Let a_n be a sequence defined on positive integers by the setting $a_n = \dfrac{4n+\sqrt{4n^2-1}}{\sqrt{2n+1}+\sqrt{2n-1}}$ then $a_1 + a_2 + a_3 + ... + a_{60}$ equals

(a) 665 (b) 364 (c) 1098 (d) 1312

6. The domain of $f(x) = \dfrac{1}{\sqrt{\log(\cot^{-1} x)}}$ is

(a) $\cot 1 < x < \infty$ (b) $0 < x < \infty$

(c) $-\infty < x < \cot 1$ (d) $-\infty < x < 0$

7. Let the normal at $P(2, 4)$ on the parabola $y^2 = 8x$ meet the parabola again at Q. Let C be the centre of the circle inscribed on PQ as a diameter. Then the image of C in the line $y = x$ is

(a) $(-4, 1)$ (b) $(-4, 10)$

(c) $(10, -4)$ (d) $(6, 10)$

8. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors. Define $\vec{v}_1, \vec{v}_2, \vec{v}_3$ by the setting

$\vec{v}_1 = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{v}_2 = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \vec{v}_3 = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, which one of the following is false?

(a) $\vec{a}\cdot\vec{v}_1 + \vec{b}\cdot\vec{v}_2 + \vec{c}\cdot\vec{v}_3 = 3$

(b) $\vec{v}_1\cdot(\vec{v}_2\times\vec{v}_3) + \vec{v}_2\cdot(\vec{v}_3\times\vec{v}_1) + \vec{v}_3\cdot(\vec{v}_1\times\vec{v}_2) = \dfrac{3}{[\vec{a}\,\vec{b}\,\vec{c}]}$

(c) $[\vec{v}_1\,\vec{v}_2\,\vec{v}_3] = \dfrac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}$

(d) $[\vec{v}_1\times\vec{v}_2\;\;\vec{v}_2\times\vec{v}_3\;\;\vec{v}_3\times\vec{v}_1] = \dfrac{2}{[\vec{a}\,\vec{b}\,\vec{c}]^2}$

9. Let $f: X \to Y$ be a function. Which of the following is true?

(a) $f^{-1} f(A) \supseteq A \;\forall\; A \subseteq X$, where equality holds iff f is one-one.

(b) $f^{-1} f(A) \supseteq A \;\forall\; A \subseteq X$, where equality holds iff f is onto.

(c) $f^{-1} f(A) \subseteq A \;\forall\; A \subseteq X$, where equality holds iff f is one-one

(d) $f^{-1} f(A) \subseteq A \;\forall\; A \subseteq X$, where equality holds iff f is onto.

SECTION-II

Assertion-Reason Type

(a) Statement-1 is true, statement-2 is true ; statement-2 is a correct explanation for statement-1

(b) Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for statement-1

(c) Statement-1 is true, statement-2 is false

(d) Statement-1 is false, statement-2 is true

10. Consider the equation $\sin x = k$, k being a parameter lying in the interval $[-1, 1]$

Statement -1 : For a given k, the equation has two solutions in $[0, 2\pi]$

because

Statement -2 : $\sin\theta = k \Leftrightarrow \sin(\pi - \theta) = k$

11. Let a, b, c, d be non-zero real numbers

Statement-1 : $(b^2 - ac)^2 + (c^2 - bd)^2 + (ad - bc)^2 = 0$ $\Rightarrow a, b, c, d$ are in G. P

because

$\Rightarrow \omega^2 = (\bar{z})^2 \Rightarrow \bar{z} = \omega$ or $\bar{z} = -\omega$

$\Rightarrow \bar{\omega} = z$ or $z = -\bar{\omega}$, out of these $z = -\bar{\omega}$ satisfies $\arg(z\omega) = \pi$.

18. Use S-34

20. $(x-1)^3 + 8 = 0 \Rightarrow x = 1 + (-8)^{1/3}$, Use S-4 to find the roots.

21. $\bar{z} + i\bar{\omega} = 0 \Rightarrow z - i\omega = 0 \Rightarrow z = i\omega$

$\arg(z\omega) = \pi \quad \Rightarrow \arg(z) + \arg\left(\frac{z}{i}\right) = \pi$

$\Rightarrow 2\arg(z) - \arg(i) = \pi \Rightarrow \arg(z) = \frac{3}{4}\pi$

ISI 2008

23. Using S-28, $|z_1 + z_2| = |z_1| + |z_2| \Rightarrow z_1, z_2, z_1 + z_2$ are collinear

24. The value of $z = \omega$ and $\left(z^a + \frac{1}{z^a}\right) = \begin{cases} -1, a \neq 3m \\ 2, a = 3m \end{cases}$

26. Use S-12

27. Use S-44

29. Use S-45

34. Using S-6, $|z+3|^2 - |z-3|^2 = 6 \Rightarrow 2\,\mathrm{Re}(z)(3-(-3)) + 3^2 - (-3)^2 = 6 \Rightarrow 12x = 6$

36. Length of each side is $|z|$. Hence it is an equilateral triangle and its area $= \frac{\sqrt{3}}{4}|z|^2$

38. Use AGP

40. $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$ are equations of circles. Using S-40, their centres are (0, 0) and (3, 4) and their radii are 12 and 5.

Now distance between their centres = 5 < difference between their radii. So second circle lies inside first circle completely.

Now, minimum value of $|z_1 - z_2|$

= Radius of bigger circle – Radius of smaller circle – Distance between their centre

$= 12 - 5 - OC = 7 - 5 = 2$

41. $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| = \left|\frac{(\beta - \alpha)\alpha}{\alpha - \alpha\bar{\alpha}\beta}\right| = \left|\frac{(\beta - \alpha)}{\alpha - \beta}\right| = 1 \quad \{\alpha\bar{\alpha} = |\alpha|^2 = 1\}$

42. Using S-4, $\alpha = p$, $\beta = p\omega$, $\gamma = p\omega^2$. Put these values in the given expression to simplify it.

43. Use S-28

44. Use S-28, $|z+1| = |z+4-3| \le |z+4| + |-3|$

45. $\sqrt{5} + i\sqrt{3} = \sqrt{8}(\cos\theta + i\sin\theta)$

$\Rightarrow (\sqrt{5} + i\sqrt{3})^{1/3} = (\sqrt{8})^{1/3}\left(\cos\frac{2k\pi + \theta}{3} + i\sin\frac{2k\pi + \theta}{3}\right)$, $k = 0, 1, 2$

Now magnitude of each value equals $(\sqrt{8})^{1/3} = \sqrt{2}$. Therefore, distance of each root from origin is $\sqrt{2}$

Olympiad Enrichment Series - VIII

useful for IIT-JEE 2009-10

This series is selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solution.

1. Let a, b, c and d be real numbers such that $(a^2+b^2-1)(c^2+d^2-1) > (ac+bd-1)^2$. Prove that $a^2+b^2>1$ and $c^2+d^2>1$.

2. Find all complex numbers z such that $(3z+1)(4z+1)(6z+1)(12z+1)=2$.

3. Let $x_1, x_2, \ldots, x_{n-1}$ be the zeros different from 1 of the polynomial $P(x)=x^n-1$, $n\geq 2$.
Prove that $\dfrac{1}{1-x_1}+\dfrac{1}{1-x_2}+\ldots+\dfrac{1}{1-x_{n-1}}=\dfrac{n-1}{2}$.

4. Let a and b be given real numbers. Solve the system of equations
$$\frac{x-y\sqrt{x^2-y^2}}{\sqrt{1-x^2+y^2}}=a,\quad \frac{y-x\sqrt{x^2-y^2}}{\sqrt{1-x^2+y^2}}=b$$
for real numbers x and y.

SOLUTIONS

1. For the sake of the contradiction, suppose that one of a^2+b^2 or c^2+d^2 is less than or equal to 1. Since $(ac+bd-1)^2\geq 0$, a^2+b^2-1 and c^2+d^2-1 must have the same sign. Thus both a^2+b^2 and c^2+d^2 are less than 1. Let $x=1-a^2-b^2$ and $y=1-c^2-d^2$.

Then $0<x, y\leq 1$. Multiplying by 4 on both sides of the given inequality gives

$$4xy > (2ac+2bd-2)^2=(2-2ac-2bd)^2$$
$$=(a^2+b^2+x+c^2+d^2+y-2ac-2bd)^2$$
$$=[(a-c)^2+(b-d)^2+x+y]^2$$
$$\geq (x+y)^2=x^2+2xy+y^2, \text{ or } 0>x^2-2xy+y^2$$
$=(x-y)^2$, which is impossible. Thus our assumption is wrong and both a^2+b^2 and c^2+d^2 are greater than 1.

2. Note that, $8(3z+1)\,6(4z+1)\,4(6z+1)\,2(12z+1)=768$,
i.e., $(24z+8)(24z+6)(24z+4)(24z+2)=768$.
Setting $u=24z+5$ and $w=u^2$ yields
$(u+3)(u+1)(u-1)(u-3)=768$,
i.e. $(u^2-1)(u^2-9)=768$, i.e. $w^2-10w-759=0$
i.e. $(w-33)(w+23)=0$.
Therefore the solutions to the given equation are
$$z=\frac{\pm\sqrt{33}-5}{24} \text{ and } z=\frac{\pm\sqrt{23}i-5}{24}.$$

3. **Alternative - 1 :** For $i=1, 2, \ldots, n$, let $a_i=1-x_i$.

Let $Q(x)=\dfrac{P(1-x)}{x}=\dfrac{(1-x)^n-1}{x}$.
Then
$$Q(x)=(-1)^n x^{n-1}+(-1)^{n-1}\binom{n}{1}x^{n-2}+\ldots+\binom{n}{2}x-\binom{n}{1}$$
and a_i's are the non zero roots of the polynomial $Q(x)$, as
$$Q(a_i)=\frac{(1-a_i)^n-1}{a_i}=\frac{x_i^n-1}{1-x_i}=0.$$
Thus the desired sum is the sum of the reciprocals of the roots of polynomial $Q(x)$, that is,
$$\frac{1}{1-x_1}+\frac{1}{1-x_2}+\ldots+\frac{1}{1-x_{n-1}}=\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_{n-1}}$$
$$=\frac{a_2a_3\ldots a_n+a_1a_3\ldots a_n+\ldots+a_1a_2\ldots a_{n-1}}{a_1a_2\ldots a_n}$$
By the Vieta's Theorem, the ratio between
$S=a_2\ldots a_n+a_1a_3\ldots a_n+\ldots+a_1a_2\ldots a_{n-1}$
and $P=a_1\ldots a_n$ is equal to the additive inverse of the ratio between the coefficient of x and the constant term in $Q(x)$, i.e., the desired value is equal to
$$\frac{S}{P}=-\frac{\binom{n}{2}}{-\binom{n}{1}}=\frac{n-1}{2}, \text{ as desired.}$$

Alternative - 2 :
For any polynomial $R(x)$ of degree $n-1$, whose zeros are $x_1, x_2, \ldots, x_{n-1}$, the following identity holds :
$$\frac{1}{x-x_1}+\frac{1}{x-x_2}+\ldots+\frac{1}{x-x_{n-1}}=\frac{R'(x)}{R(x)}.$$
For $R(x)=\dfrac{x^n-1}{x-1}=x^{n-1}+x^{n-2}+\ldots+x+1$,
$R(1)=n$ and $R'(1)=(n-1)+(n-2)+\ldots+1=\dfrac{n(n-1)}{2}$.
It follows that
$$\frac{1}{1-x_1}+\frac{1}{1-x_2}+\ldots+\frac{1}{1-x_{n-1}}=\frac{R'(1)}{R(1)}=\frac{n-1}{2}.$$

4. Let $u=x+y$ and $v=x-y$. Then
$0<x^2-y^2=uv<1$, $x=\dfrac{u+v}{2}$, and $y=\dfrac{u-v}{2}$.

Adding the two equations and subtracting the two equations in the original system yields the new system

$$u - u\sqrt{uv} = (a+b)\sqrt{1-uv} \quad v + v\sqrt{uv} = (a-b)\sqrt{1-uv}.$$

Multiplying the above two equations yields

$uv(1 - uv) = (a^2 - b^2)(1 - uv)$, hence $uv = a^2 - b^2$.

It follows that $u = \dfrac{(a+b)\sqrt{1-a^2+b^2}}{1-\sqrt{a^2-b^2}}$ and

$$v = \frac{(a-b)\sqrt{1-a^2+b^2}}{1+\sqrt{a^2-b^2}}$$

which in turn implies that

$$(x,y) = \left(\frac{a+b\sqrt{a^2-b^2}}{\sqrt{1-a^2+b^2}}, \frac{b+a\sqrt{a^2-b^2}}{\sqrt{1-a^2+b^2}}\right)$$

whenever $0 < a^2 - b^2 < 1$.

Right Act : IITs change selection mode

Move comes after IITs found it hard to explain procedure to people using Right To Information Act

THE Joint Entrance Examination (JEE), conducted by the Indian Institutes of Technology (IITs), has posed questions that have foxed many a student. Now that people using the Right To Information Act are asking it to disclose how students are selected, the IITs had to change the selection procedure.

"We have had to make the procedure much more simpler because we thought the earlier system was a little too complicated to be explained to the general public," says a highly placed source in IIT.

The JEE is arguably one of the toughest entrance examinations in the world with 60-70 students vying for each of the 5,000-odd seats on offer. The competition is so intense that many coaching institutes, and indeed entire towns like Kota, specialize in training people to pass this test. This is a trend that clearly worried the wise old men at IIT. "We are looking for innate analytical abilities rather than people who have access or the money to go through the intensive system," says an IIT director.

COURSE CORRECTION

- 60-70 students vie for each of the 5,000-odd seats in IITs every year
- Earlier, IITs chose top 84% students in each subject. Now, it will pick 80%
- Number of seats to decide cut-off marks; 5,500 students to be selected

The only way to do this to keep varying the nature of the question papers in physics, chemistry and maths – the three subjects that JEE tests – and that is where the problem of selection starts. The selection process has to make sure that students from even poor areas get to the short listing stage and also has to take into account that random nature of questions and varying ways in which students can be evaluated. Many questions in the JEE can be solved through more than one method.

For the last 10-12 years IITs have used a statistical method of coming up with a list of students from which the merit list is then made, depending on the number of seats available that year. The scores in each subject are accumulated in the form of a Bell Curve. The curve is called so because it looks like a bell and has all the average scores in the centre while exceptionally good and unusually bad scores are at the right and left extremeties, respectively.

The JEE would select all the students on right-half of the bell and about 34% of the scores on the left-half of the curve, what is technically called "one standard deviation" on the negative side. Effectively, the IITs would end up selecting the top 84% of the students in each of the subjects. The availability of seats would decide the total cut-off marks based on which the 5,000-odd students would be selected.

It is this method that has now been discontinued, ostensibly because the IITs would have found it hard to put their statistical technique up for public scrutiny. Now, the IITs have decided that the top 80% (a nice round number) would be selected. A total of 5,500 students would be selected. "This is a much simpler way and can be easily understood by everyone," says a source close to JEE.

Incidentally, some news reports of the IITs selecting people with single digit marks appears incorrect. "The numbers must be wrong. This is a wrong cut-off as the last time when we had single digit cut-offs was 15 years ago," IIT Madras director MS Ananth said.

There iş another explanation for it. According to information made available, there are two levels of cut-offs. When the IITs select 84% of the students (first cut-off), about 1,50,000 of the 2,00,000-odd students are selected and this is where students with marks low as 1, 2 or even 4 are selected but only to be put through another round of selection. A second cut-off decides the 5,000-odd people who make it through the gates of seven IITs.

By - Ramkrishna B.S. Khandeparkar, Dept. of Mathematics,
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Trigonometric Ratios (continued) :

In this article we obtain answer and the preceding rows simultaneously. We have seen the method of multiplication where we multiply from left to right. This helps to get the product in the exponential form. Similar method is followed here, getting answer and the preceding steps simultaneously. This is one of the advantage of Vedic Mathematics methods. We use this method in finding values of inverse trigonometric functions and later solving transcendental equations.

1. To find the value of $\sin^{-1} 0.3$.

Let $\sin^{-1} 0.3 = x$ then $\sin x = 0.3$. We know that,

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots$ which gives us

$$x = 0.3 + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \frac{x^{11}}{11!} - \ldots$$

Note that we have to find the value of x and the R. H. S. gives us the first digit of the value of x.

$\sin x$	0.3	0	0	0	0
$\frac{x^3}{3!}$	.	0_3	5	2_2	1_2
$-\frac{x^5}{5}$	0.			0	$3_{\bar{1}}$
x	0.3	0	5	2	4
x^2	$0.1_{\bar{1}}$	0	3	$1_{\bar{2}}$	

In this example there are five rows involved concerning the terms $\sin x, \frac{x^3}{3!}, -\frac{x^5}{5}, x$ and x^2. We know the value of $\sin x$ and the first digit of this is the first digit in the answer value and therefore we write 0.3 down into the answer in the fourth row. This digit also helps us in finding first digit of the x^2.

i.e. duplex $D<3> = 9 = 1\bar{1}$. We take $\bar{1}$ as remainder and write $1_{\bar{1}}$ in the place of first digit of x^2.

This value of x^2 will then help us to get first digit in the second row. The term of the second row can be expressed as

$\frac{x^3}{3!} = x \times x^2 \div 6$ and hence we have

$P\begin{pmatrix}3\\1\end{pmatrix} = 3, 3 \div 6 = 0$ (quotient) and remainder 3. We write 0_3 as the first digit for the term $\frac{x^3}{3!}$ in the second row. Simultaneously take 0 into the answer.

Then we find the next digit for the term x^2. We have,

i.e. duplex $D<3\ 0> = 0$

Our next step is to find the next digit for the term $\frac{x^3}{3!}$ in the second row. We have, $P\begin{pmatrix}3 & 0\\1 & 0\end{pmatrix} = 0, 0 + 30 = 30$, $30 \div 6 = 5$ (quotient) and remainder 0.5 is the second digit of the second row and also 5 is taken into the answer simultaneously.

For next digit of x^2, duplex $D<3\ 0\ 5> = 30$. We write 3 in the place of third digit of x^2.

For second row, $P\begin{pmatrix}3 & 0 & 5\\1 & 0 & 3\end{pmatrix} = 14$. $14 \div 6 = 2$ (quotient) and remainder 2. Thus we get fourth digit for the term $\frac{x^3}{3!}$. Next, $P\begin{pmatrix}0\\1\end{pmatrix} = 0$, gives zero digit for the term $-\frac{x^5}{5}$

For next digit of x^2, duplex $D<3\ 0\ 5\ 2> = 12$. We write 1_2 in the place of fourth digit of x^2.

Similarly, for second row, $P\begin{pmatrix}3 & 0 & 5 & 2\\1 & 0 & 3 & 2\end{pmatrix} = 8, 8 \div 6 = 1$ (quotient) and remainder 2. This gives the next digit for the term $\frac{x^3}{3!}$. Also, $P\begin{pmatrix}0 & 5\\1 & 0\end{pmatrix} = 5,\ 5 \div 2 = 3$ (quotient) and remainder $\bar{1}$. This gives the next digit for the term $-\frac{x^5}{5}$

2. To find the value of $\sin^{-1} 0.2195$

Let $\sin^{-1} 0.2195 = x$, then $\sin x = 0.2195$. We know that,

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots$. which gives us

$$x = 0.2195 + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \frac{x^{11}}{11!} - \ldots$$

Note that we have to find the value of x and the R. H. S. gives us the first digit of the value of x.

MOCK TEST
FOR
ISI 2008

Exam on 7th May 2008

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Each candidate applying for admission to this programme has to take a selection test comprising Objective type and Short-answer type questions in mathematics at the Higher Secondary level (10 + 2 year's programme).

The selection tests consists of

(1) A multiple choice type test having about 30 questions, and

(2) A short-answer type test having about 10 questions.

Questions will be set on the following and related topics.

Algebra : Sets, operations on sets, prime numbers, factorization of integers and divisibility, rational and irrational numbers, permutations and combinations, binomial theorem, logarithms, theory of quadratic equations, polynomial and remainder theorem, arithmetic and geometric progressions, inequalities involving A.M., G.M., and H.M., complex numbers.

Geometry : Plane geometry of class X level. Geometry of 2 dimensions with cartesian and polar co-ordinates. Concept of a locus, equation of a line, angle between two lines, distance from a point to a line. Areas of a triangle, equations of a circle, parabola, ellipse and hyperbola and equations of their tangents and normals, mensuration.

Trigonometry : Measures of angles, trigonometric and inverse trigonometric functions, trigonometric identities including addition formulae, solutions of trigonometric equations. Properties of triangles, heights and distances.

Calculus : Functions, one-one functions, onto functions, limits and continuity, derivatives and methods of differentiation, slope and curve, tangents and normals, maxima and minima, use of calculus in sketching graph of functions, methods of integration, definite and indefinite integrals, evaluation of area using integrals.

Logical Reasoning : Consistency of statements.

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MULTIPLE CHOICE TEST

1. Let $n = \underline{|2008} + 1$ then the number of primes in the list $n+1, n+2, \ldots, n+2007$ is

(a) 0 (b) at least 8

(c) exactly 8 (d) at least 3

2. Let $N = (a+\sqrt{a^2-1})^{1/n} + (a-\sqrt{a^2-1})^{1/n}, n \in N$, $n \geq 1$. It is also known that a is irrational, then which one of the following statements is true?

(a) N is a rational number for exactly two values of n.

(b) N is a rational number for exactly four values of n.

(c) N is rational for no value of n.

(d) N is a rational number for exactly six values of n.

3. If the circles $z\bar{z} + \bar{a}z + a\bar{z} + b = 0$, and $z\bar{z} + \bar{c}z + c\bar{z} + d = 0$, b and d being real cut orthogonally, then $\text{Re}(a\bar{c})$ equals

(a) $b+d$ (b) $\frac{b+d}{2}$ (c) $\frac{b-d}{2}$ (d) $b-d$

4. The equation $|z-i| - |z+i| = k$ represents hyperbola if

(a) $0 < |k| < 2$ (b) $0 < k < 2$

(c) $-2 < k < 2$ (d) $k > 2$

5. Let 72 be a divisor of the base 10 number $(x679y)_{10}$. Then the value of $x+y$ is

(a) 5 (b) 6 (c) 4 (d) 8

6. The number of positive integers n less than 1991 for which 6 divides $n^2 + 3n + 2$ is

(a) 1329 (b) 1328 (c) 1341 (d) 1342

7. The largest positive integer n such that $n + 10$ divides $n^3 + 100$ is
(a) 900 (b) 890 (c) 790 (d) 800

8. Let n be a 50-digit number in base 10. All digits except the 26^{th} (reading from the left) are 1. If n is divisible by 13, then its 26^{th} digit is
(a) 1 (b) 2 (c) 3 (d) 4

9. How many solution pairs (x, y) in integers are there to the equation $x^2(x^2 + y) = y^{n+1}$
(a) finitely many (b) infinitely many
(c) none (d) exactly one

10. Let f satisfy
$$f(x) + f(2x) + f(2-x) + f(1+x) = x$$
for all $x \in R$. Then $f(0)$ equals
(a) $-\frac{1}{4}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

11. An escalator is moving downwards with the speed of 4-steps a minute. Ram takes 6 minutes less to get down if he is going down on the moving escalator as compared to when he comes down via the stationary escalator. Shyam takes 6 minutes more to get up if he is going up on the moving escalator as compared to when he goes up via the stationary escalator. If Ram and Shyam start together from the top and bottom respectively they meet after 4 minutes on the moving escalator. How many steps are there in the escalator?
(a) 32 (b) 48
(c) 16 (d) can't be determined

12. A student holidaying for n days observed that it rained on 7 days – morning or afternoon. When it rained in the afternoon, it was clear in the morning. There were 6 clear mornings. There were 5 clear afternoons. Then n equals
(a) 8 (b) 6 (c) 12 (d) 9

13. Let $S = \{1, 2, 3, \ldots 10\}$, the number of subsets of S whose sum of elements is odd is
(a) 256 (b) 512 (c) 348 (d) 408

14. Let a circle of radius 'b' be inscribed in a rhombus of side a, then the area of the rhombus is (in sq units)
(a) $4ab$ (b) $\frac{1}{2}ab$ (c) ab (d) $2ab$

15. The least value of the expression
$2x^2 + 3y^2 - 4x - 2y + 20$, where x and y range over all real numbers is
(a) 6 (b) 8
(c) 12 (d) none of the above

16. The average weight (in kg) of all the students in a gathering equals the number of students in the gathering. The increase in the average weight when a teacher of 21 kg is included equals the decrease in average weight when a student of 19 kg is excluded. The strength of the class is
(a) 15 (b) 10 (c) 20 (d) 17

17. Let $P = \{(x, y) \mid x^2 + y^2 = 4, x, y \in z\}$
$Q = \{(x, y) \mid x^2 + y^2 = 25; x, y \in z\}$
The number of points that lie inside Q but not inside P is
(a) 38 (b) 48 (c) 60 (d) 56

18. Two circles having equal radii are drawn, without any overlap in a semicircle of radius 2 units. If these be greatest possible such circles that the semicircle can accommodate, the radius of each circle is
(a) $\sqrt{2}+1$ (b) $\sqrt{2}-1$
(c) $2\sqrt{2}+1$ (d) $2\sqrt{2}-2$

19. Let $x, y, z \in R$ satisfy
$$\frac{(x+1)^2+(y+1)^2+(z+1)^2}{x+y+z} = 4,$$ then the value of
$$\left(x+\frac{1}{y}-1\right)\left(y+\frac{1}{z}-1\right)\left(z+\frac{1}{x}-1\right)$$ is
(a) 1 (b) 0
(c) -1 (d) cannot be determined

20. The number of order pairs (a, b) of integers that satisfy $|a + 10^4| + |b - 10^3| \le 3$ equals
(a) 18 (b) 25 (c) 19 (d) 21

21. Let $n = 2^{35} \cdot 3^{23}$, the number of divisors (positive) of n^2 that are less than n and don't divide n is
(a) 1668 (b) 863 (c) 804 (d) 805

22. How many four digit positive integers are there in base 6 if one is counting the number in the same base system?
(a) 5555 (b) 6000 (c) 4555 (d) 5000

23. Let $S(n) = 3 + 8 + 15 + \ldots$ to n terms, and $S'(n) = 6 + 11 + 18 + \ldots$ to n terms, the value of $S(111) - S'(111)$ is
(a) -333 (b) 333 (c) 211 (d) -112

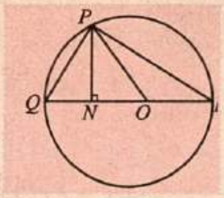
24. Suppose $a + a + a + aa + aaa = 10^k$, where a and n are single digit numbers, aa stands for 2-digit numbers and aaa for 3-digit numbers, then a cannot be
(a) 8 (b) 80 (c) 800 (d) 81

25. The number of 4-digit numbers that are divisible by 30 and 35 but not by 140 is
(a) 22 (b) 44 (c) 45 (d) 23

26. The number of 3-digit numbers such that if one of the digits is 8, the following digit is 9, is

(a) 665 (b) 666 (c) 18 (d) 19

27. In the diagram shown below, $PQ = a$, $PR = b$, O is the centre of the circle and N is a point between Q and O such that $PN \perp QR$. The length of ON equals



(a) $\dfrac{a^2-b^2}{2\sqrt{a^2+b^2}}$ (b) $\dfrac{b^2-a^2}{\sqrt{a^2+b^2}}$

(c) $\dfrac{b^2-a^2}{2\sqrt{a^2+b^2}}$ (d) $\dfrac{a^2-b^2}{\sqrt{a^2+b^2}}$

28. Let S denotes the sum of squares of the sides of a right angle triangle and P the square of the perimeter of the right angled triangle. Then the least value of $\dfrac{S}{P}$ is

(a) $6-4\sqrt{2}$ (b) $3-2\sqrt{2}$ (c) $3-\sqrt{2}$ (d) $6-\sqrt{2}$

29. The lines $y = mx + 9$ and $x + 4y = 81$, where m is a positive integer, intersect at points both of whose co-ordinates are integers. The number of values that m can take is

(a) 1 (b) 2 (c) 3 (d) 4

30. Let n be a positive integer such that $2^n - 1$ is a prime number, then n is a

(a) Prime number (b) Composite number

(c) Prime number only for finitely many values of n.

(d) None of the above.

SHORT ANSWER TYPE TEST

1. Let $f: N \to N$ be a function such that $f(n + 1) > f(f(n)) \;\forall\; n \in N$. Prove that $f(n) = n \;\forall\; n \in N$.

2. Find all polynomials $g(x)$ such that $x\,g(x - 1) = (x - 15)\,g(x)$

3. Given a set of $n + 1$ positive integers, none of which exceeds $2n$, show that at least one of the numbers of the set must divide another member of the set.

4. How many 6-digit codes are there which uses only the digits 0, 1 or 2 and in which the digit 2, whenever appears, it always does so after 1?

5. Let n be a positive integer with at least 4 divisors and let the divisors be $d_1, d_2, d_3, d_4, \ldots$ where $d_1 < d_2 < d_3 < d_4 < \ldots$ with $d_1 = 1$. Find all such n if it is known that $n = d_1^2 + d_2^2 + d_3^2 + d_4^2$.

6. A lattice point (x, y) is a point in the $x - y$ plane both of whose co-ordinates are integers. Find the number of lattice points on the hyperbola $2xy - 5x + y = 55$.

7. Let p be a prime number greater than 3. Prove that 24 divides $p^2 - 1$.

8. Let $p > 2$ be a prime number. Define $S = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{p-1}$, let $S = \dfrac{N_p}{D_p}$ where N_p are D_p are co-prime positive integers. For instance, when $p = 5$ we get $S = \dfrac{25}{12}$, given $N_p = 25$ and $D_p = 12$ observe that 5 divides N_5. Show that in general, p divides N_p.

9. P is a point inside a given triangle ABC. D, E, F are the foot of the perpendiculars from P to the lines BC, CA, AB respectively. Determine, with proof, all P for which $\dfrac{BC}{PD} + \dfrac{CA}{PE} + \dfrac{AB}{PF}$ is least.

10. Three roots of the equation $x^4 - px^3 + qx^2 - rx + s = 0$ are $\tan A$, $\tan B$, $\tan C$, where A, B, C are the angles of a triangle. Determine the fourth root as a function of (only) p, q, r and s.

SOLUTION TO MULTIPLE CHOICE TEST

1. (a) : $n+1 = \underline{|2008} + 2$

$n+2 = \underline{|2008} + 3$

$n+3 = \underline{|2008} + 4 \ldots$

....

....

$n+2007 = \underline{|2008} + 2008$

Thus 2 divides $n + 1$, 3 divides $n + 2$, ... and finally 2008 divides $n + 2007$.

Note that $\underline{|2008}$ is divisible by each of 2, 3, 4, ... and 2008. We have then shown that all the numbers in this list are composite, establisting that there are no primes in the list.

2. (c) : $N = (a + \sqrt{a^2-1})^{1/n} + (a - \sqrt{a^2-1})^{1/n}$

let $b = (a + \sqrt{a^2-1})^{1/n}$ then $N = b + \dfrac{1}{b}$

Now $b^{m+1} + \dfrac{1}{b^{m+1}} = \left(b + \dfrac{1}{b}\right)\left(b^m + \dfrac{1}{b^m}\right) - \left(b^{m-1} + \dfrac{1}{b^{m-1}}\right)$

If N is assumed to be rational, then we obtain that $b^m + \dfrac{1}{b^m}$ is rational for all $m \in N$.

But $b^n + \dfrac{1}{b^n} = a + \sqrt{a^2-1} + a - \sqrt{a^2-1} = 2a$,

an irrational number. Thus the assumption of N being rational leads to a contradiction. Hence N is irrational.

3. (b)

4. (a) : Let z be the variable point P. $-i$ and i be respectively F_1 and F_2 – two fixed points.

The equation $|z - i| - |z + i| = k$ reduces to

$PF_1 - PF_2 = k$

The locus of P, under suitable condition on k, can be a hyperbola having foci as F_1 and F_2.

By the triangle inequality $|PF_1 - PF_2| < F_1F_2$ (the difference of two sides being less than the third side).
Thus $|k| < |(-i)-(i)|$
$\Rightarrow\ |k| < |2i| \qquad \Rightarrow \qquad |k| < 2$
But observe that $|k|$ can't be zero, i.e. $k \neq 0$, for then equation would reduce to
$|z-i| = |z+i|$
the locus of which is the perpendicular bisector of the join of points $-i$ and i – clearly a straight line.
Thus for $0 < |k| < 2$ does the equation
$|z-i| - |z+i| = k$ represents a hyperbola.

5. (a) : Write 72 as a product of two numbers prime to each other $72 = 2^3 \cdot 3^2$. Since 72 divides $x679y$, it follows that 8 and 9 both divide $x679y$.
For a number to be divisible by 8, its last three digits should be divisible by 8, so 8 divides $79y$ giving $y = 2$.
Also for a number to be divisible by 9, the sum of its digits must be divisible by 9, so 9 divides $x + 6 + 7 + 9 + y = x + 24$. Thus 9 divides $x + 6$, giving $x = 3$.
The number $x679y$ therefore, is 36792 and then $x + y$ equals 5.

6. (b) : $n^2 + 3n + 2 = (n+1)(n+2)$, again $6 = 2 \times 3$. If 6 is to be a divisor of $(n+1)(n+2)$ then four possibilities arise
(a) 6 is a divisor of $n + 1$, or
(b) 6 is a divisor of $n + 2$, or
(c) 3 is a divisor of $n + 1$, and 2 is a divisor of $n + 2$, or
(d) 2 is a divisor of $n + 1$, and 3 is a divisor of $n + 2$.
The possibility (a) holds for $n = 5, 11, 17, \ldots 1991$ thus for 332 values of n.
The possibility (b) holds for $n = 4, 10, 16, \ldots, 1990$, thus for 332 values of n.
The possibility (c) holds for $n = 2, 8, 14, \ldots 1988$, thus for 332 values of n.
The possibility (d) holds for $n = 1, 7, 13, \ldots 1987$, thus for 332 values of n.
So there are in all $4 \times 332 = 1328$ values of n between 1 and 1991 for which $n^2 + 3n + 2$ is divisible by 6.

7. (b)

8. (c) : Let x be the 26th digit of n.
$$n = \underbrace{111\ldots1111}_{25\text{1's}}\,x\,\underbrace{111\ldots111}_{24\text{1's}}$$
for the divisibility by 13, we apply the alternating 3-digit sum test – remember that the sum has to be calculated starting from the right.
Let the sum be $\alpha + \overline{11x} - \beta + 11$, where
$$\alpha = \underbrace{111\ldots.111 + \ldots - 111}_{8\text{ blocks}}$$
$$\beta = \underbrace{111 - 111 + \ldots + 111}_{7\text{ blocks}}$$
we have $\alpha = 0$ and $\beta = 111$. So the sum is $\overline{11x} - 100 = \overline{1x}$
Obviously then $x = 3$. Thus the 26th digit of N is 3.
[Note that $\overline{11x}$ means the three digit number $(11x)10$]

9. (b) : The equation is $x^2(x^2 + y) = y^{n+1}$
the equation is $x^4 + x^2y = y^{n+1}$
$\Rightarrow\ 4x^4 + 4x^2y = 4y^{n+1}$
$\Rightarrow\ y^2 + 4x^4 + 4x^2y = y^2 + 4y^{n+1}$
$\Rightarrow\ (2x^2 + y)^2 = y^2 + 4y^{n+1} = y^2(1 + 4y^{n-1}) \qquad \ldots(i)$
from (i) it follow that $1 + 4y^{n-1}$ is an odd square,
let $1 + 4y^{n-1} = (2k+1)^2$
$\Rightarrow\ 1 + 4y^{n-1} = 1 + 4k + 4k^2$
$\Rightarrow\ y^{n-1} = k^2 + k = k(k+1)$
Since k and $k + 1$ are relatively prime integers, each of them must be the $(n-1)^{\text{th}}$ power of some integer. It is possible only when $n = 2$, thus given $y = k(k+1)$
from (i)
$\Rightarrow\ 2x^2 + k(k+1) = k(k+1)(2k+1)$
$\Rightarrow\ 2x^2 = k(k+1)(2k+1) - k(k+1) = k(k+1)\cdot 2k$
$\Rightarrow\ x^2 = k^2(k+1)$
So $k + 1$ should be a square $\Rightarrow k + 1 = t^2$ (say).
then $x^2 = (t^2-1)^2t^2 \quad \therefore \quad x = (t^2-1)t = t^3 - t$
Again $y = k(k+1) = (t^2-1)t^2 = t^4 - t^2$.
$\therefore\ (x, y) = (t^3 - t, t^4 - t^2)$ describes all the solution of the problem, which are obviously infinitely many.

10. (a) : The given functional relation is
$f(x) + f(2x) + f(1+x) + f(2-x) = x$
Set $x = 0$ in the above relation to get
$f(0) + f(0) + f(1) + f(2) = 0$
$\Rightarrow\ 2f(0) + f(1) + f(2) = 0 \qquad \ldots(i)$
Set $x = 1$ in the funtional relation to obtain
$f(1) + f(2) + f(2) + f(1) = 1$
$\Rightarrow\ 2f(1) + 2f(2) = 1 \Rightarrow\ f(1) + f(2) = 1/2 \qquad \ldots(ii)$
using (i) and (ii)
$$2f(0) + \frac{1}{2} = 0 \quad \therefore \quad f(0) = -\frac{1}{4}$$

11. (b) : Let n be the number of steps in the escalator and the speed of Ram and Shyam be a and b steps a minute respectively. Using the idea of relative speed three equations can be immediately written
$$a + b = \frac{n}{4} \qquad \ldots(i)$$
$$\frac{n}{a} - \frac{n}{a+4} = 6 \qquad \ldots(ii)$$
$$\frac{n}{b-4} - \frac{n}{b} = 6 \qquad \ldots(iii)$$
Now the problem will be solved with the help of 3 simultaneous equations in 3 unknowns to find n.
First Solution
From (ii) and (iii) on equating

$$\frac{n}{a}-\frac{n}{a+4}=\frac{n}{b-4}-\frac{n}{b} \Rightarrow \frac{1}{a}-\frac{1}{a+4}=\frac{1}{b-4}-\frac{1}{b}$$

$$\Rightarrow \frac{4}{a(a+4)}=\frac{4}{b(b-4)} \quad \Rightarrow a(a+4)=b(b-4)$$

$\Rightarrow a^2+4a=b^2-4b \quad \Rightarrow \quad a^2-b^2=4(a-b)$

$\Rightarrow (a-b)(a+b)=-4(a+b)$

$\Rightarrow (a+b)(a-b+4)=0$

As $a \neq -b$, we have $b=a+4$ ...(*iv*)

from (*i*) and (*ii*)

$$a+b=\frac{n}{4}$$

$$n\left(\frac{1}{a}-\frac{1}{a+4}\right)=6 \quad \Rightarrow 4(a+b)\left(\frac{1}{a}-\frac{1}{a+4}\right)=6$$

$$\Rightarrow 4(a+a+4)\left(\frac{1}{a}-\frac{1}{a+4}\right)=6$$

$$\Rightarrow 8(a+2)\cdot\frac{4}{a(a+4)}=6$$

$\Rightarrow 16(a+2)=3a(a+4)=3a^2+12a$

$\Rightarrow 3a^2-4a-32=0 \Rightarrow \quad 3a^2-12a+8a-32=0$

$\Rightarrow 3a(a-4)+8(a-4)=0 \Rightarrow (3a+8)(a-4)=0$

As a can't be negative, $a=4$. Consequently $b=8$

Total number of steps $n=4(a+b)=4(4+8)=4\cdot 12=48$

Second Solution (Elegant)

It takes some ingenuity to look at the system in a way that uses 'quadratic equation' in a subtle way.

Rewrite (*ii*) and (*iii*) as

$$\frac{n}{a}-\frac{n}{a+4}=6; \quad \frac{n}{-b}-\frac{n}{-b+4}=6$$

indicating that a and '$-b$' are the roots of the quadratic

$$\frac{n}{x}-\frac{n}{x+4}=6$$

$\Rightarrow 4n=6x(x+4) \quad \Rightarrow \quad 2n=3x(x+4)$

$\Rightarrow 3x^2+12x-2n=0$

$\Rightarrow a+(-b)=-12/3=-4$ and $a(-b)=-2n/3$

The first equation is

$$a+b=\frac{n}{4} \qquad \text{...(A)}$$

i.e. $a-(-b)=\dfrac{n}{4}$ ..(B)

using (A) and (B)

$$\{a-(-b)\}^2=\{a+(-b)\}^2-4(a)(-b)$$

$$\Rightarrow \frac{n^2}{16}=16-4\cdot\left(\frac{-2n}{3}\right)$$

$$\Rightarrow \frac{n^2}{16}=16+\frac{8n}{3} \quad \Rightarrow \quad 3n^2-128n-768=0$$

$\Rightarrow 3n^2-144n+16n-768=0$

$\Rightarrow 3n(n-48)+16(n-48)=0$

$\Rightarrow (n-48)(3n+16)=0 \qquad \Rightarrow n=48.$

12. (d) : There were 6 clear mornings mean there were $n-6$ rainy morning. Similarly, there were 5 clear afternoons mean there were $n-5$ rainy afternoons. When it rained in the afternoon it was clear in the morning implies that there is no day when it rained both in the afternoon and in the morning. The number of rainy days = the number of rainy afternoons + the number of rainy mornings

$=(n-5)+(n-6)=2n-11$

But $2n-11=7$ (given)

$\Rightarrow 2n=18 \quad \therefore \quad n=9$

13. (b) : First Solution

The set S has 5 even and 5 odd numbers. For a subset of S having the sum of elements as odd, the subset must contain an odd number of odd numbers and any number of even numbers. Thus the required number of ways

$=({}^5C_1+{}^5C_3+{}^5C_5)\times({}^5C_0+{}^5C_1+{}^5C_2+{}^5C_3+{}^5C_4+{}^5C_5)$
$=2^{5-1}\times 2^5=2^9=512.$

(Recall that $C_0+C_1+C_2+\ldots C_n=2^n$ and
$C_1+C_3+C_5+\ldots=2^{n-1}$)

Second Solution

As the set S contains an equal number of odd numbers and even numbers, there will be an equal number of subsets of S that add up to even or odd numbers. The total number of subsets $=2^{10}$.

The desired number is half of this, that is $2^9=512$.

14. (d)

15. (a) : $2x^2+3y^2-4x-12y+20$
$=2(x^2-2x)+3(y^2-4y)+20$
$=2(x^2-2x+1)+3(y^2-4y+4)+6$
$=2(x-1)^2+3(y-2)^2+6$

The least value occurs at $x=1, y=2$ and is given by 6.

16. (b) : Let the number of students in the class be n

Thus the average weight $=n$

total weight $=n\times n=n^2$

When a teacher of 21 kg is included, the increase in average weight $=\dfrac{n^2+21}{n+1}-n$

When a student of 19 kg is excluded, the decrease in averge weight $=n-\dfrac{n^2-19}{n-1}$

Both these are equal, which means

$$\Rightarrow \frac{n^2+21}{n+1}-n=n-\frac{n^2-19}{n-1} \quad \Rightarrow \frac{21-n}{n+1}=\frac{-n+19}{n-1}$$

$$\Rightarrow \frac{22-(n+1)}{n+1}=\frac{-(n-1)+18}{n-1}$$

$$\Rightarrow \frac{22}{n+1}-1=-1+\frac{18}{n-1} \quad \Rightarrow \frac{22}{n+1}=\frac{18}{n-1}=\frac{40}{2n}=\frac{4}{2}$$

(using ratio and proportion)

which gives $n=10$

17. (c) : We are looking for lattice points that lie inside Q but not inside P, *i.e* (x, y) must satisfy $4\le x^2+y^2<25$

we consider non-negative solutions and then multiply by a suitable factor to take care of change of sign or interchange of x and y.

for $x = 0, y = 2, 3, 4 \quad \Rightarrow 3 \times 2 = 6$ solutions

$x = 1, y = 2, 3, 4 \quad \Rightarrow 3 \times 4 = 12$ solutions

$x = 2, y = 0, 1, 2, 3, 4 \quad \Rightarrow 5 \times 4 - 2 = 18$ solutions

$x = 3, y = 0, 1, 2, 3 \quad \Rightarrow 4 \times 4 - 2 = 14$ solutions

$x = 4, y = 0, 1, 2, \quad \Rightarrow 3 \times 4 - 2 = 10$ solutions

we have in all $(6 + 12 + 18 + 14 + 10) = 60$ solutions

18. (d)

19. (a) : from the given relation

$$(x+1)^2 + (y+1)^2 + (z+1)^2 = 4(x+y+z)$$

$$\Rightarrow (x+1)^2 - 4x + (y+1)^2 - 4y + (z+1)^2 - 4z = 0$$

$$\Rightarrow (x-1)^2 + (y-1)^2 + (z-1)^2 = 0$$

which holds iff $x = y = z = 1$

Now $\left(x+\frac{1}{y}-1\right)\left(y+\frac{1}{z}-1\right)\left(z+\frac{1}{x}-1\right)$

$$= (1+1-1)(1+1-1)(1+1-1) = 1 \cdot 1 \cdot 1 = 1$$

20. (b) : Let $a + 10^4 = x$ and $b - 10^3 = y$, then the equation reduces to $|x| + |y| \le 3$.

The left hand side can take values 0, 1, 2 or 3.

$|x| + |y| = 0$ has just one solution (0, 0)

$|x| + |y| = 1$ has 4 solutions (1, 0), (–1, 0), (0, 1), (0, –1)

$|x| + |y| = 2$ has 8 solutions (0, 2), (0, – 2), (2, 0), (–2, 0), (1, 1), (1, – 1), (–1, 1), (– 1, – 1)

$|x| + |y| = 3$ has 12 solutions.

The total number of solutions = 1 + 4 + 8 + 12 = 25.

Corresponding to every (x, y) that is a solution, we have a corresponding solutions (a, b). Thus there are 25 solutions.

21. (d)

22. (d) : Suppose one counts in base 10, then there are 5000 four digit numbers. If counting happens in base 6, there could be 5000 4-digit numbers. Note that 9 and 5 are respectively the greatest digit in base 10 and base 6.

23. (a) : $S(n) - S'(n) = (3 + 8 + 15 + ...) - (6 + 11 + 18 + ...)$

$= -(3 + 3 + ...$ to n terms$) = -3n$

we have $S(111) - S'(111) = -3 \times 111 = -333$

24. (d) : $aaa + aa + a + a + a = 10^k$

$\Rightarrow aaa + aa + 3a = 10^k \quad \Rightarrow a(111 + 11 + 3) = 10^k$

$\Rightarrow 125a = 10^k$

As 125 must divide 10^k, then the min. value of k is 3.

For $k = 3, a = 8$

$k = 4, a = 80$

$k = 5, a = 800$

25. (a) : The number divisible by 30 and 35 both is also divisible by their LCM, *i.e.* 210. If a number is divisible by 210, but not by 140, then it is not divisible by their LCM, i.e. 420. The least and the greatest 4-digit number divisible by 210 are 1050 and 9870 respectively.

The number of numbers divisible by 210

$$= \frac{9870-1050}{210} + 1 = \frac{8820}{210} + 1 = 43$$

Similarly the number of 4-digit number divisible by 420

$$= \frac{9660-1260}{420} + 1 = 21$$

Hence the number of numbers divisible by 210 but not divisible by 420 = 43 – 21 = 22

26. (a) : If 8 is not one of the 3-digits, we have $8 \times 9 \times 9 = 648$ 3-digit numbers.

With 8 in tens place and 9 in units place we have 8 3-digit numbers - viz 189, 289, ... 789, 989 (Note that 889 is excluded).

With 8 in hundreds place and 9 in tens place we have 9 3-digit numbers - viz 890, 891, ... 897, 899 (Note that 898 is excluded)

The total number of numbers = 648 + 8 + 9 = 665.

27. (c) **28. (a)**

29. (c) : $y = mx + 9$ and $x + 4y = 81$ together give

$$\frac{81-x}{4} = mx + 9 \quad \Rightarrow \quad 81 - x = 4mx + 36$$

$\Rightarrow 45 = x(4m + 1) \quad \Rightarrow \quad x(4m + 1) = 45$

As m is a positive integer so is $4m + 1$ and then so is x. The above equation is to be read as 45 being the product of two positive integers.

$4m + 1$ can be 5, 9, 45

So m can be 1, 2, 9. With these values of m, both x and y turn out to be integers.

30. (a) : We will show that if $2^n - 1$ is a prime number then so is n. Assume n is not a prime number, then $n = ab$ for some positive integers $a, b > 1$.

Now $2^n - 1 = 2^{ab} - 1 = (2^a)^b - 1$

$$= (2^a - 1)\{(2^a)^{b-1} + (2^a)^{b-2} + ... + 2^a + 1\}$$

and this factorisation establishes that $2^n - 1$ is not a prime number. contradiction!

Thus n is a prime number.

SOLUTIONS TO SHORT ANSWER TYPE TEST

1. Let d be the least element of the range of f; *i.e.* $d = \min\{f(n) : n \in N\}$

By the well ordering principle, such an element d exists and it is unique. Let $m \in N$ be such that $d = f(m)$. If $m > 1$, then we have $d = f(m) > f(f(m-1))$. Thus we get a new element $f(m-1)$ whose f-value is smaller than d. But this contradicts the choice of d as the least element of $\{f(n) : n \in N\}$. So $m = 1$.

Now think of the set $\{f(n) : n \ge 2\}$. Reasoning as we did before, we can conclude that this set has the least element $f(2)$. Also $f(1) < f(2)$. For $f(1) = f(2)$ could give $f(1) > f(f(1))$ contradicting the choice of $f(1)$. This chain can be continued to give

$$f(1) < f(2) < f(3) ... < f(n) < ... \quad ...(A)$$

Note that $f(1) \ge 1$. This alone with (A) shows that

$f(k) \geq k$ for all natural k. Let $f(k) > k$ for some k, then $f(k) \geq k+1$. Using (A) we have $f(k+1) \leq f(f(k))$.
But this contradicts the given condition $f(k+1) > f(f(k))$.
We have $f(k) = k$ for all natural numbers k.

2. From $x\,g(x-1) = (x-15)\,g(x)$ we observe that x divides $g(x)$. Thus $g(x) = xg_1(x)$, where $g_1(x)$ is another polynomial. Putting this in the original equality

$$x\cdot(x-1)g_1(x-1) = (x-15)xg_1(x)$$

Cancelling the common factor

$$(x-1)\,g_1(x-1) = (x-15)g_1(x) \quad \ldots(i)$$

Thus $(x-1)$ divides $g_1(x)$. So we can write

$$g_1(x) = (x-1)\,g_2(x)$$

Putting this in (i) we have

$$(x-1)(x-2)\,g_2(x-1) = (x-15)(x-1)\,g_2(x)$$

Cancelling the common factor

$$(x-2)\,g_2(x-1) = (x-15)\,g_2(x)$$

Thus $(x-2)$ divides $g_2(x)$

Continuing in this manner, we have a polynomial $g_{15}(x)$ such that

$$x(x-1)(x-2)\ldots(x-15)g_{15}(x-1) = (x-15)g(x)$$
$$= x(x-1)(x-2)\ldots(x-15)g_{15}(x)$$

Thus $g_{15}(x-1) = g_{15}(x)$

If $g_{15}(x)$ is not a constant polynomial, then $g_{15}(x) = 0$ has a root α in C. But then $g_{15}(\alpha-1) = g_{15}(\alpha) = 0$, so that $\alpha - 1$ is also a root of $g_{15}(x) = 0$. We see that α, $\alpha-1$, $\alpha-2$, ... are all roots of $g_{15}(x) = 0$.

This is clearly impossible, since the equation $g_{15}(x) = 0$ can have only finitely many roots. The only possibility is that $g_{15}(x) = $ constant.

Thus $g(x) = kx(x-1)(x-2)\ldots(x-14)$ for some constant k.

3. Let the given set be $S = \{a_1, {}_2, \ldots a_{n+1}\}$

Suppose $a_i = 2^{n_i} b_i$

Where n_i is a non-negative integer and b_i is odd. What we have done is to write each a_i as the product of the highest power of 2 present in it and an odd number. For instance $48 = 2^4\cdot 3$, $32 = 2^5\cdot 1$. Corresponding to each a_i we have a b_i, in effect, we get a total of $(n+1)$ odd numbers b_1, b_2, ... b_{n+1}. Also all these odd numbers are less than $2n$. But there are only n odd numbers less than $2n$. So by the **Pigeon-hole principle** some two of them, say b_j and b_k must be equal. *i.e.* $b_j = b_k = b$

Then $a_j = 2^{n_j}\cdot b_j$ and $a_k = 2^{n_k}\cdot b_k$
$= 2^{n_j} b \qquad = 2^{n_k}\cdot b$

If $n_j \leq n_k$ then a_j divdes a_k and if $n_j > n_k$ then a_k divides a_j. Thus in either case we have two numbers in S, one of which divides the other.

4. (This problem can't be solved by ordinary principles of counting. Through this problem we introduce the reader to a powerful counting strategy – recursion.)

Denote by a_n the number of n-digit codes made up of 0, 1 or 2 and satisfying the conditions of the problem. a_2, the number of 2-digit codes can be obtained by simply counting them all : 00, 01, 02, 10, 11, 12, 22, 20.

Let $x = x_1x_2 \ldots x_n$ be a sequence belonging to a_n. All the codes fall in two mutually exclusive cases.

(i) If x starts with 2 *i.e.* $x_1 = 2$ then each of x_2, x_3, ... x_n can be 0 or 2. Then there are 2^{n-1} such sequences. Remember that 1 cannot appear to the right of 2.

(ii) If x starts with 0 or 1, *i.e.* $x_1 = 0$ or 2, then $x_2x_3 \ldots x_n$ is a sequence of $(n-1)$ digits satisfying the conditions of the problem. Thus there are $2a_{n-1}$ such sequences.

$$a_n = 2^{n-1} + 2a_{n-1},\ n \geq 2$$

Also $a_2 = 8$.

$$a_3 = 2^2 + 2a_2 = 4 + 2\cdot 8 = 20$$
$$a_4 = 2^3 + 2a_3 = 8 + 2\cdot 20 = 48$$
$$a_5 = 2^4 + 2a_4 = 16 + 2\cdot 48 = 112$$
$$a_6 = 2^5 + 2a_5 = 32 + 2\cdot 112 = 256$$

5. $d = 1$ (given)

If n were odd, then all its divisors would be odd and in that case $1 + d_2^2 + d_3^2 + d_4^2$ – *a* sum of four odd numbers, could be odd. Contradiction, so n is even, which gives that $d_2 = 2$.

Now $n = d_1^2 + d_2^2 + d_3^2 + d_4^2 = 1 + 4 + d_3^2 + d_4^2$
$\Rightarrow \quad n = 5 + d_3^2 + d_4^2$

with d_3, d_4 both dividing n and $2 < d_3 < d_4$.

Suppose d_3 is even then $d_3 = 4$(the only possibility), so $n = 21 + d_4^2$ and since d_4 must be odd, it is the least odd prime divisor of n. As $2d_4$ divides n, it follows that $2d_4$ divides $21 + d_4^2$, So d_4 divides 21 and thus $d_4 = 3$ or 7, which mean that $n = 30$ or 70. But neither 30 nor 70 satisfies the condition of the problem.

So d_3 is odd, and d_4 is even. If d_4 be 4, then $d_3 = 3$ and $n = 5 + 3^2 + 4^2 = 30$. But we have ruled out 30. Hence $d_4 > 4$. As d_4 is the first even divisor of n after 2, we have $d_4 = 2d_3$. So $n = 5 + d_3^2 + (2d_3)^2 = 5(1 + d_3^2)$.

As d_3 divides n we must have $d_3 / 5(1 + d_3^2)$. Also d_3 and $1 + d_3^2$ have no factors in common, we got that d_3 divides 5, and then $n = 5 \times 26 = 130$.It can be checked that 130 satisfies the conditions of the problem. The first four divisors of 130 are 1, 2, 5, 10 and we have in fact $130 = 1^2 + 2^2 + 5^2 + 10^2$. So there is exactly one such n.

6. Our task is to find all pairs (x, y) of integers such that $2xy - 5x + y = 55$.

Rewrite the equation as $y(2x+1) = 5x + 55$

$$y = \frac{5x+55}{2x+1} = \frac{2(2x+1)+x+53}{2x+1} = 2 + \frac{x+53}{2x+1}$$

Since y is to be an integer, $2x + 1$ divides $x + 53$.

Thus $2x + 1$ also divides

$2(x + 53) = 2x + 106 = (2x + 1) + 105$.

Consequently $2x + 1$ divides 105

$$105 = 3\cdot 5\cdot 7$$

The divisors for 105 are $\pm 1, \pm 3, \pm 5, \pm 7, \pm 15, \pm 21, \pm 35$ and ± 105. As $(2x + 1)$ may assume any of these values, the possible values of x are $0, -1, 1, -2, 2, -3, 3, -4, 7, -8, 10, -11, 17, -18, 52$ and -53. The values of y are

then found by $y=\dfrac{5x+55}{2x+1}$.

We obtain the following pairs (x, y) that satisfy the equation.

(52, 3), (17, 4), (10, 5), (7, 6), (3, 10) (2, 13), (1, 20), (0, 55), (−1, − 50), (−2, − 15), (−3, −8), (−4, −5), (−8, − 1), (−11, 0), (−18, 1), (− 53, 2).

These are all the 16 pairs that satisfy the equation.

7. Suppose p is a prime number greater than 3, then p must be congruent to 1, 5, 7, or 11 (mod 12). This can be checked by writing any positive integer n mod 12.

$n=12k-5 \qquad n=12k-4$
$n=12k-3 \qquad n=12k-2$
$n=12k-1 \qquad n=12k$
$n=12k+1 \qquad n=12k+2$
$n=12k+3 \qquad n=12k+4$
$n=12k+5$

of these only the number $12k-5$, $12k-1$, $12k+1$, $12k+5$ are prime *i.e.* $n=1, 5, 7, 11 \pmod{12}$

Thus $p=12k\pm 1$ or $12k\pm 5$

$p^2=(12k\pm 1)^2$ or $(12k\pm 5)^2$

$=144k^2\pm 24k+1$ or $144k^2\pm 120k+25$

In either case we have $p^2=1 \pmod{24}$. Thus p^2-1 is divisible by 24.

8. Note that $(p-1)$ is an even number, so we can pair the numbers 1, 2, 3, ... $p-1$ as $\{1, p-1\}$, $\{2, p-2\}$, ... the sum of the numbers in each pair being p.

Now $\dfrac{1}{1}+\dfrac{1}{p-1}=\dfrac{p}{p-1}$; $\qquad \dfrac{1}{2}+\dfrac{1}{p-2}=\dfrac{p}{2(p-2)}$

and, thus for any k, $1\le k\le p-1$

$$\frac{1}{k}+\frac{1}{p-k}=\frac{p}{k(p-k)}$$

$$S=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{p-1}=\left(1+\frac{1}{p-1}\right)+\left(\frac{1}{2}+\frac{1}{p-2}\right)+...$$

$$=\frac{p}{p-1}+\frac{p}{2(p-2)}+\frac{p}{3(p-3)}$$

Then S is a sum of $\dfrac{p-1}{2}$ factions in all, each with a numerator p. The least common multiple of the denominators of these fractions is not divisible by p, as the numbers $p-1$, $2(p-1)$, $3(p-3)$, ... are not divisible by p (each is product of numbers less than p, therefore not divisible by p). So p in the numerator remains, it doesn't get cancelled with anything in the denominator. Thus the numerator of S, *i.e.* Np, contains a factor 'p', establishing that p divides Np.

9. Denote the lengths of segments PP, PE, PF by x, y and z. The area of the triangle ABC is

$$A=1/2(ax+by+cz) \qquad ...(i)$$

We seek to minimize $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}$ subject to constraint (i).

Recall the cauchy's inequality

$$(u_1v_1+u_2v_2+u_3v_3)^2\le(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)$$

Take u's as $\sqrt{ax}, \sqrt{by}, \sqrt{cz}$ and v's as $\sqrt{\dfrac{a}{x}}, \sqrt{\dfrac{b}{y}}, \sqrt{\dfrac{c}{z}}$.

Then cauchy's inequality yields

$$\left(\sqrt{ax}\cdot\sqrt{\frac{a}{x}}+\sqrt{by}\cdot\sqrt{\frac{b}{y}}+\sqrt{cz}\cdot\sqrt{\frac{c}{z}}\right)^2$$

$$\le(ax+by+cz)\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)$$

$$\Rightarrow \quad (a+b+c)^2\le 2A\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)$$

$$\Rightarrow \quad \left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)\ge\frac{1}{2A}(a+b+c)^2$$

The equality holds if and only if the triplets

$\left(\dfrac{a}{x}, \dfrac{b}{y}, \dfrac{c}{z}\right)$ and (ax, by, cz) are proportional

$\dfrac{ax}{a/x}=\dfrac{by}{b/y}=\dfrac{cz}{c/z}$ *i.e.* $x^2=y^2=z^2\Rightarrow x=y=z$.

Geometrically the minimum value of the expression

$$\frac{BC}{PD}+\frac{CA}{PE}+\frac{AB}{PF}$$

occurs when p is the incentre of ΔABC.

10. Recall that in a triangle ABC, we have

$$\tan A\tan B\tan C=\tan A+\tan B+\tan C$$

(It can be proved in a single way)

Let $\tan A\tan B\tan C=a=\tan A+\tan B+\tan C$

Let $\tan A\tan B+\tan B\tan C+\tan C\tan A=b$

$(x-\tan A)(x-\tan B)(x-\tan C)=x^3-ax^2+bx-a \quad ...(A)$

Denote the remaining zero of the polynomial

$$x^4-px^3+qx^2-rx+s$$

by α, so that we have

$$(x^3-ax^2+bx-a)(x-\alpha)=x^4-px^3+qx^2-rx+s$$

Equating coefficient of x^3, x^2, x, 1 in both expression

$a+\alpha=p \qquad ...(1)$

$b+a\alpha=q \qquad ...(2)$

$a+b\alpha=r \qquad ...(3)$

$a\alpha=s \qquad ...(4)$

our aim is to eliminate a and b and solve for α.

from (1) and (3) on subtraction $\alpha(1-b)=p-r$

from (2) and (4) on subtraction $b=q-s$

If $b\ne 1$ we have $\alpha=\dfrac{p-r}{1-b}=\dfrac{p-r}{1-(q-s)}=\dfrac{p-r}{1-q+s}$

If $b=1$ the right side of A becomes

$$x^3-ax^2+x-a=(x-a)(x^2+1)$$

As this polynomial has only one real zero, while the hypothesis of the problem says that it has at least three, this case cannot occur.

Thus the fourth root is $\dfrac{p-r}{1-q+s}$.

MOCK TEST

FOR

ISI 2009

Exam on 10th May 2009

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Questions will be set on the following and related topics.

Algebra : Sets, operations on sets, prime numbers, factorization of integers and divisibility, rational and irrational numbers, permutations and combinations, binomial theorem, logarithms, theory of quadratic equations, polynomial and remainder theorem, arithmetic and geometric progressions, inequalities involving A.M., G.M., and H.M., complex numbers.

Geometry : Plane geometry of class X level. Geometry of 2 dimensions with cartesian and polar co-ordinates. Concept of a locus, equation of a line, angle between two lines, distance from a point to a line. Areas of a triangle, equations of a circle, parabola, ellipse and hyperbola and equations of their tangents and normals, mensuration.

Trigonometry : Measures of angles, trigonometric and inverse trigonometric functions, trigonometric identities including addition formulae, solutions of trigonometric equations. Properties of triangles, heights and distances.

Calculus : Functions, one-one functions, onto functions, limits and continuity, derivatives and methods of differentiation. slope and curve, tangents and normals, maxima and minima, use of calculus in sketching graph of functions, methods of integration, definite and indefinite integrals, evaluation of area using integrals.

Logical Reasoning : Consistency of statements.

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MULTIPLE CHOICE TEST

1. If $\log_4(x + 2y) + \log_4(x - 2y) = 1$, then the minimum value of $|x| - |y|$ is

(a) $\sqrt{2}$ (b) $\sqrt{3}$

(c) $\sqrt{4}$ (d) $\sqrt{5}$

2. Let a, b, c, d be positive integers and $\log_a b = \frac{3}{2}, \log_c d = \frac{5}{4}$. If $a - c = 9$, then $b - d =$

(a) 92 (b) 93

(c) 94 (d) 95.

3. It is given that $f(x)$ is a function defined on R, satisfying $f(1) = 1$ and for any $x \in R, f(x+5) \geq f(x)+5$ and $f(x + 1) \leq f(x) + 1$. If $g(x) = f(x) + 1 - x$, then $g(2009) =$

(a) 0 (b) 1 (c) -1 (d) 2009.

4. The area of the region bounded by $x^2 + y^2 \leq \pi^2$ and $y \geq \sin x$ is

(a) $\frac{\pi^2}{2}$ (b) $\frac{\pi^2}{3}$ (c) $\frac{\pi^2}{4}$ (d) $\frac{\pi^2}{6}$

5. Let k be a real number such that the inequality $\sqrt{x-3} + \sqrt{6-x} \leq k$ has a solution. The maximum value of k is

(a) $\sqrt{6} - \sqrt{3}$ (b) $\sqrt{6} + \sqrt{3}$

(c) $\sqrt{3}$ (d) $\sqrt{6}$.

6. Let $a_0, a_1, a_2, \ldots, a_n, \ldots$ be a sequence of numbers satisfying $(3 - a_{n+1})\cdot(6 + a_n) = 18$ and $a_0 = 3$ then $\sum_{i=0}^{n} \frac{1}{a_i} =$

(a) $\frac{1}{3}(2^{n+2} - n - 3)$ (b) $\frac{1}{3}(2^{n+2} + n - 3)$

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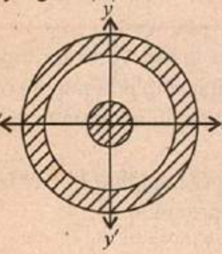
(c) $\frac{1}{3}(2^{n+2}-n+3)$ (d) $\frac{1}{3}(2^{n+2}+n+3)$.

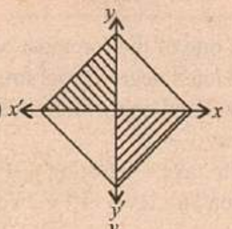
7. The equation $x^3+ax^2-b=0$ $(b>0, a, b \in R)$ has

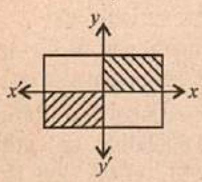
(a) only one positive root (b) two positive roots

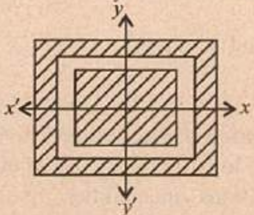
(c) three positive roots (d) no positive root.

8. The region described by the complex number z satisfying $\sin|z|>0$ is

(a) 

(b) 

(c) 

(d) 

9. If M is a 3×3 matrix where $MM'=I$ and $\det(M)=1$, then $\det(M-I)=$

(a) -1 (b) 0

(c) 1 (d) None of these.

10. Let a be the coefficient of x^{1000} in $(1-x^2+x^5)^{2005}$ and b be the coefficient of x^{1000} in $(1+x^2-x^5)^{2005}$ then

(a) $a>b$ (b) $a<b$

(c) $a=b$ (d) None of these.

11. The number of (different) ways in which the words in this problem be ordered (including the words in brackets) is

(a) $\frac{38\times 18!}{9}$ (b) $\frac{38\times 17!}{9}$

(c) $\frac{36\times 17!}{9}$ (d) $\frac{36\times 18!}{9}$.

12. The area enclosed by the graph of $|x|+|y|+|x+y|=1$ is (in sq. unit)

(a) $\frac{1}{4}$ (b) $\frac{2}{4}$ (c) $\frac{3}{4}$ (d) $\frac{4}{4}$.

13. A quadrilateral in the XY plane is formed by lattice vertex (x, y) such that $x^2+y^2=50$. The number of parallelograms in such quadrilaterals are

(a) 6 (b) 9

(c) 15 (d) None of these.

Note : Lattice vertex $(x, y) \Rightarrow x, y \in$ integers.

14. The curve represented by

$$\frac{x^2}{\sin\sqrt{2}-\sin\sqrt{3}}+\frac{y^2}{\cos\sqrt{2}-\cos\sqrt{3}}=1 \text{ is}$$

(a) an ellipse with foci on x-axis

(b) an ellipse with foci on y-axis

(c) a hyperbola with foci on x-axis

(d) a hyperbola with foci on y-axis.

15. For any complex numbers z_1 and z_2, which one of the following statements is false?

(a) $|z_1+z_2| \le |z_1|+|z_2|$ (b) $|z_1-z_2| \ge ||z_1|-|z_2||$

(c) $\left|\frac{z}{|z|}-1\right| \le (\arg z)$

(d) $|z-1| \ge ||z|-1|+|z|(\arg z)$

16. Points $A_1, A_2, A_3, \ldots$ are placed on a circle with centre O such that $\angle OA_nA_{n+1}=35°$ and $A_n \ne A_{n+2}$ $\forall$ positive integers n. The smallest $n(>1)$ for which $A_1=A_n$ is

(a) 35 (b) 36 (c) 37 (d) 38

17. A point in the xy plane is called lattice point if its co-ordinates are both integers. The maximum possible area of a circle in the plane that does not contain any lattice point in its interior is

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{8}$.

18. Three students A, B and C are in the same class of 50 students, but are not friends with each other. A is friend with 24 students, B is friend with 39 students and C is friend with 20 students. The greatest number of students that could be friends with all three of them are

(a) 18 (b) 19

(c) 20 (d) None of these.

19. A collection P of points (x, y) of the plane is said to be convex, if whenever two points $A(u, v)$ and $B(s, t)$ belong to P, every point on the line segment AB also belongs to P. Let P_1 be the collection of all points (x, y) for which $2<x^2+y^2<3$ and let P_2 be the collection of all points (x, y) for which x and y have same sign. Then

(a) P_1 is convex and P_2 is not convex

(b) P_2 is convex and P_1 is not convex

(c) Both P_1 and P_2 are convex

(d) Neither P_1 nor P_2 is convex.

20. A new sequence is obtained from the sequence of positive integers {1, 2, 3, ... } by deleting all perfect squares. Then the $(2003)^{rd}$ term is

(a) 2047 (b) 2048 (c) 2049 (d) 2050.

SHORT ANSWER TYPE TEST

1. Prove that $n^4+4^n, n \in N$ is never a prime number.

2. Suppose that the numbers $x_1, x_2, \ldots x_n$ all satisfy, $-1 \le x_i \le 1 (1 \le i \le n)$ and $\sum_{i=1}^{n} x_i^3 = 0$. Prove that $\sum_{i=1}^{n} x_i \le \frac{n}{3}$.

3. Let $\{x_n\}$ be a sequence satisfying the recurrence relation $x_{n+1}=\frac{\sqrt{3}x_n-1}{\sqrt{3}+x_n}$, $n \ge 1$. Prove that the sequence

is periodic and also find its period.

4. If the inequality $\sin^2 x + a\cos x + a^2 \geq 1 + \cos x$ holds for any $x \in R$, find the range of values for negative a.

5. A plane is roled with parallel straight lines at equal distances of $2a$. A needle $2l$ long $(l < a)$ is thrown on the plane at random. Find the probability that the needle will hit any of the lines.

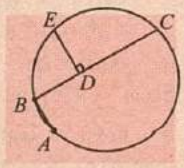
6. Find the sum of the last 100 digits of the number $A = 1\cdot 2\cdot 3\cdot 4 \,.....\, 2007\cdot 2008 + 2009$.

7. Let $a > 0$ and b, be real parameters, and suppose that f is a function, $R \to R$ and $f(a^3x^3 + 3a^2bx^2 + 3ab^2x) \leq x \leq a^3(f(x))^3 + 3a^2b(f(x))^2 + 3ab^2f(x)\ \forall\ x \in R$. Prove that f is a bijective function.

8. Let $A = \{x \mid x^2 - 4x + 3 < 0, x \in R\}$
$B = \{x \mid 2^{1-x} + a \leq 0, x^2 - 2x(a+7) + 5 \leq 0, x \in R\}$
If $A \subseteq B$, then find the range of real number a.

9. An elastic ball whose dimensions may be neglected is found inside a round billiard table at a point A different from centre O. Find the locus of points A from which this ball can be directed so that after two successive boundary reflections, by passing the centre of the table, it finds itself at the point A. Take R to be the radius of the table.

10. In the figure aside, E is the midpoint of the arc $ABEC$ and the segment ED is perpendicular to chord BC at D. If the length of the chord AB is l_1 and that of segment BD is l_2, determine the length of DC is terms of l_1 and l_2.



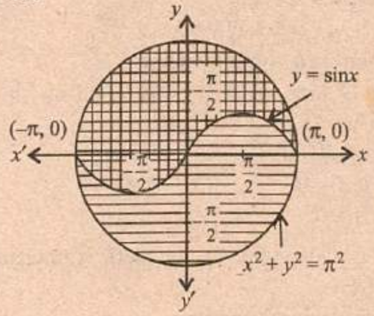
SOLUTIONS TO MULTIPLE CHOICE TYPE TEST

1. **(b) :** $x + 2y > 0, x - 2y > 0$
$\Rightarrow\ x > 2|y| \geq 0$
and $(x + 2y)(x - 2y) = 4 \ \Rightarrow\ x^2 - 4y^2 = 4$ ...(i)
Let $x - y = a$ or $x = a + y$
so, now (i) becomes $(a + y)^2 - 4y^2 = 4$
i.e., $3y^2 - 2ay + (4 - a^2) = 0$
for $y \in R, D \geq 0 \ \Rightarrow a \geq \sqrt{3}$
so, minimum value of $|x| - |y| = \sqrt{3}$.

2. **(b) :** $b = a^{3/2}, d = c^{5/4}$. Let $x^2 = a, y^4 = c$
$x, y \in R^+$
$a - c = x^2 - y^4 = (x - y^2)(x + y^2) = 9$
$\Rightarrow\ (x - y^2, x + y^2) = (1, 9)$
i.e., $x = 5, y = 2$
so, $b - d = x^3 - y^5 = 93$.

3. **(b) :** From the given conditions
$f(x) + 5 \leq f(x + 5) \leq f(x + 4) + 1 \leq f(x + 3) + 2$
$\leq f(x + 2) + 3 \leq f(x + 1) + 4 \leq f(x) + 5$
$\Rightarrow$ The equality holds for all the above relations.
i.e., $f(x + 1) = f(x) + 1$ and so $f(x) = x$.
$\therefore\ g(x) = f(x) + 1 - x = x + 1 - x = 1$
$\therefore\ g(x)$ is a constant function
$\therefore\ g(2009) = 1$.

4. **(a) :** $(x, y) : x^2 + y^2 \leq \pi^2$ represent all the points inside or on the circle, which is represented by horizontal lines $(x, y) : y \geq \sin x$ represent all the points which is lying on or above the curve $y = \sin x$, which is represented by vertical lines.



So common region which satisfies both the inequations is represented by vertical and horizontal lines.
$\therefore$ Required area
$$= \int_{-a}^{a} \sqrt{\pi^2 - x^2}\,dx - \int_{0}^{\pi} \sin x\,dx + \left| \int_{-\pi}^{0} \sin x\,dx \right| = \frac{\pi^2}{2}.$$
or symmetry of the curves, required area is half the circular area.

5. **(d) :** Let $y = \sqrt{x - 3} + \sqrt{6 - x},\ x \in [3, 6]$
$y^2 = (x - 3) + (6 - x) + 2\sqrt{(x - 3)(6 - x)}$
$y^2 \leq 2[(x - 3) + (6 - x)] = 6$
so, $0 < y \leq \sqrt{6}$ so, $k_{\text{max.}} = \sqrt{6}$.

6. **(a) :** Let $b_n = \dfrac{1}{a_n}$ then the given relation becomes
$$b_{n+1} + \frac{1}{3} = 2\left(b_n + \frac{1}{3}\right)$$
i.e., the sequence $\left\{b_n + \dfrac{1}{3}\right\}$ is a G.P. with common ratio 2.
$$\text{So, } b_n + \frac{1}{3} = 2^n\left(b_0 + \frac{1}{3}\right)$$
$$\Rightarrow\ b_n = \frac{1}{3}(2^{n+1} - 1)$$
$$\text{so, reqd. } \sum_{i=0}^{n} \frac{1}{a_i} = b_0 + b_1 + b_2 + \ldots + b_n$$
$$= \frac{1}{3}(2 - 1) + \frac{1}{3}(2^2 - 1) + \frac{1}{3}(2^3 - 1) + \ldots + \frac{1}{3}(2^{n+1} - 1)$$
$$= \frac{1}{3}\left[\frac{2(2^{n+1} - 1)}{2 - 1} - (n + 1)\right] = \frac{1}{3}(2^{n+2} - n - 3).$$

7. **(a) :**

8. **(a) :** Let $z = x + iy$, $|z| = x^2 + y^2$
so, $\sin|z| > 0$
$\Rightarrow\ 0 < |z| < \pi$ or $2\pi < |z| < 3\pi$, etc.

9. **(b) :** For any matrix M there exists a real number λ and a non-zero column vector X such that

$$(M - \lambda I)X = 0 \ \Rightarrow\ MX = \lambda IX = \lambda X \qquad ...(1)$$

Taking transpose of both sides of (1), we get

$$(MX)' = (\lambda X)' \Rightarrow X'M' = \lambda X' \qquad ...(2)$$

Multiplying (1) and (2), we get

$$X'M'MX = \lambda X'\lambda X$$

$$\Rightarrow\ X'IX = \lambda^2 X'X \Rightarrow X'X = \lambda^2 X'X$$

as X is a non zero column vector

$\therefore\ X'X \neq O \ \Rightarrow\ \lambda^2 = 1 \ \Rightarrow\ \lambda = 1$ or -1

$\therefore$ when $\lambda = 1$

$(M - \lambda I)X = 0$

$\Rightarrow\ (M - I)X = 0 \ \Rightarrow\ |(M - I)X| = 0$

$\Rightarrow\ |M - I| = 0.$

10. **(b) :** x^{1000} is even, so, its coeff. is unchanged if we replace x by $-x$
so, under this transformation
$(1 - x^2 + x^5)^{2005} \to (1 - x^2 - x^5)^{2005}$
$(1 + x^2 - x^5)^{2005} \to (1 + x^2 + x^5)^{2005}$ and
so, in the second series x^{1000} will have +ve coeff.

11. **(b) :**

12. **(c) :** The figure formed is a hexagon with sides

$$x + y = \frac{1}{2},\ x + y = \frac{-1}{2},\ x = \pm\frac{1}{2},\ y = \pm\frac{1}{2}.$$

13. **(a) :** All such parallelograms have two vertices with $y > 0$. There are 6 such points. So, there are ${}^6C_2 = 15$ ways to choose them. Each such choice, the other 2 points of the parallelogram are determined uniquely. Hence, the answer is 15.

14. **(b) :** Note that $\sin\sqrt{2} > \sin\sqrt{3}$ and $\cos\sqrt{2} > 0$, $\cos\sqrt{3} < 0$ and

$$(\sin\sqrt{2} - \sin\sqrt{3}) - (\cos\sqrt{2} - \cos\sqrt{3})$$

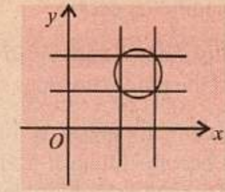
$$= 2\sqrt{2}\sin\left(\frac{\sqrt{2}-\sqrt{3}}{2}\right)\cdot\sin\left(\frac{\pi}{4} + \frac{\sqrt{2}+\sqrt{3}}{2}\right)$$

$$\frac{-\pi}{2} < \frac{\sqrt{2}-\sqrt{3}}{2} < 0 \ \text{and}\ \frac{\pi}{2} < \frac{\sqrt{2}+\sqrt{3}}{2} < \frac{3\pi}{4}.$$

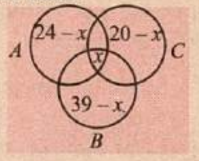
15. **(d) :**

16. **(c) :** Note that the minor arcs between A_n and A_{n+1} are all congruent with measure 110°. So, the solution is the smallest n for which $110°(n - 1)$ is a multiple of 2π. *i.e.*, 37.

17. **(b) :** A circle passing through lattice point vertices of a unit square will be the required area.



18. **(a) :** $x + (24 - x) + (20 - x) + (39 - x) = 47$, since they are 3 members of the class.



19. **(d) :**

20. **(b) :** Since
$[\sqrt{2046}] = [\sqrt{2047}] = [\sqrt{2048}] = [\sqrt{2049}] = 45$ and
$2003 + 45 = 2048.$

SOLUTIONS TO SHORT ANSWER TYPE QUESTIONS

1. If $n \in$ even, then
$n^4 + 4^n$ = even and > 2. So, never a prime.
If $n \in$ odd, let $n = 2m + 1$
then $n^4 + 4^n = (2m + 1)^4 + 4^{2m+1}$
$= (2m + 1)^4 + 4\cdot(2^m)^4$
$= a^4 + 4b^4$ where $a = 2m + 1$. $b = 2^m$
$= a^4 + 4b^4 + 4a^2b^2 - 4a^2b^2$
$= (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$ not a prime.
So, $n^4 + 4^n$ is never a prime, $n \in N$.

2. Let $x_i = \cos\theta_i$, $\theta_i \in [0, \pi]$
then ATQ, $\Sigma\cos^3\theta_i = 0$ and

$$\sum x_i = \sum\cos\theta_i = \frac{1}{3}[4\Sigma\cos^3\theta_i - \Sigma\cos 3\theta_i]$$

$$= \frac{-1}{3}\Sigma\cos 3\theta_i \leq \frac{n}{3}.$$

3. $x_{n+1} = \dfrac{x_n - \frac{1}{\sqrt{3}}}{1 + \frac{x_n}{\sqrt{3}}}$ let $x_n = \tan\alpha$

then $x_{n+1} = \tan\left(\alpha - \dfrac{\pi}{6}\right)$

$\Rightarrow\ \tan^{-1}x_{n+1} = \tan^{-1}x_n - \dfrac{\pi}{6}$

similarly $\tan^{-1}x_{n+2} = \tan^{-1}x_{n+1} - \dfrac{\pi}{6}$ etc.

so, adding such 6 recurrsive relations

$\tan^{-1}x_{n+6} = \tan^{-1}x_n - \pi$

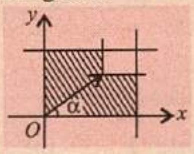
$\Rightarrow\ x_{n+6} = x_n$

Hence, $\{x_n\}$ is periodic with period = 6.

4. When $x = 0$, $a + a^2 \geq 2$

$\Rightarrow\ a \le -2$ (since $a < 0$)
when $a \le -2$, we have
$a^2 + a\cos x \ge a^2 + a \ge 2 \ge \cos^2 x + \cos x = 1 + \cos x - \sin^2 x$
i.e., $\sin^2 x + a^2 + a\cos x \ge 1 + \cos x$.
so, for negative a, $a \le -2$.

5. We consider the more general limiting case.
Needle = 1 unit, grid = 1 × 1



$$\text{Probability} = \frac{\text{Shaded area}}{\int_0^{\pi/2} d\alpha} = \frac{3}{\pi}$$

Shaded area $= 1 - (1 - \sin\alpha)(1 - \cos\alpha)$
where unshaded area $= (1 - \sin\alpha)(1 - \cos\alpha)$

6. The total number of 5's in 2008! are

$$\left[\frac{2008}{5}\right] + \left[\frac{2008}{5^2}\right] + \left[\frac{2008}{5^3}\right] + \left[\frac{2008}{5^4}\right] + \left[\frac{2008}{5^5}\right]$$
$$= 401 + 80 + 16 + 3 + 0 = 500$$

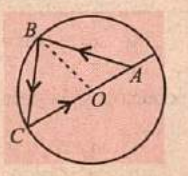
so, 2008! will have last 500 digits as 0 and hence the reqd. sum of 100 digits
$= 2 + 0 + 0 + 9 = 11$.

7. Let $g(x) = a^3x^3 + 3a^2bx^2 + 3ab^2x = (ax + b)^3 - b^3$.
so, g is a one-one function
let h be the inverse of g.
then $h(g(x)) = g(h(x)) = x,\ \forall\ x \in R$.
The given condition is
$f(g(x)) \le x \le g(f(x)),\ \forall x \in R$.
But $f(x) = f(g(h(x))) \le h(x)$ ...(1)
from the left inequality and
$x \le g(f(x))$
$\Rightarrow\ h(x) \le h(g(f(x)))$, since h is increasing
i.e., $h(x) \le f(x)$ ...(2)
so, from (1) and (2), $f(x) = h(x)$

and so, $f(x) = \dfrac{\sqrt[3]{x + b^3} - b}{a}$, which is clearly a bijective function.

8. Note that $A = (1, 3)$
Let $f(x) = 2^{1-x} + a$, $g(x) = x^2 - 2x(a + 7) + 5$
so, $f(x)$, $g(x)$ both are ≤ 0 for all $x \in (1, 3)$ when $f(1) \le 0$, $f(3) \le 0$, $g(1) \le 0$, $g(3) \le 0$.
Note that f is a monotonically decreasing function
so, solving these above four inequalities, we have
$-4 \le a \le -1$.

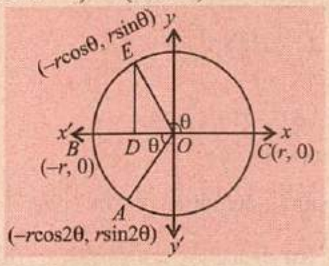
9. Note that C and A should lie on a diameter.
Let $OA = a$, $OB = OC = R$
let $\angle BCO = \theta$ then $\angle OBC = \theta$
so, $\angle ABO = \theta$ [angle of incidence = angle of reflection]
and $\angle AOB = 2\theta$.
Applying Sine rule to ΔAOB,



$$\frac{OA}{\sin\theta} = \frac{OB}{\sin(\pi - 3\theta)} \quad \Rightarrow \frac{a}{\sin\theta} = \frac{R}{\sin 3\theta}$$
or $\quad 4\sin^2\theta = 3 - \dfrac{R}{a} > 0 \quad \Rightarrow\ a > \dfrac{R}{3}$.

so, the locus is all points situated outside the circle of radius $R/3$ centred at the centre of the table.

10. Let radius of circle $= r$
$D \equiv (-r\cos\theta, 0)$
Let $CD = l_3 = r + r\cos\theta$ and $l_2 = r - r\cos\theta$
and $l_1 = (r\cos 2\theta - r)^2 + (r\sin 2\theta)^2$



Simplifying, $l_1 = 2r\sin\theta$
so, using the identity, $\sin^2\theta + \cos^2\theta = 1$

we have, $\left(\dfrac{l_1}{2r}\right)^2 + \left(\dfrac{l_2 - r}{-r}\right)^2 = 1$

i.e., $r = \dfrac{l_1^2 + 4l_2^2}{8l_2}$

so, $l_3 = 2r - l_2 = \dfrac{2(l_1^2 + 4l_2^2)}{8l_2} - l_2$ *i.e.*, $l_3 = \dfrac{l_1^2}{4l_2}$.

DIFFERENTIAL CALCULUS FOR STATES' ENGG. ENTRANCE EXAMS

1. The domain of definition of $f(x)=\sqrt{\log_e(x^2-6x+6)}$ is

(a) $(-\infty,3-\sqrt{3})\cup[3+\sqrt{3},\infty)$
(b) $(-\infty,1]\cup[5,\infty)$
(c) $(-\infty,3-\sqrt{3})\cup(3+\sqrt{3},\infty)$
(d) $(-\infty,1)\cup(5,\infty)$

2. The domain of definition of $\sin^{-1}\left[\log_3\left(\frac{x}{3}\right)\right]$ is

(a) $[-9,-1]$ (b) $[-1,9]$ (c) $[1,9]$ (d) $[-9,1]$

3. The function $f(x)=\sin\left[\log(x+\sqrt{1+x^2})\right]$ is

(a) even (b) odd
(c) neither even nor odd (d) none of these.

4. If $f(x)=\cos[\pi^2]x+\cos[-\pi^2]x$, then $f\left(\frac{\pi}{2}\right)$ equals

(a) 1 (b) 2 (c) -1 (d) 0

5. If $f(a+b,a-b)=ab$, then A.M. of $f(a,b)$ and $f(b,a)$ is

(a) 0 (b) 1 (c) a (d) b

6. Let $f(x)=a^x$ $(a>0)$ be written as $f(x)=g(x)+h(x)$ where $g(x)$ is an even function and $h(x)$ is an odd function. Then $g(x+y)+g(x-y)$ equals

(a) $g(x)\,h(x)$ (b) $2g(x)$
(c) $g(x)+g(y)$ (d) $2g(x+y)g(x-y)$

7. If $f(x+y)=f(x)+f(y)$ and $f(1)=7$, then the value of $\sum_{r=1}^{n}f(r)$ is

(a) $\frac{7n}{2}$ (b) $\frac{7n(n+1)}{2}$
(c) $\frac{7(n+1)}{2}$ (d) $7n\,(n+1)$

8. Let $f(x)$ be a polynomial of degree 2 & if $f(1)=f(-1)$ and l, m, n are in A.P, then $f'(l), f'(m), f'(n)$ are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these.

9. The range of $f(x)=\frac{x^2+x+2}{x^2+x+1}$ is

(a) $[1,\infty)$ (b) $[1, 7/5]$ (c) $(1,\infty)$ (d) $(1, 7/3]$

10. If $f(2)=4$ and $f'(2)=4$ then $\lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2}$ equal

(a) -2 (b) 2 (c) 3 (d) -4

11. If $f(x)=\cot^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ and $g(x)=\cot^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, then $\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$ equals

(a) $-\frac{3}{2}$ (b) $\frac{3}{2}$
(c) $\frac{3}{2(1+a^2)}$ (d) $-\frac{3}{2(1+a^2)}$

12. If $l(x)$ is the least integer not less than x and $g(x)$ is the greatest integer not greater than x, then $\lim_{x\to\pi+e}[l(x)+g(x)]$ equals

(a) 11 (b) 10 (c) 9 (d) 12

13. Which of the following statements is false?

(a) a polynomial function is always continuous
(b) a differentiable function is always continuous
(c) a continuous function is always differentiable
(d) $\log_e x$ is continuous for all $x>0$.

14. The function $f(x)=a[x+1]+b[x-1]$, where [.] is the greatest integer function, is continuous at $x=1$ if

(a) $a=0$ (b) $b=0$ (c) $a+b=0$ (d) $a-b=0$

15. If $f(x)=x\sin l/x$, when $x\neq 0$ and 0 when $x=0$, then

(a) $f(x)$ is differentiable but not continuous at $x=0$
(b) continuous and differentiable at $x=0$
(c) not continuous at $x=0$
(d) continuous but not differentiable at $x=0$

16. $f(x)=e^{2x}-1$ when $x\le 0$ and $ax+bx^2/2$ when $x>0$ is continuous and differentiable for

(a) $a=1, b=2$ (b) $a=2, b\in R$
(c) $a=2, b=4$ (d) $a\in R, b=4$

17. If $f(x)=\frac{(4^x-1)^3}{\sin\frac{x}{a}\log\left(1+\frac{x^2}{3}\right)}$ when $x\neq 0$ and $9(\log_e 4)^3$ when $x=0$ is continuous at $x=0$, then a equals

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(a) 1 (b) 2 (c) 4 (d) 3

18. If $f(x) = ax^2 + 1$ when $x \le 1$ and $x + a$ when $x > 1$, also $f(x)$ is differentiable at $x = 1$, then a equals

(a) 1/2 (b) –1/2 (c) 2 (d) –2

19. If $g(x) = \dfrac{\sin 5x}{x^2 + 2x}$ when $x \ne 0$ and $k + \dfrac{1}{2}$ when $x = 0$, also $g(x)$ is continuous at $x = 0$, then k equals

(a) 3/2 (b) –2 (c) 1 (d) 2

20. If f and g are continuous and real valued function on an interval, then

(i) $f + g$ is continuous in the interval

(ii) fg is continuous in the interval

(iii) f/g is continuous in the interval

(a) (i), (ii), (iii) are correct (b) (i) & (ii) are correct

(c) (i) & (iii) are correct (d) (ii) & (iii) are correct

21. The differential coefficient of the function $f(x) = |x - 1| + |x - 3|$ at the point $x = 2$ is

(a) –2 (b) 2

(c) 0 (d) does not exist.

22. If $f(x) = |x - 3| + |x - 4|$, then in $0 \le x \le 5, f(x)$ is

(a) differentiable at $x = 3$ (b) differentiable at $x = 4$

(c) not continuous in $0 \le x \le 5$

(d) not differentiable at $x = 3$ & 4

23. Let $f(x)$ be a differentiable function and $f'(1) = 4$ $f'(2) = 6$, then $\lim\limits_{h \to 0} \dfrac{f(2 + 2h + h^2) - f(2)}{f(1 + h - h^2) - f(1)}$ equals

(a) 3/2 (b) 3

(c) –3 (d) does not exist

24. If $f(x + y) = f(x)\, f(y)$ & $f(x) \ne 0$. If $f(x)$ is differentiable at $x = 0$ & $f'(0) = 2$, then $f'(x)$ equals

(a) $f(x)$ (b) $2f(x)$ (c) $-f(x)$ (d) $1/2\, f(x)$

25. If $f(x)$ is differentiable at $x = a$, then

$\lim\limits_{x \to a} \dfrac{(x + a) f(x) - 2a\, f(a)}{x - a}$ equals

(a) $f(a) + af'(a)$ (b) $2af'(a)$

(c) $f(a) + 2af'(a)$ (d) $2f(a) + f'(a)$

26. If $y = \cot^{-1} \dfrac{b - ax}{a + bx}$, then $\dfrac{dy}{dx}$ equals

(a) 1 (b) $\dfrac{1}{1 + x^2}$ (c) –1 (d) $-\dfrac{1}{1 + x^2}$

27. If $f(x)$ is differentiable and $f'(4) = 5$, then

$\operatorname{lt}\limits_{x \to 2} \dfrac{f(4) - f(x^2)}{x - 2}$ equals

(a) 20 (b) 10 (c) –20 (d) –10

28. If $f(x + y + z) = f(x)\, f(y)\, f(z) \ne 0$ and $f(2) = 5$, $f'(0) = 2$, then $f'(2)$ equals

(a) 0 (b) ± 2 (c) ± 5 (d) ± 10

29. $f(x) = xe^{-\left[\frac{1}{|x|} + \frac{1}{x}\right]}$ when $x \ne 0$ and 0, when $x = 0$, then $f(x)$ is

(a) discontinuous every where

(b) continuous & differentiable at $x = 0$

(c) continuous for all x but not differentiable at $x = 0$

(d) neither continuous nor differentiable at $x = 0$

30. The value of $\lim\limits_{x \to \tan^{-1} 3} \dfrac{\tan^2 x - 2\tan x - 3}{\tan^2 x - 4\tan x + 3}$ equals

(a) 0 (b) 1 (c) 1/2 (d) 2.

31. If $x = e^{\tan^{-1}\left(\frac{y - x^2}{x^2}\right)}$, then $\dfrac{dy}{dx}$ equals

(a) $2x\,[1 + \tan(\log x)] + x\sec^2(\log x)$

(b) $x\,[1 + \tan(\log x)] + \sec^2(\log x)$

(c) $2x\,[1 + \tan(\log x)] + x^2\sec^2(\log x)$

(d) $2x\,[1 + \tan(\log x)] + \sec^2(\log x)$

32. If $y = \sqrt{\sin\sqrt{x}}$, then $\dfrac{dy}{dx}$ equals

(a) $\dfrac{1}{2\sqrt{\sin\sqrt{x}}}$ (b) $\dfrac{\sqrt{\cos\sqrt{x}}}{2x}$

(c) $\dfrac{1}{2\sqrt{\cos\sqrt{x}}}$ (d) $\dfrac{\cos\sqrt{x}}{4\sqrt{x}\sqrt{\sin\sqrt{x}}}$

33. If $f(x) = \cos^{-1}\left[\dfrac{1 - (\log x)^2}{1 + (\log x)^2}\right]$, then $f'(e)$ equals

(a) $\dfrac{2}{e}$ (b) $\dfrac{1}{e}$ (c) 1 (d) $\dfrac{1}{e^2}$

34. If $\sin y + e^{-x\cos y} = e$, then $\dfrac{dy}{dx}$ at $(1, \pi)$ is

(a) 0 (b) 1 (c) e (d) –1

35. If $f(x)$ is differentiable even function, then

(a) $f'(x)$ is an even function

(b) $f'(x)$ is an odd function

(c) $f'(x)$ may be even or odd

(d) $f'(x)$ is neither odd nor even.

36. If $y = x + x^2 + x^3 + \ldots$ to ∞, where $|x| < 1$, then for $|y| < 1$ the value of $\dfrac{dx}{dy}$ equals

(a) $1 - 2y + 3y^2 - \ldots$ to ∞ (b) $y + y^2 + y^3 + \ldots$ to ∞

(c) $1 - y + y^2 - y^3 + \ldots$ to ∞ (d) $1 + 2y + 3y^2 + \ldots$ to ∞

37. If $2y = (x - a)\sqrt{2ax - x^2} + a^2 \sin^{-1} \dfrac{x - a}{a}$, then $\dfrac{dy}{dx}$ equals

(a) $\sqrt{ax - x^2}$ (b) $\sqrt{x^2 - ax}$

(c) $\sqrt{x^2 - 2ax}$ (d) $\sqrt{2ax - x^2}$

38. If $y=\log_a x+\log_x a+\log_x x+\log_a a$, then $\frac{dy}{dx}$ equals

(a) $\frac{\log a}{x}+\frac{x}{\log a}$ (b) $\frac{1}{x\log a}-\frac{\log a}{x(\log x)^2}$

(c) $\frac{1}{x\log a}$ (d) $\frac{1}{x}+x\log a$

39. If $y=\sin\left(\frac{\pi}{6}e^{xy}\right)$, then $\frac{dy}{dx}$ at $x=0$ is

(a) $\frac{\sqrt{3}}{24}$ (b) $\frac{\sqrt{3}\pi}{24}$ (c) $\frac{\sqrt{3}}{12}$ (d) $\frac{\sqrt{3}\pi}{12}$

40. If $x\sqrt{1+y}+y\sqrt{1+x}=0$, then $\frac{dy}{dx}$ equals

(a) $\frac{1}{1+x^2}$ (b) $-\frac{1}{1+x^2}$

(c) $\frac{1}{(1+x)^2}$ (d) $-\frac{1}{(1+x)^2}$

41. If $|x-1|+|x-2|+|x-3|\geq 6$, then

(a) $x\geq 0$ and $x\geq 4$ (b) $0\leq x\leq 4$

(c) $x\leq -2$ and $x\geq 4$ (d) none of these.

42. Which of the following statements is true?

(a) $|x-y|=|x|-|y|$ (b) $|x-y|\geq|x|-|y|$

(c) $|x+y|\leq|x|-|y|$ (d) none of these

43. If $|x-a|\leq b$, then

(a) $(x-a)\leq b$ (b) $a-b\leq x\leq a+b$

(c) $x\leq a-b$ & $a+b\leq x$ (d) none of these.

44. If $\frac{dx}{dy}=u$ and $\frac{d^2x}{dy^2}=v$, then $\frac{d^2y}{dx^2}$ equals

(a) $-\frac{v}{u^2}$ (b) $\frac{v}{u^2}$ (c) $-\frac{v}{u^3}$ (d) $\frac{v}{u^3}$

45. If $\log x=z$, then $x^2\frac{d^2y}{dx^2}$ equals

(a) $\frac{d^2y}{dz^2}$ (b) $\frac{d^2y}{dz^2}+\frac{dy}{dz}$

(c) $\frac{d^2y}{dz^2}-\frac{dy}{dz}$ (d) $\frac{d^2y}{dz^2}-2\frac{dy}{dz}$

46. If $x=\frac{1}{z}$, $y=f(x)$ and $\frac{d^2y}{dx^2}=kz^3\frac{dy}{dz}+z^4\frac{d^2y}{dz^2}$, then k equals

(a) -1 (b) 1 (c) 2 (d) -2

47. If $xy=ax^2+\frac{b}{x}$, then elimination of a & b gives

(a) $x^2y_2+2xy_1-y=0$ (b) $x^2y_2+2xy_1+y=0$

(c) $x^2y_2+2xy_1+2y=0$ (d) $x^2y_2+2xy_1-2y=0$

48. If $x=\sec\theta-\cos\theta$, $y=\sec^n\theta-\cos^n\theta$ and $(x^2+4)\left(\frac{dy}{dx}\right)^2=k(y^2+4)$, then k equals

(a) n^2 (b) $2n$ (c) $-n^2$ (d) $-2n$

49. If $y=(\sin^{-1}x)^2+(\cos^{-1}x)^2$, then

(a) $(1-x^2)y_2+xy_1+4=0$ (b) $(1+x^2)y_2-xy_1-4=0$

(c) $(1-x^2)y_2-xy_1-4=0$ (d) $(1-x^2)y_2-xy_1+4=0$

50. When the tangent to the curve $y=x\log x$ is parallel to the chord joining the points $(1, 0)$ and (e, e), the abscissa is

(a) $e^{\frac{1}{e-1}}$ (b) $e^{\frac{e-1}{2e-1}}$ (c) $e^{\frac{2e-1}{e-1}}$ (d) $\frac{e-1}{e}$

51. If $2a+3b+6c=0$, then the equation $ax^2+bx+c=0$ having at least one root in the interval

(a) $(0, 1)$ (b) $(1, 2)$

(c) $(2, 3)$ (d) none of these.

52. In the interval $[1, 3]$, the function $f(x)=x^3-6x^2+ax+b$ satisfies the rolle's theorem at $c=2+\frac{1}{\sqrt{3}}$, then

(a) $a=11, b=6$ (b) $a=11, b\in R$

(c) $a=-11, b=6$ (d) none of these.

53. If $f(x)=\sqrt{x}$ and $g(x)=\frac{1}{\sqrt{x}}$, then c in MVT is

(a) A.M. between a & b (b) G.M. between a & b

(c) H.M. between a & b (d) none of these.

54. If $a+b+c=0$, then the equation $3ax^2+2bx+c=0$

(a) having at least one real root in $(0, 1)$

(b) one root in $(-1, 0)$ and another in $(1, 2)$

(c) both roots are imaginary

(d) roots are equal.

55. If $f(x)=Ax^2+Bx+c$ at $c\in(a, b)$ satisfies the lagrange's MVT,then

(a) c divideds the interval (a, b) in the ratio 2 :1

(b) c divides the interval (a, b) in the ratio 1 : 2

(c) c is the mid point of (a, b)

(d) none of these.

56. The value of $\lim_{x\to 0}\frac{\int_0^{x^2}\sec^2 t\,dt}{x\sin x}$ equals

(a) 1 (b) 2 (c) 3 (d) 0

57. The value of $\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$ equals

(a) 2 (b) -2 (c) 1 (d) -1

58. If $\lim_{x\to 0}\frac{\sin 2x+k\sin x}{x^3}$ is finite, then k equals

(a) 1 (b) -1 (c) 2 (d) -2

59. The integer n for which $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite non zero number, is

(a) 1 (b) 2 (c) 3 (d) 4

60. If $\lim_{x\to\infty}\left[\frac{x^3+1}{x^2+1}-(ax+b)\right]=2$, then the values of a

& b are

(a) $1, -2$ (b) $1, 1$ (c) $1, -1$ (d) $1, 2$

61. If $f(x) = kx^3 - 9x^2 + 9x + 3$ is an increasing function, then

(a) $k < 3$ (b) $k \le 3$

(c) $k > 3$ (d) k is indeterminate

62. Let $f(x) = x^3 + 6x^2 + px + 2$; if the largest possible interval in which $f(x)$ is a decreasing function is $(-3, -1)$ then P equals

(a) 3 (b) 9

(c) -2 (d) none of these.

63. The length of the longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing is

(a) $\pi/2$ (b) π (c) $3\pi/2$ (d) $\pi/3$

64. The tangent to the graph of the function $y = f(x)$ makes angles 45° and 60° with the x axis at $x = 2$ and $x = 4$ respectively, then $\int_2^4 f'(x)f''(x)\,dx$ equals

(a) $f(4)$ (b) $f(2)$ (c) 0 (d) 1

65. The rate of change of surface area of a sphere of radius r when the radius is increasing at the rate of 2cm/sec is proportional to

(a) $1/r^2$ (b) r^2 (c) r (d) $1/r$

66. If minimum value of $x^2 + 2bx + 2c^2$ is greater then maximum value of $g(x) = -x^2 - 2cx + b^2$, then

(a) $\sqrt{2}|c| > |b|$ (b) $|c| > \sqrt{2}|b|$

(c) $0 < c < 2b$ (d) none of these.

67. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is

(a) $\left(-\frac{9}{8}, \frac{9}{2}\right)$ (b) $(2, -6)$ (c) $(2, 6)$ (d) $\left(\frac{9}{8}, \frac{9}{2}\right)$

68. If PQ and PR are the two sides of a triangle, then the angle between them which gives maximum area of triangle is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

69. Let a and b are the roots of the equation $x^2 + (3 - k)x - k = 0$; then the value of k for which $a^2 + b^2$ is minimum, is

(a) 0 (b) 1 (c) 3 (d) 2.

70. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$ has

(a) no maxima or minima

(b) one maximum & one minimum

(c) two maxima (d) two minima.

71. The maximum distance from the origin of a point on the curve $x = a\sin t - b\sin\left(\frac{at}{b}\right)$, $y = a\cos t - b\cos\left(\frac{at}{b}\right)$, then $a, b > 0$ is

(a) $a - b$ (b) $a + b$

(c) $\sqrt{a^2 + b^2}$ (d) $\sqrt{a^2 - b^2}$

72. The points of extrema of $f(x) = \int_0^x \frac{\sin t}{t}$ in the domain of $x > 0$, are

(a) $(2n+1)\frac{\pi}{2}$ (b) $n\pi$

(c) $(4n+1)\frac{\pi}{2}$ (d) $(2n+1)\frac{\pi}{4}$

73. The function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ where $a > 0$ attains its maximum & minimum at $x = p$ & $x = q$ respectively such that $p^2 = q$, then a equals

(a) 1/2 (b) 3 (c) 1 (d) 2

74. If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2}\,dt$, then the interval in which $f(x)$ increasing is

(a) $(-\infty, 0)$ (b) $(0, \infty)$ (c) $[-2, 2]$ (d) $[3, 5]$

75. Let $P(a \sec\theta, b \tan\theta)$ & $Q(a \sec\phi, b \tan\phi)$ where $\theta + \phi = \pi/2$ be two points on the hyperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If (h, k) be the point of intersection of the normals at P & Q, then K equals

(a) $\frac{a^2+b^2}{a}$ (b) $-\frac{a^2+b^2}{a}$

(c) $\frac{a^2+b^2}{b}$ (d) $-\frac{a^2+b^2}{b}$

76. The curves $y^2 = 4x$ & $xy = k$ cut orthogonally, then the value of k^2 will be

(a) 16 (b) 32 (c) 36 (d) 8

77. If the st. line $y = 4x - 5$ touches the curve $y^2 = ax^3 + b$ at $(2, 3)$, then

(a) $a = 2, b = -7$ (b) $a = 2, b = 7$

(c) $a = -2, b = -7$ (d) $a = -2, b = 7$

78. The equation of tangent to curve $xy^2 = 4(4 - x)$ where it meets the line $y = x$ is

(a) $x + y + 4 = 0$ (b) $x + y = 4$

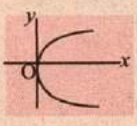
(c) $x - y = 2$ (d) $x - y + 2 = 0$

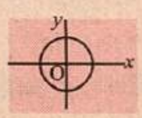
79. The locus of the point of intersection of two perpendicular tangents to the ellipse is

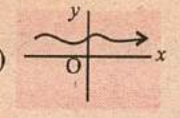
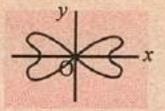
(a) straight line (b) circle

(c) parabola (d) none of these.

80. Which of the following graphs is a function?

(a) 

(b) 

(c)  (d) 

ANSWER

1. (b) :
2. (c) : Hints : The function will be defined when $-1 \le \log_3\left(\frac{x}{3}\right) \le 1$ & $\frac{x}{3} > 0$

$3^{-1} \le \frac{x}{3} \le 3^1$ & $x > 0 \quad \Rightarrow 1 \le x \le 9$

3. (b) :
4. (c) : Hints : $[\pi^2] = 9$ & $[-\pi^2] = -10$
5. (a) : Hints :- Let $m = a + b$ & $n = a - b$

$\therefore a = \frac{m+n}{2}, b = \frac{m-n}{2}$

$\therefore f(m,n) = \frac{1}{4}(m^2 - n^2)$ (Proceed)

6. (c) : Hints :- $f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{(even part)}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{(odd part)}}$

$\therefore g(x) = \frac{1}{2}[f(x)+f(-x)] = \frac{1}{2}[a^x + a^{-x}]$

7. (b) : Hints :- Putting $x = 1, y = 1, f(2) = 2\cdot f(1) = 2\cdot 7$ Putting $x = 2, y = 1, f(3) = 3\cdot 7, f(4) = 4\cdot 7, f(5) = 5\cdot 7$ $\therefore f(n) = n\cdot 7$
8. (a) : Hints : - Let $f(x) = ax^2 + bx + c$
9. (d) 10. (d)
11. (a) : Hints : Put $x = \tan\theta$
12. (a) : $\pi + e = 3.14..... + 2.718 ..... = 5.859.....$

$\therefore \lim\limits_{x\to\pi+e} l(x) + \lim\limits_{x\to\pi+e} g(x) = 6 + 5 = 11$

13. (c) 14. (c) 15. (d) 16. (b)
17. (d) 18. (a) 19. (d) 20. (b)
21. (c) : Hints : case – I $f(x) = -(x-1)-(x-3) = -2x+4$ when $x < 1$. case – II $f(x) = (x-1)-(x-3) = 2$ when $1 < x < 3$. case – III $f(x) = x-1+x-3 = 2x-4$ when $x > 3$.
22. (d) : Hints : The function $|x - x_1| + |x - x_2| + |x - x_3| +...... + |x - x_n|$ is continuous at each of $x_1, x_2, ... x_n$ but not differentiable at $x_1, x_2, ... x_n$.
23. (b) 24. (b) 25. (c) 26. (b)
27. (c)
28. (d) : Hints : Put $x = y = z = 0$ $\therefore f(0) = [f(0)]^3$ or $f(0) = \pm 1$. Again put $x = 2, y = h$ & $z = 0$ $\therefore f(2 + h) = 5f(h)\ (\pm 1)$
29. (c) 30. (d) 31. (a) 32. (d)
33. (b) 34. (c) 35. (c) 36. (a)
37. (d) 38. (b) 39. (b) 40. (d)
41. (a) : Hints : Proceed like (21)
42. (b) 43. (b) 44. (c) 45. (c)
46. (c) 47. (d) 48. (a) 49. (c)
50. (a)
51. (a) : Hints : Let $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$, then $f(x)$ is continuous & differentiable for all x and $f(0) = f(1) = 0$

$\therefore$ By Rolle's theorem $f'(c_1) = 0$ or $ac_1^2 + bc_1 + c = 0$. By Algebraic interpretation of Rolle's theorem we know $f'(x) = 0$ has at least one root between the roots of $f(x) = 0$

52. (b)
53. (b) : Hints : By lagrange's MVT $(b-a)f'(c) = f(b) - f(a)$ and $(b-a)g'(c) = g(b) - g(a)$. Now divide the relations.
54. (a) 55. (c)
56. (a) : Hints : Newton leibnitz formula

$\frac{d}{dx}\left[\int_{\theta(x)}^{\phi(x)} f(t)dt\right] = f\{\phi(x)\}\ \phi'(x) - f\{\theta(x)\}\ \theta'(x)$. Given limit is 0/0 form $\therefore \lim\limits_{x\to 0}\frac{\sec^2(x^2)2x}{x\cos x + \sin x}$ [By L' Hospital's rule]

57. (a) 58. (d) 59. (c) 60. (a)
61. (c) 62. (b) 63. (d)
64. (d) : Hints : $f'(2) = \tan\frac{\pi}{4} = 1$ and $f'(4) = \tan\frac{\pi}{3} = \sqrt{3}$

$$\int f'(x)f''(x)dx = \int f'(x)d\,[f'(x)] = \frac{1}{2}[f'(x)]^2$$

65. (c) 66. (b) 67. (d) 68. (c)
69. (d) 70. (b) 71. (b)
72. (b) : Hints : Proceed like 56.
73. (d)
74. (a) : Hints : Proceed like 56.
75. (d)
76. (c) : Hints: Let $c_1 = y^2 = 4x$ & $c_2 = xy = k$ then $\left[\frac{dy}{dx}\right]_{c_1} \times \left[\frac{dy}{dx}\right]_{c_2} = -1$
77. (a) 78. (b)
79. (b) : Hints : The equation of the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $y = mx \pm \sqrt{a^2m^2 + b^2}$.

Let they meet at (x_1, y_1)

$\therefore (y_1 - mx_1)^2 = a^2m^2 + b^2$.

or $m^2(x_1^2 - a^2) - 2mx_1y_1 + y_1^2 - b^2 = 0$, now $m_1 \times m_2 = -1$,

$\frac{y_1^2 - b^2}{x_1^2 - a^2} = -1 \therefore$ locus of (x_1, y_1) is $x_1^2 + y_1^2 = a^2 + b^2$

i.e., $x^2 + y^2 = a^2 + b^2$ which is a circle.

80. (c)

Understanding concepts

Limits and Differential Equations

Learn fast will aid students in quick understanding of concepts on the above topics.

LIMITS

What is limits ?

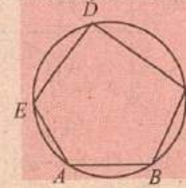
Let us first try to understand what is limit? May be the given examples are best to illustrate limits. Assume you are travelling from point A to point B while passing through point C. The average speed from A to B can be calculated by simply taking the ratio between the distance from A to B and total time taken to travel from point A to B. But this has no physical meaning. Indeed, let us suppose that a policeman is standing at point C checking for speeders through C. Then the policeman does not care about the average speed. He is only interested about the speed that you see on the speedometer, the one that the car actually has when crossing C. Now the question arose "*How to compute this instantaneous speed*? But this is not easy at all. The one natural way to do this is to compute the average speed from C to points close to C. In this case the distance as well as the time from these points to C is very small. And hence the average speed is calculated using these small values. These average speeds over small distances get close to a certain value and that value should be called the instantaneous speed at C.

- Now let us express this more mathematically.

If $f(t)$ is a function that determines the position of the moving object, and assume that at time t_0, the moving object is at C. At $t_0 + \Delta t$ we are at some point close to C. Then the average speed between these two points is

$$\frac{f(t_0+\Delta t)-f(t_0)}{\Delta t}$$

- Then we study these numbers when Δt gets smaller and smaller. This is exactly the idea behind limits.
- We will write $\lim_{\Delta t\to 0}\frac{f(t_0+\Delta t)-f(t_0)}{\Delta t}$ to indicate the instantaneous speed at C.
- Consider another example. Let a regular polygon be inscribed in a circle of given radius. We may notice the following points.



(i) The area of the polygon cannot be greater than the area of the circle however large the number of sides may be.

(ii) As the number of sides of the polygon increases indefinitely, the area of the polygon continually approaches the area of the circle.

(iii) Finally the difference between the area of the circle and the area of the polygon can be made as small as possible by sufficiently increasing the number of sides of the polygon.

- This is expressed by saying that the limits of the area of the polygon inscribed in a circle as the number of sides increases indefinitely, is the area of the circle.
- **Meaning of $x \to a$:** Let x be a variable and a be a constant. Since x is a variable hence it can be changed to assume any value. It can be so changed that its value come nearer and nearer to a. Then we say that x approaches a and express it by the notation $x \to a$. More mathematically,

Given a number $\delta > 0$ however small, if x takes up values such that $0 < |x-a| < \delta$, then x is said to tend to a and is symbolically written as $x \to a$.

- **Limit of a function:** Let $f(x)$ be any function of x. If x approaches a fixed a, $f(x)$ approaches a fixed and finite quantity l then limit of $f(x)$ is said to exist at $x = a$ and is equal to l.
- Symbolically $\lim_{x\to a} f(x) = l$ $= l$ or $f(x) \to l$ as $x \to a$.
- The formal rigorous definition of limit of a function is given as:

Let $f(x)$ be a function of x. If for every positive number $\varepsilon > 0$, however small it may be, there exists a positive number δ such that whenever $0 < |x-a| < \delta$ we have $|f(x) - l| < \varepsilon$, then we say $f(x)$ tends to the limit l as 'x tend to a' and write $\lim_{x\to a} f(x) = l$.

Right hand and left hand limits :

- Sometimes the function is not defined around the point a but only to the left or right of a. Then we have concept of left hand limit and right hand limit at a.
- **Right hand limit :** If $f(x)$ approaches l_1 when x approaches a from right side of a, i.e. $x > a$ then l_1 is called right hand limit of $f(x)$ at $x = a$.

Symbolically $\lim_{x\to a^+} f(x) = l_1 = \lim_{h\to 0} f(a+h)$ where $h > 0$.

- **Left hand limit :** If $f(x)$ approaches l_2 when x approaches a from left sides of a *i.e.* $x < a$, then l_2 is called left hand limit of $f(x)$ at $x = a$, symbolically $\lim_{x\to a^-} f(x) = l_2 = \lim_{h\to 0} f(a-h)$ where $h > 0$.

- **Note:** A function $f(x)$ is said to posses the limiting value only if its right hand limit equals left hand limit.
- **Indeterminate forms:** Now the question arises why to calculate $\lim_{x \to a} f(x)$ but not $f(a)$. For answering this let us consider one example.

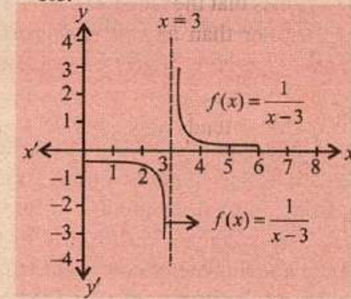
If we assume $f(x) = \frac{1-x^2}{1-x}$ then $f(1) = \frac{0}{0}$ (not defined).
Hence the value of $f(x)$ at $x = 1$ cannot be determined.

- Although we may not be able to report a exact value but we can report a limiting value of $f(x)$ at $x = 1$ although $f(1)$ theoretically is indeterminate. Not only 0/0, there are several other forms which are known to be indeterminate like (i) 0/0, (ii) ∞/∞ (iii) $\infty - \infty$ (iv) $\infty \times 0$ (v) ∞^0 (vi) 0^0 (vii) x^∞ where $x \to 1$.
- In all these, only the limiting value can be determined. To calculate the limiting value for such cases in general a rule is used known as *L'* Hospital rule which states that "If $f(x)$ and $g(x)$ are both zero at some point x_0, and f and g are both differentiable at every point except possibly x_0 of an open interval (a, b) that contains x_0 and $g'(x_0) \neq 0$; $x \in (a, x_0) \cup (x_0, b)$ then
- The rule holds good if both $f(x)$ and $g(x)$ tends to ∞ as $x \to x_0$.
- Although this rule is not directly applicable to other indeterminate forms, it can be applied after reducing the given form either to 0/0 or to ∞/∞.

Some basic properties of limits :

1. To start with various properties of the limit let us note that the limit, when it exist, is unique.
i.e. if $\lim_{x \to a} f(x) = K$ and also $\lim_{x \to a} f(x) = L$, then $K = L$.
2. The limit of the sum of two functions is equal to the sum of their individual limits.
i.e. $\lim_{x \to a}(f+g)x = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.
3. Similarly the limit of the difference of two functions is equal to the difference of their limits.
i.e. $\lim_{x \to a}(f-g)x = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$.
4. The limit of the product of two functions is equal to the product of their limits.
i.e. $\lim_{x \to a}(fg)x = \lim_{x \to a} f(x) . \lim_{x \to a} g(x)$.
5. The limit of quotient of two functions is equal to the quotient of their limits provided the limit of the divisor is not zero.
i.e. $\lim_{x \to a}(f \div g)x = \lim_{x \to a} f(x) / \lim_{x \to a} g(x)$, where $\lim_{x \to a} g(x) \neq 0$.
6. **The pinching or Sandwich theorem:** Assume that $h(x) \leq f(x) \leq g(x)$ for any interval around the point a. If $\lim_{x \to a} h(x) = l$ and $\lim_{x \to a} g(x) = l$ then $\lim_{x \to a} f(x) = l$.

- **Limits and infinity:** Concept of infinity is one of the mysteries of mathematics. So what is ∞? It is simply a symbol that represents large numbers. Do not treat $\pm\infty$ as ordinary numbers. These symbols do not obey the usual rules of arithmetic for instance $\infty + 1 = \infty$, $\infty - 1 = \infty$, $2 . \infty = \infty$ etc.

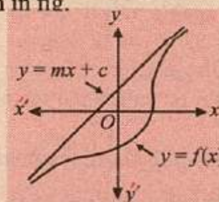


- Consider the function $f(x) = \frac{1}{x-3}$ then as $x \to 3$, $x - 3 \to 0$ as shown in fig.

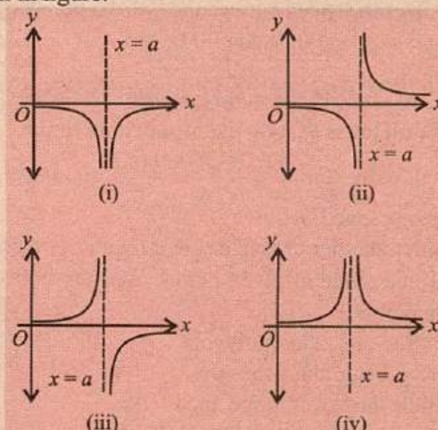
$\lim_{x \to 3^-} f(x) = \frac{1}{0^-} = -\infty$ and $\lim_{x \to 3^+} f(x) = \frac{1}{0^+} = +\infty$.

- Note that when x get closer to 3, then the points on the graph get closer to the vertical line $x = 3$. Such a line is called a vertical asymptotes.

A line $y = mx + c$ is said to be an asymptote of the curve $y = f(x)$ if it touches the given curve either at ∞ or at $-\infty$ or at both as shown in fig.



Thus for a given function $f(x)$, there are four cases in which vertical asymtotes can present themselves are shown in figure.



(i) $\lim_{x \to a^-} f(x) = -\infty$; $\lim_{x \to a^+} f(x) = -\infty$

(ii) $\lim_{x \to a^-} f(x) = -\infty$; $\lim_{x \to a^+} f(x) = +\infty$

(iii) $\lim_{x\to a^-} f(x) = +\infty$; $\lim_{x\to a^+} f(x) = -\infty$

(iv) $\lim_{x\to a^-} f(x) = +\infty$; $\lim_{x\to a^+} f(x) = +\infty$

• Next we investigate the behaviour of functions when $x \to \pm\infty$, $x \to \infty$ means that the value of x so changes that it can be made greater than any pre-assigned positive number however large. Now if $f(x)$ be a function of x, and if $f(x) \to l$ as $x \to \infty$, then we say that the limiting value of $f(x)$ is l when x tends to infinity. In symbol it is written as $\lim_{x\to\infty} f(x) = l$.

Consider $\lim_{x\to\infty} \frac{1}{x}$.

Some useful facts about the above function are:

(i) As x approaches 0 from the right, $1/x$ tends to ∞.

(ii) As x approaches 0 from left, $1/x$ tends to $-\infty$.

(iii) As x tends to ∞, $1/x$ approaches to 0.

(iv) As x tends to $-\infty$, $1/x$ approaches to 0.

• Thus to find $\lim_{x\to\infty} f(x)$, divide the numerator and denominator of the function by the highest power of x present in the fraction and then use the idea of $\frac{1}{x}, \frac{1}{x^2}, \ldots \to 0$ as $x \to \infty$.

Some standard results for evaluating trigonometric limits

$\left(\lim_{x\to 0} \sin x = 0\right)$ $\left(\lim_{x\to 0} \cos x = 1\right)$

$\left(\lim_{x\to 0} \frac{\sin x}{x} = 1\right)$ $\left(\lim_{x\to 0} \frac{\tan x}{x} = 1\right)$

Note that in $\lim_{x\to 0} \frac{\sin x}{x} = 1$, x is expressed in radians. Whenever degree measure of an angle is given, it should be converted into radian measure by the conversion formula $x^\circ = \pi x/180$ radians. Moreover, x should tend to zero, and the angle x in the numerator, x in the denominator should be same.

Evaluation of exponential and logarithmic limits

(1) $\lim_{x\to 0} (1+x)^{1/x} = e$ (ii) $\lim_{x\to\infty} \left(1+\frac{\lambda}{x}\right)^x = e^\lambda$

(iii) $\lim_{x\to 0} (1+\lambda x)^{1/x} = e^{1/\lambda}$ (iv) $\lim_{x\to 0} \frac{e^x-1}{e^x} = 1$

(v) $\lim_{x\to 0} \frac{a^x-1}{x} = \log_e a \ (a > 0)$

(vi) $\lim_{x\to 0} \log_e (1+x)^{1/x} = \log_e e = 1$

(vii) $\lim_{x\to a} [f(x)]^{g(x)} = e^{\lim_{x\to a} [f(x)-1]g(x)}$

where $g(x) \to 0$ and $f(x) \to 1$, as $x \to a$.

The following illustrations will make the conecpt more clear.

Illustration 1. Find

$$\lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{2/x} \quad (a, b, c > 0).$$

1. Let $y = \lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{2/x}$ (1^∞ form)

Then $\log y = \lim_{x\to 0} \frac{2}{x} \log\left(\frac{a^x+b^x+c^x}{3}\right)$

$$= 2\lim_{x\to 0} \frac{\log(a^x+b^x+c^x) - \log(3)}{x} \quad (0/0 \text{ form})$$

$$= 2\lim_{x\to 0} \left[\frac{1}{a^x+b^x+c^x}\right]\left[\frac{a^x \log a + b^x \log b + c^x \log c}{1}\right]$$

(By L. Hospital rule

$$= \frac{2}{3}(\log a + \log b + \log c) = \frac{2}{3}\log(abc) = \log(abc)^{2/3}$$

$\Rightarrow y = (abc)^{2/3}$.

Illustration 2. $\lim_{x\to\infty} \frac{x^n}{e^x} = ?$ (n integer), for all values of n.

Soln.: If n is –ve integer, then $n = -m$, where $m \in N$.

$\therefore \lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{x^{-m}}{e^x} = \lim_{x\to\infty} \frac{1}{x^m e^x} = \frac{1}{\infty} = 0$

If $n = 0$, then $\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$

If $n \in N$, then $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$.

Illustration 3. If x is real number in [0, 1], then find the value of $\lim_{m\to\infty} \lim_{n\to\infty} [1+\cos^{2m}(n!\pi x)]$

Soln.: If $x \in Q$, then $n!\pi x$ will be an integral multiple of π for large value of n. Therefore $\cos(n!\pi x)$ will be either 1 or –1 and so $\cos^{2m}(n!\pi x) = 1$.

$\therefore \lim_{m\to\infty} \lim_{n\to\infty} [1+\cos^{2m}(n!\pi x)] = 1+1 = 2$

If $x \notin Q$, $n!\pi x$, will not be an integral multiple of π and so $\cos(n!\pi x)$ will lie between –1 and 1.

Thus $\lim_{m\to\infty} \cos^{2m}(n!\pi x) = 0$

$\Rightarrow \lim_{m\to\infty} \lim_{n\to\infty} [1+\cos^{2m}(n!\pi x)] = 1$.

Illustration 4. Determine the value of $\lim_{x\to -\pi} \frac{|x+\pi|}{\sin x}$.

Soln.: $\because \lim_{x\to -\pi^-} f(x) = \lim_{h\to 0} f(-\pi-h)$

$$= \lim_{h\to 0} \frac{|-\pi-h+\pi|}{\sin(-\pi-h)} = -\lim_{h\to 0} \frac{h}{\sin(\pi+h)} = \lim_{h\to 0} \frac{h}{\sin h} = 1$$

and $\lim_{x\to -\pi^+} f(x) = \lim_{h\to 0} f(-\pi+h) = \lim_{h\to 0} \frac{|-\pi+h+\pi|}{\sin(-\pi+h)}$

$$= -\lim_{h\to 0} \frac{h}{\sin h} = -1$$

Since R.H.L. $\neq$ L.H.L.

$\therefore \lim_{x\to -\pi} f(x)$ does not exist.

Illustration 5. Find the value of $\lim_{x\to 0^+} x^m(\log x)^n$, $m, n \in N$.

Soln.: $\lim_{x\to 0^+} x^m(\log x)^n = \lim_{x\to 0^+} \frac{(\log x)^n}{x^{-m}}$ (∞/∞ form)

By L. Hospital's rule, $\lim_{x\to 0^+} \frac{n(\log x)^{n-1}(1/x)}{-mx^{-m-1}}$

$$= \lim_{x\to 0^+} \frac{n(\log x)^{n-1}}{-mx^{-m}} \ (\infty/\infty \text{ form})$$

$$= \lim_{x\to 0^+} \frac{n(n-1)(\log x)^{n-2}\times(1/x)}{(-m)^2 x^{-m-1}}$$

$$= \lim_{x\to 0^+} \frac{n(n-1)(\log x)^{n-2}}{m^2 x^{-m}} \ (\infty/\infty \text{ form})$$

(Repeatedly differentiating numerator and denominator n times) we get,

$$\lim_{x\to 0^+} \frac{n!}{(-m)^n x^{-m}} = 0.$$

Illustration 6. If $f(x) = \frac{\sin[x]}{[x]}, [x]\neq 0, = 0, [x]=0$

where $[x]$ denotes the greatest integer less than or equal to x, then $\lim_{x\to 0} f(x)$ equals

(a) 1 (b) 0

(c) -1 (d) none of these.

Soln.: First note that by def. of $[x]$, we have

$[x] = -1$ when $-1 \le x < 0$

and $[x] = 0$ when $x \le 0 < 1$

Hence by def. of

$f(x) = \frac{\sin(-1)}{-1} = \sin 1$ when $-1 \le x < 0$

and $f(x) = 0$ when $0 \le x < 1$.

$\therefore \quad f(0-0) = \lim_{h\to 0} \sin 1 = \sin 1.$

and $f(0+0) = \lim_{h\to 0} 0 = 0$

since $f(0-0) \neq f(0+0)$ the limit of $f(x)$ at $x = 0$ does not exist.

Illustration 7. $\lim_{h\to 0} \frac{(a+h)^2\sin(a+h) - a^2\sin a}{h} =$

(a) $a^2\cos a + a\sin a$ (b) $a^2\cos a + 2a\sin a$

(c) $2a^3\cos a + a\sin a$ (d) None of these.

Soln.: (b) $\lim_{h\to 0} \frac{(a+h)^2\sin(a+h) - a^2\sin a}{h}$

$$= \lim_{h\to 0} \frac{(a+h)^2[\sin(a+h) - \sin a] + \sin a[(a+h)^2 - a^2]}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}(a+h)^2 2\cos\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right) + \lim_{h\to 0}\frac{1}{h}(\sin a)[a^2 + 2ah + h^2 - a^2]$$

$$= \lim_{h\to 0}(a+h)^2\cos\left(\frac{2a+h}{2}\right)\frac{\sin(h/2)}{(h/2)} + \lim_{h\to 0}(\sin a)(2a+h)$$

$$= a^2\cos a.1 + (\sin a)(2a) = a(a\cos a + 2\sin a).$$

Illustration 8. $\lim_{x\to\infty}[x - \sqrt{(x^2+x)}] =$

(a) 1/2 (b) 1 (c) $-1/2$ (d) 0

Soln.: (c) Put $x = 1/h$, then $h \to 0$ as $x \to \infty$

Then $\lim_{x\to\infty} [x - \sqrt{(x^2+x)}]$

$$= \lim_{h\to 0}\left\{\frac{1}{h} - \sqrt{\left(\frac{1}{h^2}+\frac{1}{h}\right)}\right\} = \lim_{h\to 0}\frac{1-\sqrt{(1+h)}}{h}$$

$$= \lim_{h\to 0}\frac{1-(1+h)}{h[1+\sqrt{(1+h)}]} = \lim_{h\to 0}\frac{-h}{h[1+\sqrt{(1+h)}]}$$

$$= \lim_{h\to 0} -\frac{1}{1+\sqrt{(1+h)}} = \frac{-1}{1+1} = -\frac{1}{2}.$$

Illustration 9. $\lim_{x\to 0}\frac{\tan x - \sin x}{x^3} =$

(a) 1 (b) 2 (c) 1/2 (d) -1

Soln.: (c) $\lim_{x\to 0}\frac{\tan x - \sin x}{x^3} = \lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3\cos x}$

$$= \lim_{x\to 0}\frac{2\sin(x/2)\cos(x/2)2\sin^2(x/2)}{x^3\cos x}$$

$$= \lim_{x\to 0}\frac{4\sin^3(x/2)}{x^3}\cdot\frac{\cos(x/2)}{\cos x}$$

$$= \lim_{x\to 0}\frac{1}{2}\left(\frac{\sin(x/2)}{x/2}\right)^3\frac{\cos(x/2)}{\cos x} = \frac{1}{2}.1.1 = \frac{1}{2}$$

Alternative Method : By expansion

$$= \lim_{x\to 0}\frac{\tan x - \sin x}{x^3} = \lim_{x\to 0}\frac{\left(x+\frac{x^3}{3}+...\right)-\left(x-\frac{x^3}{3!}+...\right)}{x^3}$$

$$= \lim_{x\to 0}\frac{\left(\frac{1}{3}+\frac{1}{6}\right)x^3 + \text{..........}}{x^3} = \frac{1}{2}.$$

Illustration 10. Suppose $f: \mathbf{R} \to \mathbf{R}$ is a differentiable function and $f(1) = 4$. Then the value of

$\lim_{x\to 1}\int_4^{f(x)}\frac{2t}{(x-1)}dt$ is

(i) $8f'(1)$ (b) $4f'(1)$ (c) $2f'(1)$ (d) $f'(1)$

Soln.: (a) $\lim_{x\to 1}\int_4^{f(x)}\frac{2t}{x-1}dt = \lim_{x\to 1}\frac{1}{x-1}[t^2]_4^{f(x)}$

$$= \lim_{x\to 1}\frac{1}{x-1}[f^2(x) - 16] \ [\text{form } 0/0, \text{ since } f(1) = 4]$$

$$= \lim_{x\to 1}\frac{2f(x)f'(x)}{1} = 2f(1)f'(1)$$

$$= 2\times 4f'(1) = 8f'(1).$$

[Note that since $f(x)$ is differentiable, $f'(1)$ exists.]

DIFFERENTIAL EQUATIONS

Differential Equations : A differential equation is an equation which contain dependent and independent variable and different derivatives of the dependent variables.

For example : $\frac{dy}{dx} = xy + 1$ or $\frac{d^2y}{dx^2} = 2x + 2y + 4$

Order and degree of differential equation : The order of the highest derivative appearing in a differential equation is called the order of that differential equation. The power of the highest order derivative appearing in a differential equation is called degree of a differential equation.

ILLUSTRATIONS

Differential Equation	**order**	**degree**
(1) $\frac{dy}{dx} + 2xy = x^3$	1	1
(2) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$	2	1
(3) $\left(\frac{d^3y}{dx^3}\right)^2 + 6y = 0$	3	2

General solution of a differential equation : A solution of a differential equation means a relation between the variables x, y and z etc. and some constants, which satisfy the given differential equation.

For example : $y = \sin x$ is solution of $\frac{d^2y}{dx^2} + y = 0$.

The solution of a differential equation which contains a number of arbitrary constants equal to the order of the differential equation is called the general solution of the given differential equation.

Note that the general solution of a n^{th} order differential equation has n arbitrary constants. It must be remembered that the first order differential equation has only one arbitrary constant similarly, a second order differential equation has two arbitrary constants and so on.

Types of differential equations :

I. A differential equation of the form $F\left(x, y, \frac{dy}{dx}\right) = 0$ is said to be of first order and first degree. Its general solution will contain only one arbitrary constant.

II. A differential equation of the form

$$F\left(x, y, \frac{dy}{dx}, \left(\frac{dy}{dx}\right)^2, \ldots\ldots \left(\frac{dy}{dx}\right)^n\right) = 0$$

$F(x, y, p, p^2 \ldots\ldots p^n) = 0$ where $p = \frac{dy}{dx}$

is of order one and degree n. Its general solution will contain only one arbitrary constant.

III. A differential equation of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_{n-1}\frac{dy}{dx} + a_n y = X$$

is called an n^{th} order linear differential equation with $a_1, a_2, \ldots, a_n$ as its constants coefficients.

IV. A differential equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_{n-1} x \frac{dy}{dx} + a_n y = X$$

is called an n^{th} order homogeneous linear differential equation.

V. A differential equation of the form

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ where P, Q and R are functions of x alone is called a second order linear differential equation. Its general solution will contain two arbitrary constants.

VI. A pair of differential equations

$$\frac{dx}{dt} + a_1 x + b_1 y = f(t); \ \frac{dy}{dt} + a_2 x + b_2 y = g(t)$$

(where a_1, a_2, b_1 and b_2 are constants) is called simultaneous linear differential equations with constant coefficients.

VII. Another form of simultaneous equations is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where P, Q and R are functions of x, y and z.

VIII. A very special and important class of first order differential equations known as total differential equations is $Pdx + Qdy + Rdz = 0$.

Here P, Q and R are functions of x, y, z.

Its general solution consists of only one arbitrary constant.

Methods for solving some particular type of differential equation of first order and first degree

All the differential equations of first order and first degree cannot be always solved. Here we will discuss some special particular type of the differential equations and various methods to solve them.

I. Equation with separated variables : A differential equation of the form $f(x)dx + g(y)dy = 0$ is said to have separated variables.

Its solution is

$\int f(x)dx + \int g(y)dy = c$ where c is the constant of integration.

The following illustrations will make the concept more clear.

The following illustrations will make the concept more clear.

Illustration 1. Solve $3e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$

Soln.: we have $\frac{3e^x}{1 - e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0$

Integrating on both sides, we get

$-3\log(1 - e^x) + \log \tan y = \log c$

$\Rightarrow \log((1 - e^x)^{-3} \tan y) = \log c$

$\Rightarrow \frac{\tan y}{(1 - e^x)^3} = c$ is the required solution.

Illustration 2. Solve $y - x\frac{dy}{dx} = 3\left(1 + x^2\frac{dy}{dx}\right)$

Soln.: $y - 3 = x(3x+1)\frac{dy}{dx}$

$\Rightarrow \int \frac{dy}{y-3} = \int \frac{dx}{x(3x+1)}$

$\Rightarrow \log(y-3) = \int \frac{dx}{x} - \int \frac{3dx}{3x+1} + \log c$

$\Rightarrow \log(y-3) = \log x - \log(3x+1) + \log c$

$\Rightarrow \log(y-3) + \log(3x+1) = \log c + \log x$

$\Rightarrow \log[(3x+1)(y-3)] = \log(xc)$

$\Rightarrow (3x+1)(y-3) = cx$

$\therefore\ cx = (3x+1)(y-3)$ is the solution.

II. Homogeneous Equation : A differential equation of the form

$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$, where $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions of x and y of the same degree is called a homogeneous equation.

Working Rule : To solve a homogeneous differential equation, we follow the following procedure :

Step 1 : Put $y = vx$ so that $v + x\frac{dv}{dx} = \frac{dy}{dx}$

Step 2 : The equation thus obtained will be of the form in which variables v and x are seperable i.e. we get $f_1(v)dv + f_2(x)dx = 0$.

Step 3 : After integrating this, replace v by $\frac{y}{x}$.

The next few illustrations will make the concept more clear.

Illustration 3. Solve $x\frac{dy}{dx} - y = \sqrt{x^2+y^2}$

Soln.: The given equation may be written as

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x} \quad ...(1)$$

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Thus equation (1) reduces to

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2+v^2x^2}}{x} = \frac{v+\sqrt{1+v^2}}{1}$$

$$\therefore\ x\frac{dv}{dx} = \sqrt{1+v^2} \Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating both sides, we get

$$\log[v + \sqrt{1+v^2}] = \log x + \log c \quad \Rightarrow v + \sqrt{1+v^2} = xc$$

Put $v = \frac{y}{x} \Rightarrow \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = xc$

$\Rightarrow y + \sqrt{x^2+y^2} = x^2c$ is the required solution of given differential equation.

Illustration 4. Solve $y - xp = x + yp$ $\left(\text{where } p = \frac{dy}{dx}\right)$

Soln.: The given equation can be written as

$\frac{dy}{dx} = \frac{y-x}{y+x}$ or $v + x\frac{dv}{dx} = \frac{v-1}{v+1}$ (By putting $y = vx$)

or $x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{-(1+v^2)}{v+1}$

$\Rightarrow \int\left(\frac{v+1}{1+v^2}\right)dv + \int\frac{dx}{x} = \log c$

$\Rightarrow \frac{1}{2}\log(1+v^2) + \tan^{-1}v + \log x = \log c$

$\Rightarrow \log\frac{x\sqrt{1+v^2}}{c} = -\tan^{-1}v;$ Put $v = \frac{y}{x}$

Hence $\sqrt{x^2+y^2} = -ce^{-\tan^{-1}\frac{y}{x}}$ is the required solution.

III. Non-Homogeneous Equations : Non-homogeneous equations of the first degree in x and y are of the form :

$$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+d} \quad ...(1)$$

The above equation can be reduced to the homogeneous form as follows :

Put $x = X + h$, $y = Y + k$; where h and k are arbitrary constants.

$\therefore\ \frac{dx}{dX} = 1;\ \frac{dy}{dY} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{dY}\cdot\frac{dY}{dX}\cdot\frac{dX}{dx} = 1\cdot\frac{dY}{dX}\cdot 1 = \frac{dY}{dX}$

equation (1) thus reduced to

$$\frac{dY}{dX} = \frac{aX + bY + (ah+bk+c)}{AX+BY+(Ah+Bk+d)} \quad ...(2)$$

we choose the constants h and k in such a way that

$ah + bk + c = 0$ and $Ah + Bk + d = 0$ ...(3)

From (2) and (3) we finally get

$$\frac{dY}{dX} = \frac{aX+bY}{AX+BY}\left(\frac{a}{A} \neq \frac{b}{B}\right) \quad ...(4)$$

Equation (4) is a homogeneous equation in X and Y, which can be solved by substitution $Y = VX$.

Illustration 5. Solve : $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$

Soln.: Put $x = X + h$, $y = Y + k$ thus the given equation reduced to

$$\frac{dY}{dX} = \frac{(X+h) - 2(Y+k) + 5}{2(X+h) + (Y+k) - 1}$$

$$\therefore\ \frac{dY}{dX} = \frac{X - 2Y + (h - 2k + 5)}{2X + Y + (2h + k - 1)} \quad ...(1)$$

Choose h, k so that $h - 2k + 5 = 0$ and $2h + k - 1 = 0$

Thus $h = \frac{-3}{5};\ k = \frac{11}{5}$ ...(2)

$$\therefore\ \frac{dY}{dX} = \frac{X-2Y}{2X+Y} \quad ...(3)$$

is a homogeneous equation. We substitute $Y = VX$ so that

$$\frac{dY}{dX} = V + X\frac{dV}{dX}$$

Putting in (3) we get; $V + X\dfrac{dV}{dX} = \dfrac{1-2V}{2+V}$

or $\quad X\dfrac{dV}{dX} = \dfrac{1-2V}{2+V} - V = \dfrac{1-4V-V^2}{2+V}$

$\Rightarrow \int \dfrac{V+2}{V^2+4V-1}dV + \int \dfrac{dX}{X} = \log C$

put $V^2 + 4V - 1 = t$

$\Rightarrow \dfrac{1}{2}\int \dfrac{dt}{t} + \log X = \log C \quad \Rightarrow \quad \dfrac{1}{2}\log t + \log X = \log C$

$\Rightarrow tX^2 = C^2 \quad \Rightarrow \quad (V^2 + 4V - 1)X^2 = C^2$

Put $X = x + \dfrac{3}{5} \quad Y = y - \dfrac{11}{5}$

$\therefore\ x^2 - y^2 - 4xy + 10x + 2y = a$ (for some constant a) is the required solution.

IV. Linear Differential Equations

(1) A differential equation of the form $\dfrac{dy}{dx} + Py = Q$, where P and Q are functions of x only or constant is called a linear differential equation.

Working rule for solving linear differential eqns :

Step 1 : Write the given equation in form $\dfrac{dy}{dx} + Py = Q$, P and Q are functions of x only or constant.

Step 2 : The solution of the differential equation is $y\cdot(\text{I.F.}) = \int Q\cdot(\text{I.F})dx + C$

The next few illustrations will make the procedure more clear.

Illustration 6 : Solve $\dfrac{dy}{dx} + \dfrac{y}{(1-x^2)^{3/2}} = \dfrac{x+\sqrt{1-x^2}}{(1-x^2)^2}$

Soln.: we see that $\int P\,dx = \int \dfrac{dx}{(1-x^2)^{3/2}}$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

Thus we get;

$$\int P\,dx = \int \frac{\cos\theta d\theta}{\cos^3\theta} = \int \sec^2 d\theta = \tan\theta = \frac{x}{\sqrt{1-x^2}}$$

$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{x/\sqrt{1-x^2}}$

Thus the solution is given by

$$y\,e^{x/\sqrt{1-x^2}} = \int e^{x/\sqrt{1-x^2}} \cdot \frac{x+\sqrt{1-x^2}}{(1-x^2)^2}dx + C$$

$$= \int e^{\tan\theta} \frac{(\sin\theta + \cos\theta)\cos\theta}{\cos^4\theta} d\theta + C$$

$$= \int e^{\tan\theta}(\sec^2\theta\tan\theta + \sec^2\theta)d\theta + C$$

put $\tan\theta = t$

$$= \int e^t(t+1)dt + C \quad \text{(integrating by parts)}$$

$$= te^t + C = \tan\theta\, e^{\tan\theta} + C$$

$$\Rightarrow \quad y e^{x/\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}} e^{x/\sqrt{1-x^2}} + C$$

$\Rightarrow \quad y = Ce^{-x/\sqrt{1-x^2}} + \dfrac{x}{\sqrt{1-x^2}}$ is the required solution.

(2) Linear equation of the form $\dfrac{dx}{dy} + Rx = S$ where R, S are the functions of y alone or are constants. In such type of equations the integrating factor will be $\text{I.F.} = e^{\int R\,dy}$ and hence the solution of the equation is given by $x(\text{I.F.}) = \int S(\text{I.F.})dy + C$.

Illustration 7. Solve $(1+y^2)dx + (x - e^{-\tan^{-1}y})dy = 0$

Soln.: we have $\dfrac{dx}{dy} + \dfrac{x - e^{-\tan^{-1}y}}{1+y^2} = 0$

or $\quad \dfrac{dx}{dy} + \dfrac{1}{1+y^2}x = \dfrac{e^{-\tan^{-1}y}}{1+y^2} \qquad ...(1)$

$\text{I.F.} = e^{\tan^{-1}y} \quad \left(\because \int \dfrac{dy}{1+y^2} = \tan^{-1}y\right)$

Thus the solution is given

$$x\,e^{\tan^{-1}y} = \int \frac{e^{-\tan^{-1}y}}{1+y^2}\, e^{\tan^{-1}y}dy + C$$

Hence $xe^{\tan^{-1}y} = \tan^{-1}y + C$ is the required solution.

(3) Equations Reducible to the linear form: consider the differential equation of the form

$$\frac{dy}{dx} + Py = Qy^n \qquad ...(1)$$

where P and Q are functions of x. We can reduce (1) to the linear form as follows. Dividing both sides of (1) by y^n, we get

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q \qquad ...(2)$$

Put $y^{-n+1} = z \Rightarrow (-n+1)y^{-n}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

Substituting in (2), we obtain

$$\frac{dz}{dx} + (1-n)Pz = (1-n)Q$$

which is now a linear differential equation with z as dependent variable.

Illustration 8. Solve $x\dfrac{dy}{dx} + y^2x = y$

Soln.: Dividing by x, we get

$\dfrac{dy}{dx} - \dfrac{y}{x} = -y^2$ dividing by $(-y^2)$, we get

$$-\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = 1 \qquad ...(1)$$

Put $\dfrac{1}{y} = z \Rightarrow -\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

Thus equation (1) reduces to

$$\frac{dz}{dx} + \frac{1}{x}\cdot z = 1 \qquad ...(2)$$

$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Thus the solution of (2) is

$z\cdot x = \int 1\cdot x dx + C \Rightarrow \dfrac{1}{2}x^2 + C = zx$

$\Rightarrow \quad \dfrac{x}{y} = \dfrac{1}{2}x^2 + C$ is the required solution.

BEAT THE TIME TRAPS

Binomial Theorem

Rule – 1 : Binomial theorem for positive integral index:
If n is a positive integer and $x, y \in C$ then
$(x+y)^n = {}^nC_0x^{n-0}y^0 + {}^nC_1x^{n-1}y^1 + ... + {}^nC_rx^{n-r}y^r + ... + {}^nC_{n-1}xy^{n-1} + {}^nC_nx^0y^n$ ...(i)
Here ${}^nC_0, {}^nC_1, {}^nC_2 ... {}^nC_n$ are called binomial coefficients
If we replace y by $-y$ then
$(x-y)^n = {}^nC_0x^ny^0 - {}^nC_1x^{n-1}y^1 + ... + (-1)^n\,{}^nC_nx^0y^n$ ...(ii)

Rule – 2 : Adding (i) and (ii)
$(x+y)^n + (x-y)^n = 2\{x^n + {}^nC_2x^{n-2}y^2 + ...\}$
$= 2\{\text{Sum of odd places terms}\}$

Rule – 3 : Subtracting (ii) from (i)
$(x+y)^n - (x-y)^n = 2\{{}^nC_1x^{n-1}y^1 + {}^nC_3x^{n-3}y^3 + ...\}$
$= 2\ \{\text{Sum of even places terms}\}$
If we replace x by 1 and y by x in (i)

Rule – 4 : $(1+x)^n = {}^nC_0x^0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$
or $\quad (1+x)^n = \sum\limits_{r=0}^{n} {}^nC_rx^r$
If we replace x by 1 and y by $-x$ in (i)

Rule – 5 : $(1-x)^n = {}^nC_0x^0 - {}^nC_1x^1 + {}^nC_2x^2 ... + {}^nC_n(-1)^nx^n$
or $\quad (1-x)^n = \sum\limits_{r=0}^{n} (-1)^r\,{}^nC_rx^r$

Rule – 6 : General term
The term ${}^nC_rx^{n-r}y^r$ is the $(r+1)^{\text{th}}$ term from the beginning in the expansion of $(x+y)^n$. It is called the general term and is denoted by T_{r+1}.
$T_{r+1} = {}^nC_rx^{n-r}y^r$

Rule – 7 : p^{th} term from the end is equal to the $(n-r+2)^{\text{th}}$ term from the beginning,
i.e., ${}^nC_{n-r+1}x^{r-1}y^{n-r+1}$

Rule – 8 : $(r+1)^{\text{th}}$ term from the end in the expansion of $(a+b)^n$ is same as the $(r+1)^{\text{th}}$ term from the beginning in $(b+a)^n$.

Rule – 9 : Middle term
When n is an even : Then total no. of terms in the expansion of $(x+y)^n$ is $n+1$ (odd)
So there is only one middle term. *i.e.* $\left(\frac{n}{2}+1\right)^{\text{th}}$ term is the middle term when n is even

$$T_{\frac{n}{2}+1} = {}^nC_{n/2}x^{n/2}y^{n/2}$$

When n is an odd : Then total no. of terms in the expansion of $(x+y)^n$ is $n+1$ (even).
So there are two middle terms *i.e.* $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$ are two middle terms.

They are given by
${}^nC_{\frac{n-1}{2}}\, x^{\frac{n+1}{2}}\, y^{\frac{n-1}{2}}$ and ${}^nC_{\frac{n+1}{2}}\, x^{\frac{n-1}{2}}\, y^{\frac{n+1}{2}}$

Rule – 10 : Greatest term
To find the greatest term (numerically) in the expansion of $(1+x)^n$.
(i) At first, calculate $m = \dfrac{|x|(n+1)}{|x|+1}$
(ii) If m is integer, then T_m and T_{m+1} are equal and both are greatest terms.
(iii) If m is not integer, then $T_{[m]+1}$ is the greatest term, where [.] denotes the greatest integral part.

Rule – 11 : Greatest coefficient
(i) If n is an even, then greatest coefficient $= {}^nC_{n/2}$
(ii) If n is an odd, then greatest coefficients are
${}^nC_{\frac{n-1}{2}}$ and ${}^nC_{\frac{n+1}{2}}$

Rule – 12 : Properties of binomial coefficients
(i) Sum of the binomial coefficients in the expansion of $(1+x)^n$ is 2^n :
$\because \ (1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$
Putting $x = 1$
$2^n = C_0 + C_1 + C_2 + ... + C_n.$
$\therefore \ \sum\limits_{r=0}^{n} C_r = 2^n$

Rule – 13 : The sum of the coeffcients of the odd terms in the expansion of $(1+x)^n$ is equal to the sum of the coefficients of the even terms and each is equal to 2^{n-1} :
Since $(1+x)^n = C_0 + C_1x + ... + C_nx^n$
Putting $x = -1$
$0 = C_0 - C_1 + C_2 - C_3 + ... + (-1)^nC_n$ ...(i)
and $2^n = C_0 + C_1 + C_2 + ... + C_n$ ...(ii)
Adding (i) and (ii)
$2^n = 2(C_0 + C_2 + C_4 + ...)$
$\Rightarrow \ C_0 + C_2 + C_4 + ... = 2^{n-1}$
Subtracting (i) from (ii)
Note : $C_1 + C_3 + C_5 + ... = 2^{n-1}$

Rule – 14 : To find sum of coefficients, put $x = 1$ in the given binomial or multinomial term. Multinomial theorem (for a positive integral index) :-
If n is a positive integer and $a_1, a_2, a_3 ..., a_n \in C$ then
$(a_1 + a_2 + ... + a_m)^n = \sum \dfrac{n!}{n_1!n_2!...n_m!} a_1^{n_1}a_2^{n_2}...a_m^{n_m}$

MOCK TEST PAPER FOR ISI 2009

Exam on 10th May 2009

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Questions will be set on the following and related topics.

Algebra : Sets, operations on sets, prime numbers, factorization of integers and divisibility, rational and irrational numbers, permutations and combinations, binomial theorem, logarithms, theory of quadratic equations, polynomial and remainder theorem, arithmetic and geometric progressions, inequalities involving A.M., G.M., and H.M., complex numbers.

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Trigonometry : Measures of angles, trigonometric and inverse trigonometric functions, trigonometric identities including addition formulae, solutions of trigonometric equations. Properties of triangles, heights and distances.

Calculus : Functions, one-one functions, onto functions, limits and continuity, derivatives and methods of differentiation, slope and curve, tangents and normals, maxima and minima, use of calculus in sketching graph of functions, methods of integration, definite and indefinite integrals, evaluation of area using integrals.

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MULTIPLE CHOICE QUESTION TYPE

1. The number of solutions to the system of equations $[x] + 3\{y\} = 3.9$ and $\{x\} + 3[y] = 3.4$ is

(a) 1 (b) 2
(c) 3 (d) None of these

2. Suppose that $x, y \in R$ for which $x^3 + 3x^2 + 4x + 5 = 0$ and $y^3 - 3y^2 + 4y - 5 = 0$, then $(x+y)^{42} =$

(a) 0 (b) 1
(c) 2^{42} (d) None of these

3. The number of sets (x, y, z) of real numbers for which $x + y = 2$ and $xy - z^2 = 1$ (is/are)

(a) 1 (b) 2
(c) 3 (d) None of these

4. The number of integers for which $(4-x)^{4-x} + (5-x)^{5-x} + 10 = 4^x + 5^x$ (is/are)

(a) 1 (b) 2
(c) 3 (d) None of these

5. The eliminant of θ in, $x = \cot\theta + \tan\theta$, $y = \sec\theta - \cos\theta$ is

(a) $x^2y^4 = 1 + x^4y^2 + 3x^2y^2$ (b) $x^4y^2 = 1 + x^2y^4 + 3x^2y^2$
(c) $x^4y^4 = 1 + 3x^2y^2$
(d) $x^4y^4 + x^2y^2 = 1 + x^2y^4 + x^4y^2$

6. Let a and b be real parameters. One of the roots of the equation $x^{12} - abx + a^2 = 0$ is greater than 2, then

(a) $|b| > 64$ (b) $|b| < 64$
(c) $|b| = 64$ (d) $|b| < 32$

By : Er. Tapas Kumar Yogi, Bhubaneshwar

7. Let $a, b \in R^+$ for which $60^a = 3$ and $60^b = 5$, then $12^{\frac{1-a-b}{2(1-b)}}$ is

(a) 2 (b) 3 (c) 6 (d) 12

8. For each positive integer n, the expression $(3-2\sqrt{2})(17+12\sqrt{2})^n + (3+2\sqrt{2})(17-12\sqrt{2})^n - 2$ is

(a) square of an integer, when n is odd
(b) square of an integer, when n is even
(c) square of an integer, for all positive integer n
(d) not a square for any n.

9. If $0 < a < b$ and $\sqrt{\frac{t^3+a^3}{t+a}} + \sqrt{\frac{t^3+b^3}{t+b}} = \sqrt{\frac{a^3-b^3}{a-b}}$ then $t =$

(a) $\sqrt{ab}$ (b) $\frac{a+b}{2}$ (c) $\frac{ab}{a+b}$ (d) $\frac{a+b}{ab}$

10. The number of solutions to the equation $5\sin x + \frac{5}{2\sin x} - 5 = 2\sin^2 x + \frac{1}{2\sin^2 x}, x \in [0, \pi]$ is/are

(a) 2 (b) 3 (c) 4 (d) 5

11. The number of solutions to the equation $\tan^2 2x = 2\tan 2x \tan 3x + 1, x \in [0, 2\pi]$ is / are

(a) 0 (b) 1 (c) 2 (d) 3

12. Let $P(x)$ be the polynomial $P(x) = x^{15} - 2004x^{14} + 2004x^{13}$ ...... $- 2004x^2 + 2004x$ then $P(2003)$

(a) 2001 (b) 2002 (c) 2003 (d) 2004

13. Let $f: R \to R, f(x) - x^3$ is an increasing function, then $f(x) - x - x^2$ is

(a) always an increasing function
(b) always a decreasing function
(c) always a constant function
(d) increasing or decreasing depending on $f(x)$

14. Let a, b, c be three real numbers for which $0 \le c \le b \le a \le 1$ and let w be a complex root of $z^3 + az^2 + bz + c = 0$, then $|w| \le 1$,

(a) true (b) false
(c) depends on a, b
(d) nothing can be said even if a and b are given

15. Let A and B be two points on a parabola with vertex V such that VA is perpendicular to VB and θ is the angle made by chord VA with the axis of the parabola, then $\frac{|VA|}{|VB|} =$

(a) $\sin^3\theta$ (b) $\text{cosec}^3\theta$ (c) $\tan^3\theta$ (d) $\cot^3\theta$

16. The interval on which the function $f(x) = \log_{1/2}(x^2 - 2x - 3)$ is monotonically increasing is

(a) $(-\infty, -1)$ (b) $(-\infty, 1)$
(c) $(1, \infty)$ (d) $(3, \infty)$

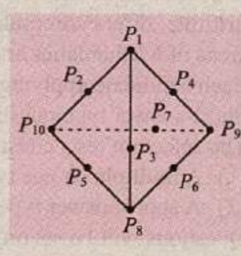
17. If real numbers x and y satisfy $(x+5)^2 + (y-12)^2 = (14)^2$, then minimum value of $x^2 + y^2$ is

(a) 2 (b) 1 (c) $\sqrt{2}$ (d) $\sqrt{3}$

18. It is given that complex numbers z_1 and z_2 satisfy $|z_1| = 2, |z_2| = 3$. If the included angle of their corresponding vectors is 60°, then $\left|\frac{z_1+z_2}{z_1-z_2}\right|$ is

(a) $\frac{\sqrt{133}}{7}$ (b) $\frac{7}{\sqrt{133}}$ (c) $\frac{6}{\sqrt{130}}$ (d) $\frac{\sqrt{130}}{6}$

19. As shown in the diagram, points $P_1, P_2, \ldots\ldots, P_{10}$ are either the vertices or the midpoints of the edges of a tetrahedron respectively. Then the number of groups of four points (P_1, P_i, P_j, P_k) $(1 < i < j < k \le 10)$ on the same plane are



(a) 30 (b) 33 (c) 36 (d) 39

20. The minimum number of points required to determine a parabola (uniquely) is

(a) 2 (b) 3 (c) 4 (d) 5

SHORT ANSWER SUBJECTIVE TYPE

1. Define the real sequences $\{a_n : n \ge 1\}$ and $\{b_n : n \ge 1\}$ by $a_1 = 1, a_{n+1} = 5a_n + 4$ and $5b_n = a_n + 1$ for $n \ge 1$.

(a) Determine $\{a_n\}$ as a function of n.
(b) Prove that $\{b_n\}$ is a geometric sequence.

2. In a tetrahedron $ABCD$, prove that $|AB| \cdot |CD| + |AC| \cdot |BD| \ge |AD| \cdot |BC|$

3. Let z_1, z_2, z_3, z_4 be distinct complex numbers for which $|z_1| = |z_2| = |z_3| = |z_4|$. Suppose further that there is a real number $t(\ne 1)$ for which $|tz_1 + z_2 + z_3 + z_4| = |z_1 + tz_2 + z_3 + z_4| = |z_1 + z_2 + tz_3 + z_4|$. Show that in the complex plane z_1, z_2, z_3 and z_4 form a rectangle.

4. Suppose that $x, y \in R^+$. Find all real solutions of the equation $\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} = \frac{x+y}{2} + \sqrt{xy}$

5. Two real non-negative numbers a and b satisfy the inequality $ab \ge a^3 + b^3$. Prove that $a + b \le 1$

6. In a 8×8 matrix, each entry is either $+1$ or -1. Let A_k be the product of all the numbers in the k^{th} row and B_k, the product of all the numbers in the k^{th} column. Prove that the number $A_1 + A_2 + \ldots. A_8 + B_1 + B_2 + \ldots + B_8$ is a multiple of 4.

7. A square is partitioned into non-overlapping rectangles. Consider the circumcircles of all the rectangles. Prove that, if the sum of the areas of all these circles is equal to the area of the circumcircle of the square, then all rectangles must be squares, too.

8. Prove that there does not exist polynomials $f(x)$ and $g(x)$ with complex coefficients for which $\log_b x = \dfrac{f(x)}{g(x)}$, $(b > 1)$.

9. Prove that for any complex numbers z and w,
$$(|z| + |w|)\left|\frac{z}{|z|} + \frac{w}{|w|}\right| \le 2|z + w|.$$

10. Let x, y, z be positive real numbers. Prove that
$$\sqrt{x^2 - xy + y^2} + \sqrt{y^2 - yz + z^2} \ge \sqrt{x^2 + xz + z^2}$$

SOLUTIONS

Multiple Choice Question Type

1. **(c)** : $[x] + 3\{y\} = 3.9$, $\{y\} \in [0, 1)$
$\Rightarrow [x] = 1, 2, 3.$
Similarly from the second eqn. $\{x\} \in [0, 1)$
$\Rightarrow [y] = 1.$
And $x = [x] + \{x\}, y = [y] + \{y\}$
gives three solutions.
$$\left(\frac{7}{5}, \frac{59}{30}\right), \left(\frac{12}{5}, \frac{49}{30}\right), \left(\frac{17}{5}, \frac{39}{30}\right).$$

2. **(a)** : Note that both the equations are increasing functions. So, they have only one solution, common to them. Further notice that if $x = u$ is a real solution then $y = -u$ is also a solution. Hence $x + y = 0$.

3. **(a)** : $z^2 = xy - 1 = x(2 - x) - 1 = -(x - 1)^2$
$\Rightarrow z^2 + (x - 1)^2 = 0 \Rightarrow z = 0, x = 1$
So, $(x, y, z) \equiv (1, 1, 0)$.

4. **(a)** : If $x < 0$, then L.H.S. = integer but R.H.S. is positive and less than $\frac{1}{4} + \frac{1}{5}$ *i.e.*, < 1.
If $x > 5$, then L.H.S. $< 1/4$ while R.H.S. is positive integer. So, the only possible solutions are integers between $x = 0$ and $x = 5$, inclusive. Checking, $x = 2$ is the only solution.

5. **(b)** : Put any value of θ in question and in options.

6. **(a)** : Clearly $a \neq 0$, $b = \dfrac{x^{12} + a^2}{ax}$

If $x > 2$, then $|b| = \left|\dfrac{x^{12} + a^2}{ax}\right| = \dfrac{x^{12} + a^2}{|a|x} \ge \dfrac{2|a|x^6}{|a|x}$
[Using $A.M. \ge G.M.$]

i.e., $|b| \ge 2x^5 = 64$.

7. **(a)**

8. **(c)** : Note that $(1 \pm \sqrt{2})^2 = 3 \pm 2\sqrt{2}$ and $(3 \pm 2\sqrt{2})^2 = 17 \pm 2\sqrt{2}$.
The given expression is $\left[(1+\sqrt{2})^{2n-1} + (1-\sqrt{2})^{2n-1}\right]^2$, which is clearly the square of an integer, by expanding using binomial expansion.

9. **(c)** : The given expression is
$$\sqrt{t^2 - at + a^2} + \sqrt{t^2 - bt + b^2} = \sqrt{a^2 + ab + b^2}$$
squaring and simplifying further, $t = \dfrac{ab}{a + b}$

10. **(b)** : Let $U = \sin x$, the given equation reduces to $(U \neq 0)$,
$U^4 - 10U^3 + 10U^2 - 5U + 1 = 0$
i.e., $U(U - 1)(2U - 1)[U^2 + (U - 1)^2] = 0$
or $U = 0, 1, 1/2 \quad \Rightarrow x = \pi/6, \pi/2, 5\pi/6$.

11. **(a)** : Let $\tan x = t$, then $\tan 2x = \dfrac{2t}{1 - t^2}$ and $\tan 3x = \dfrac{3t - t^3}{1 - 3t^2}$.
The given equation simplifies to $(t^2 + 1)^3 = 0$
$\Rightarrow$ No real solution.

12. **(c)** : Note that $x^{n+2} - 2004\, x^{n+1} + 2003\, x^n$
$= x^n(x - 1)(x - 2003)$.

13. **(d)** : Consider $f(x) = x + x^3$.

14. **(a)** : Let $w = u + iv$, then w, $\bar{w}$ and r be the three roots of the equation, then $a = -2u - r$, $b = |w|^2 + 2ur$ and $c = -|w|^2 r$.
So, rearranging, $|w|^6 - b|w|^4 + ac|w|^2 - c^2 = 0$
So that $|w|^2$ is a non-negative root of the cubic equation $q(t) = t^3 - bt^2 + act - c^2 = 0$ *i.e.*, $(t - b)t^2 + c(at - c) = 0$.
Suppose $t > 1$, then $t > b$ and $at > c$. So, $q(t) > 0$
$\Rightarrow |w| \le 1$

15. **(d)**

16. **(a)**

17. **(b)** : Put $x + 5 = 14\cos\theta$ and $y - 12 = 14\sin\theta$.

18. **(a)** : Use cosine rule in the two triangles, $(z_1, z_2, z_1 + z_2)$ and $(z_1, z_2, z_1 - z_2)$.
$|z_1 + z_2| = \sqrt{19}$, $|z_1 - z_2| = \sqrt{7}$.

19. **(b)** : On each lateral face of the tetrahedron other than P_1 there are 5 points. Take any 3 out of these 5 and add P_1 to it. (*e.g.*, $P_1P_3P_4P_6$). So, there are in all $3 \times {}^5C_3$ groups of lateral faces. Apart from these, there are 3 points on each edge containing P_1. When we add

a midpoint taking from the edge on the base which is not on the same plane with the edge above, we obtain another required group. (e.g. $P_1P_2P_6P_{10}$). There are 3 groups like this.

So, total required groups $= 3 \times {}^5C_3 + 3 = 33$.

20. (c)

Short Answer Subjective Type

1. $a_{n+1} + 1 = 5(a_n + 1) = 5.5(a_{n-1} + 1) = \ldots\ldots$
$\ldots\ldots = 5^n(a_1 + 1) = 2 \cdot 5^n$.

So, $a_n = 2 \cdot 5^{n-1} - 1$ and $b^n = 2 \cdot 5^{n-2}$, which is clearly a geometric sequence.

2. Let $\vec{u}, \vec{v}, \vec{w}$ be unit vectors and b, c, d be scalar such that $\overrightarrow{AB} = b\vec{u}$, $\overrightarrow{AC} = c\vec{v}$, $\overrightarrow{AD} = d\vec{w}$

So, $|\overrightarrow{AB}|\ |\overrightarrow{CD}| + |\overrightarrow{AC}|\ |\overrightarrow{BD}|$

$= b\,|d\vec{w} - c\vec{v}| + c\,|d\vec{w} - b\vec{u}|$

$= b\,|d\vec{v} - c\vec{w}| + c\,|d\vec{u} - b\vec{w}|$

[Using $|p\vec{u} + q\vec{v}| = |p\vec{v} + q\vec{u}|$]

$= |bd\vec{v} - bc\vec{w}| + |bc\vec{w} - cd\vec{u}|$

$\geq |bd\vec{v} - bc\vec{w} + bc\vec{w} - cd\vec{u}|$

$\geq d\,|b\vec{v} - c\vec{u}| = |\overrightarrow{AD}| \cdot |\overrightarrow{BC}|$.

3. Let $S = z_1 + z_2 + z_3 + z_4$.

So, the given condition becomes

$|S - (1-t)z_1| = |S - (1-t)z_2| = |S - (1-t)z_3|$. So, S is equidistant from the three distinct points $(1-t)z_1$, $(1-t)z_2$, $(1-t)z_3$, but these three points lie on a circle with centre at O and radius $(1-t)z_1$. So, $S = 0$.

Since, $z_1 - (-z_2) = z_1 + z_2 = -z_3 - z_4 = (-z_4) - z_3$ and $z_2 - (-z_3) = z_2 + z_3 = -z_4 - z_1 = (-z_4) - z_1$.

$\Rightarrow$ $z_1, -z_2, z_3$ and z_4 are the vertices of a parallelogram inscribed in a zero centred circle, and hence diagonals intersect at O. Therefore $-z_2$ is opposite to either z_1, z_3 or $-z_4$. Since z_2 is unequal to z_1 and z_3, we must have $z_2 = -z_4$. So, $z_1 = -z_3$. Hence z_1, z_2, z_3, z_4 form a rectangle.

4. Let $a = \dfrac{x+y}{2}$, $g = \sqrt{xy}$, $h = \dfrac{2xy}{x+y}$

and $r = \sqrt{\dfrac{x^2+y^2}{2}}$ (RMS). So, $g^2 = ah$ and

ATQ, $h + r = a + g$ and $r^2 = 2a^2 - g^2$

$\Rightarrow$ $(a + g - h)^2 = 2a^2 - g^2$

Solving, $a = h$ $\Rightarrow$ $\boxed{x = y}$

5. $1 - (a+b) = 1 - \dfrac{a^3 + b^3}{a^2 - ab + b^2} \geq 1 - \dfrac{ab}{a^2 - ab + b^2}$

$= \dfrac{a^2 - 2ab + b^2}{a^2 - ab + b^2}$ *i.e.,* $1 - (a+b) \geq \dfrac{(a-b)^2}{a^2 - ab + b^2} \geq 0$.

Note that $a^2 - ab + b^2$ is always positive.

Hence, $a + b \leq 1$.

6. Assume that p of the 8 A_k's have the value 1 and the rest $(8-p)$ have value -1. Similarly, suppose that q of the 8 B_k's have the value 1 and the rest $(8-q)$ have value -1. Then each product is the product of all entries, the product $A_1A_2\ldots\ldots A_8$ and $B_1B_2\ldots\ldots B_8$ are equal.

So, $(-1)^{8-p} = (-1)^{8-q}$

$\Rightarrow$ p and q both are odd or both are even.

So, $A_1 + A_2 + \ldots\ldots + A_8 + B_1 + B_2 + \ldots\ldots + B_8$

$= p + (8-p)(-1) + q + (8-q)(-1) = 2(p+q) - 16$

Since, $p + q$ is even, both R.H.S. terms are divisible by 4.

7. Let x be the side length of the square and (a_i, b_i) be the dimensions of the i^{th} rectangle.

Then $x^2 = \sum a_i b_i$

Area of circumcircle of square $= \dfrac{\pi x^2}{2}$

Area of circumcircle of i^{th} rectangle $= \dfrac{\pi}{4}(a_i^2 + b_i^2)$

Now, according to the question,

$$\sum \frac{\pi}{4}(a_i^2 + b_i^2) = \frac{\pi x^2}{2} = \frac{\pi}{2}\sum a_i b_i$$

$\Rightarrow$ $\sum(a_i^2 + b_i^2) = 2\sum a_i b_i$ *i.e.,* $AM = GM$ $\Rightarrow$ $a_i = b_i$

So, the rectangles are all squares.

8. Assume that the given equation is possible. Then, for any positive integer n, we must have,

$$\frac{f(x^n)}{g(x^n)} = \log_b x^n = n\log_b x = \frac{nf(x)}{g(x)}$$

$\therefore$ $f(x^n)g(x) = nf(x)\,g(x^n)$.

By comparing the leading coefficients, we see that this is not possible.

9. $$(|z| + |w|)\left|\frac{z}{|z|} + \frac{w}{|w|}\right| = \left|z + w + \frac{|z|\,w}{|w|} + \frac{|w|\,z}{|z|}\right|$$

$$\leq |z + w| + \frac{1}{|z|\,|w|}\,|z\bar{z}w + \bar{w}zw|$$

$$= |z + w| + \frac{|z|\,|w|}{|z|\,|w|}\,|\bar{z} + \bar{w}| = 2\,|z + w|$$

10. Let ABC be a triangle with $AB = x$, $AC = z$ and $\angle BAC = 120°$. Let AD be the bisector of $\angle BAC$ and $AD = y$.

Then, applying cosine rule to Δ's, ABC, ABD and ACD, we have

$BC = \sqrt{x^2 + xz + z^2}$, $BD = \sqrt{x^2 - xy + y^2}$,

$CD = \sqrt{y^2 - yz + z^2}$

Since, $BD + CD \geq BC$, we have the required result.

12. Let the in-circle of the ΔABC touches its sides BC, CA and AB at A_1, B_1 and C_1 respectively. If r_1, r_2 and r_3 are the circumradii of the triangles B_1IC_1, C_1IA_1 and A_1IB_1 respectively, then find the value of $\frac{Rr^2}{r_1r_2r_3}$ (where R is circumradius and r is the inradius of the ΔABC).

13. The sides of a triangle ABC, inscribed in a hyperbola $xy = c^2$, makes angles α, β, γ with an asymptote. If the normals at A, B, C will meet in a point, then find the value of $\cot 2\alpha + \cot 2\beta + \cot 2\gamma$.

SECTION - III

Matrix-Match type

The section consists of 3 questions has two columns with 4 entries in each column. Entries of column I are to be matched with entries of column II. One entry of column I may have more than one matching in column II. Each question carries +6 marks if all correct matching are indicated.

IS 2009

14.

	Column I		Column II
(A)	The area of the region bounded by the curves $y = x^2$, $y = \|2 - x^2\|$ and $y = 2$ which lies to the right of the line $x = 1$ is	(P)	$\frac{4}{3}$
(B)	The area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 + 3 = 4x$ is	(Q)	$\frac{9}{8}$
(C)	The area bounded by the curve $x^2 = 4y$ and straight line $x = 4y - 2$ is	(R)	$\frac{1}{3}(20-12\sqrt{2})$
(D)	A curve passes through (2, 0) and the slope of tangent at point $P(x, y)$ equals $\frac{(x+1)^2+y-3}{x+1}$. The area enclosed by the curve and the x-axis in the fourth quadrant is	(S)	$\frac{1}{3}$

15.

	Column I		Column II
(A)	The value of $\tan^{-1}([\pi]) + \tan^{-1}([-\pi]+1) =$ where [.] is the greatest integer.	(P)	2
(B)	The number of solutions of the equation $\tan x + \sec x = 2\cos x$ in the interval $[0, 2\pi]$ is	(Q)	3
(C)	The number of roots of the equation $x + 2\tan x = \pi/2$ in the interval $[0, 2\pi]$ is	(R)	0
(D)	The number of solutions of the equation $x^3 + x^2 + 4x + 2\sin x = 0$ in $0 \le x \le 2\pi$ is	(S)	1

16.

	Column I		Column II
(A)	A directed line makes angle 60° and 45° with the axes of x and y respectively. The angle it make with the axis of z is	(P)	$\frac{3\pi}{4}$
(B)	The angle (θ) of line of intersection of the planes $\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})=0$ and $\vec{r}.(3\hat{i}+3\hat{j}+\hat{k})=0$ with $\hat{j}$, then $\cos\theta$ is	(Q)	$\frac{\pi}{3}$
(C)	The acute angle between the lines whose direction cosines are given by the relations $l+m+n=0$ and $l^2+m^2-n^2=0$ is	(R)	$\sqrt{\frac{2}{3}}$
(D)	Unit vectors $\hat{a}$ and $\hat{b}$ are $\perp^r$ to each other and the unit vector $\hat{c} = m(\hat{a}+\hat{b}) + n(\hat{a}\times\hat{b})$ is inclined at an angle θ to both $\hat{a}$ and $\hat{b}$, then maximum value of θ is	(S)	$\frac{2\pi}{3}$

SOLUTIONS

Paper – I

1. (a) 2. (d) 3. (b) 4. (b) 5. (a) 6. (c) 7. (b) 8. (a) 9. (d) 10. (a) 11. (c) 12. (d) 13. (a,b) 14. (a,b) 15. (b) 16. (c) 17. (a, d) 18. (a, b, d) 19. (a,b) 20. (c, d) 21. (a, b) 22. (b, d) 23. (a, b) 24. (a, b)

Paper - II

1. (d) 2. (d) 3. (b) 4. (c) 5. (b) 6. (a) 7. (d) 8. (c) 9. (b, c) 10. (a)
11. (0000) 12. (0002)
13. (0000)
14. (A) → (R); (B) → (S); (C) → (Q); (D) → (P)
15. (A) → (R); (B) → (P); (C) → (Q); (D) → (S)
16. (A) → (Q, S); (B) → (R); (C) → (Q); (D) → (P).

MOCK TEST PAPER FOR ISI 2009

Exam on 10th May 2009

The **Indian Statistical Institute** (ISI), Kolkata, is considered as one of the foremost centres in the world for training and research in statistics and the related sciences. The B.Stat (Hons) degree program, the flagship programme of the institute, offers comprehensive instruction in the theory, method and application of statistics, in addition to several areas of Mathematics and some basic areas of computer science.

Each candidate applying for admission to this programme has to take a selection test comprising Objective type and Short-answer type questions in mathematics at the Higher Secondary level (10 + 2 year's programme).

The selection tests consist of

(1) A multiple choice type test having about 30 questions, and

(2) A short-answer type test having about 10 questions.

15/2/2009

Questions will be set on the following and related topics.

Algebra : Sets, operations on sets, prime numbers, factorization of integers and divisibility, rational and irrational numbers, permutations and combinations, binomial theorem, logarithms, theory of quadratic equations, polynomial and remainder theorem, arithmetic and geometric progressions, inequalities involving A.M., G.M., and H.M., complex numbers.

Geometry : Plane geometry of class X level. Geometry of 2 dimensions with cartesian and polar co-ordinates. Concept of a locus, equation of a line, angle between two lines, distance from a point to a line. Areas of a triangle, equations of a circle, parabola, ellipse and hyperbola and equations of their tangents and normals, mensuration.

Trigonometry : Measures of angles, trigonometric and inverse trigonometric functions, trigonometric identities including addition formulae, solutions of trigonometric equations. Properties of triangles, heights and distances.

Calculus : Functions, one-one functions, onto functions, limits and continuity, derivatives and methods of differentiation, slope and curve, tangents and normals, maxima and minima, use of calculus in sketching graph of functions, methods of integration, definite and indefinite integrals, evaluation of area using integrals.

Logical Reasoning : Consistency of statements.

In response to growing demand from students preparing for the ISI, we bring to you the first Mock ISI paper, which closely simulates the real exam.

MULTIPLE CHOICE QUESTION TYPE

1. The fourth and fifth terms of a sequence are 4 and 5 and the n^{th} term is given as $t_n = 2t_{n-1} - t_{n-2}, n \geq 3$ then, the sum to 2009 terms is

(a) 2013021 (b) 2017036

(c) 2019045 (d) 2018040

2. The number of ordered pairs (m, n) of integers that satisfy $n^2 = 1 + m + m^2$ is

(a) exactly two (b) exactly three

(c) exactly four (d) exactly eight

3. 8 points are lying in a plane with no three of them collinear. Let m and M denote respectively the minimum and maximum possible number of distinct circles that can be determined using these points, then M-m equals

(a) 55 (b) 45 (c) 47 (d) 38

4. Two circle with radii 3 cm and 8 cm touch each other externally. Three common tangents intersecting each other at three points, are drawn to these two circles. The area of triangle formed by joining these three intersection points is (in sq cm)

(a) $48\frac{\sqrt{6}}{5}$ (b) $96\frac{\sqrt{6}}{5}$ (c) $32\frac{\sqrt{6}}{5}$ (d) $24\frac{\sqrt{6}}{5}$

5. Let a function f be defined on the set of positive integers as

$$f(n) = \begin{cases} (n-3), & \text{for } n > 99 \\ f(f(n+5)), & \text{for } n < 100 \end{cases}$$

Then $f(14)$ equals

By : Alok Kumar, Rao IIT Academy, Kota (4th Rank, INMO'91 winner)

(a) 99 (b) 97 (c) 100 (d) 98

6. If $abc \neq 0$ and they satisfy

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{a+x}=0$$

$$\frac{1}{a}+\frac{1}{c}+\frac{1}{a+y}=0$$

$$\frac{1}{a}+\frac{1}{x}+\frac{1}{y}=0,$$

then $a+b+c$ equals

(a) -1 (b) 0

(c) 1 (d) cannot be determined

7. The number $A837B$ is divisible by 88. The sum $(A+B)$ equals

(a) 12 (b) 14 (c) 13 (d) 15

8. Let $ABCD$ be a cyclic quadrilateral. Denote $AB = a$, $BC = b$, $CD = c$ and $\angle ABC = 120°$, $\angle ABD = 30°$. Then which of the following statement is true?

(a) $\left|\sqrt{c+a}-\sqrt{c+b}\right|=\sqrt{c-a-b}$ and $c<a+b$

(b) $\left|\sqrt{c+a}-\sqrt{c+b}\right|>\sqrt{c-a-b}$ and $c=a+b$

(c) $\left|\sqrt{c+a}-\sqrt{c+b}\right|=\sqrt{c-a-b}$ and $c\geq a+b$

(d) $\left|\sqrt{c+a}-\sqrt{c+b}\right|<\sqrt{c-a-b}$ and $c=a+b$

9. The number of positive integer n such that n is not the square of any integer and $(\sqrt{n})^3$ divides n^2, is (Here (x) denote the integer that is less than or equal to x)

(a) 4 (b) 3

(c) 6 (d) none of these

10. Suppose P is an interior point of a triangle ABC and let AP, BP and CP cut the opposite sides BC, CA, AB in D, E and F respectively. Then which of the following statements is true?

(a) $\frac{AF}{FB}+\frac{AE}{EC}>\frac{AP}{PD}$ (b) $\frac{AF}{FB}+\frac{AE}{EC}=\frac{AP}{PD}$

(c) $\frac{AF}{FB}+\frac{AE}{EC}<\frac{AP}{PD}$ (d) None of these

11. The cubic polynomial $p(x)$ satisfies the condition that $(x-1)^2$ is a factor of $p(x)+2$ and $(x+1)^2$ is a factor of $p(x)-2$. Then $p(3)$ equals

(a) 27 (b) 18 (c) 12 (d) 6

12. The last three digits of 7^{9999} are (in that order)

(a) 523 (b) 143 (c) 343 (d) 263

13. Each number from 1 through 100 *i.e.* 1,2 .....100 (decimal system) is written in base 6 and their product is also written in base 6. Then the number of zeroes at the end of this product is

(a) 24 (b) 18 (c) 48 (d) 97

14. Given any n real numbers, there always exist two of them, say a and b such that $0 < a - b < 1 + ab$. The least value of n for which this assertion is true, is

(a) 4 (b) 5 (c) 6 (d) 7

15. The remainder when 19^{92} is divided by 92, is

(a) 41 (b) 49 (c) 32 (d) 44

16. Let f be a real valued function such that for any real x

$f(15+x)=f(15-x)$ and $f(30+x)=-f(30-x)$

Then which of the following statements is true?

(a) f is odd and periodic (b) f is odd but not periodic

(c) f is even and periodic

(d) f is even but not periodic.

17. The in-radius and circumradius of a right angled triangle are 7 cm and 32.5 cm. The area of the triangle is (in sq cm)

(a) 168 (b) 126 (c) 504 (d) 252

18. In the triangle PQR, PQ = 9 cm, PR = 15 cm and QR = 18cm. The radius of the semicircle inscribed in the triangle PQR whose diameter lies on QR and that is tangent to PQ and PR is

(a) $3\sqrt{7/2}$ (b) $3\sqrt{7}$ (c) $\frac{3}{2}\sqrt{7}$ (d) $6\sqrt{7}$

19. The greatest common divisor of

$3^{3^{333}}+1$ and $3^{3^{334}}+1$ is

(a) 28 (b) 2 (c) $3^{3^{333}}+1$ (d) 82

PASSAGE COMPREHENSION

Passage question 20-21

A 'pure' divisor of a positive integer n is a divisor of n having exactly one prime divisor. Let S_n be the set of all the pure divisors of n that are coprime with all other pure divisors of n.

20. How many elements does $S_{\lfloor 6}$ have?

(a) one (b) two

(c) three (d) None of these

21. Suppose $S_n = \{2,3\}$ for some positive integer n. The number of different values of n between 1 and 1000, inclusive, is

(a) 4 (b) 5 (c) 6 (d) 7

22. Consider the set of points

$S = \{(x, y) : |x+y| = 1 \text{ and } |x| = 1, x, y \in R\}$

Then which of the following statements is true?

(a) S determines a parallelogram whose area is 4

(b) S determines a square whose area is 4

(c) S determines a parallelogram whose area is 2.

(d) S determines a square whose area is 2.

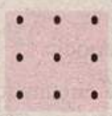
23. T is a set of triangles no two elements of which are similar. The degree measure of angle of any triangle in T are integers. The number of triangles in T is

(a) 15931 (b) 2611 (c) 2700 (d) 16471

24. Let $S = \{1, 2, 3, ..., 1000\}$. Let A be a subset of S such that any two elements in A are coprime to each other and no element of A is prime. The maximum possible number of elements in A is

(a) 31 (b) 11 (c) 12 (d) 30

25. The number of right angled triangles with all their vertices belonging to the 3×3 array of points as given below,



is (The points are evenly spaced)

(a) 36 (b) 40 (c) 44 (d) 52

26. The number of polynomials $p(x)$ satisfy

$$p(x^2) + 2x^2 + 10x = 2xp(x+1) + 3, \text{ is}$$

(a) 0 (b) 1 (c) 2 (d) infinite

27. $$\int \frac{\left(x+\frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6}\right) - 2}{\left(x+\frac{1}{x}\right)^3 + x^3 + \frac{1}{x^3}}\, dx$$

equals (k being a constant of integration)

(a) $\ln\left\{\left(x+\frac{1}{x}\right)^3 + \left(x^3 + \frac{1}{x^3}\right)\right\} + k$

(b) $\left(x+\frac{1}{x}\right)^3 + \ln\left\{2\left(x^3+\frac{1}{x^3}\right) + 3\left(x+\frac{1}{x}\right)\right\} + k$

(c) $x^3 + \frac{1}{x^3} + \ln\left\{\left(x+\frac{1}{x}\right)^3 + x^3 + \frac{1}{x^3}\right\} + k$

(d) $3\left(\frac{x^2}{2} + \ln|x|\right) + k$

28. Let f be a function defined on pair of non negative integers by

$$f(m,n) = \left(1-\frac{m}{n}\right)\left(1-\frac{m}{n+1}\right) + f(m-1, n) \text{ for } n > m \geq 1$$

$f(0, n) = 1, n \geq 1$

The value of $f(2009, 2010)$ equals

(a) $\frac{(\underline{|2009})^2}{(2009)^{2008}\cdot(2011)^{2009}}$ (b) $\frac{(\underline{|2010})^2}{(2010)^{2008}\cdot(2011)^{2008}}$

(c) $\frac{(\underline{|2009})^2}{(2010)^{2008}\cdot(2011)^{2009}}$ (d) $\frac{(\underline{|2010})^2}{(2010)^{2008}\cdot(2011)^{2008}}$

29. The sum to first n terms of a geometric sequence is given by

$$S_n = a\left(\frac{1}{3}\right)^n + b, n \geq 1$$

Suppose the sum to infinity be 1, then which of the following is the sum to first 3^{33} term.

(a) $\left(1-\frac{1}{3^{3^{33}}}\right)\left\{1+\frac{1}{33^{33}}+\left(\frac{1}{3^{3^{33}}}\right)^2\right\}$

(b) $\left(1-\frac{1}{3^{3^{32}}}\right)\left\{1+\frac{1}{3^{3^{32}}}+\left(\frac{1}{3^{3^{32}}}\right)^2\right\}$

(c) $\left(1+\frac{1}{3^{3^{32}}}\right)\left\{1-\frac{1}{3^{3^{32}}}+\left(\frac{1}{3^{3^{32}}}\right)^2\right\}$

(d) $\left(1+\frac{1}{3^{3^{33}}}\right)\left\{1-\frac{1}{3^{3^{33}}}+\left(\frac{1}{3^{3^{33}}}\right)^2\right\}$

30. Let f be a real valued function of a real variable Define $g(x) = f(x) - (f(x))^2 + (f(x))^3 - (f(x))^4 + (f(x))^5$

$h(x) = f(x) - (f(x))^2 + (f(x))^3 - (f(x))^4$

Then which of the following statement is false?

(a) If f is increasing then so is g.

(b) If f is decreasing then so is g.

(c) h is neither increasing nor decreasing

(d) If f is increasing then the so is h.

SHORT ANSWER QUESTIONS

31. Given a line segment AB and a straight line l not containing it, find the point D on l at which AB subtends the greatest angle.

32. Which integers in the set $S = \{1, 2, 3, ..., 100\}$ can be written as difference of two squares? Find their number.

33. Evaluate $\int_0^2 \max\{\log((1+x^2), 1\}\, dx$

34. Let $f : (0, \infty) \to R$ be a function such that

(a) $f(x)$ is strictly increasing

(b) $f(x) - \frac{1}{x}$ for all $x > 0$

(c) $f(x) f\left(f(x) + \frac{1}{x}\right) = 1$ for all $x > 0$.

Determine f, with proof.

35. If a, b, c, are positive real numbers such that $abc = 1$ prove that

$$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \leq 1$$

36. Consider the quadratic equation $f(x) = 90x^2 + 20x + 1$. Find the sum of the digits of the number $f(1111111)$, when written in decimal notation.

37. Let n points be given on the periphery of a circle such that, when all the connecting chords are drawn, no these are concurrent. Let $p(x)$ denote the number of pieces into which the circle is cut by these chords, find an expression for $p(x)$.

38. In a triangle ABC, prove that

$$24\,Rr - 12r^2 \le a^2 + b^2 + c^2$$

Where R and r are the circumradius and inradius respectively and a, b, c the sides of the triangle.

39. Let a, b be two positive numbers, and let $f:(a, b) \to R$ be a continuous function differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that

$$\frac{1}{a-b}\left(a\,f(b) - b\,f(a)\right) = f(c) - c\,f'(c)$$

40. The sequence $\{a_n\}, n \ge 1$ is defined by $a_1 = 1, a_2 = 2, a_3 = 24$ and for $n \ge 4$, $a_n = \dfrac{6a_{n-1}^2\,a_{n-3} - 8a_{n-1}\,a_{n-2}^2}{a_{n-2}\,a_{n-3}}$.

Show that n divides a_n for all $n \in N$.

SOLUTIONS

Multiple Choice Question Type

1. **(c) :** We have $t_n = 2t_{n-1} - t_{n-2}, n \ge 3$

$\Rightarrow\ t_n - t_{n-1} = t_{n-1} - t_{n-2}$

Thus $a_n = a_{n-1}$, where $a_n = t_n - t_{n-1}, n \ge 3$

Thus $\{a_n\}$ is a constant sequence, we have

$a_n = a_{n-1} = a_{n-2} = \ldots = a_6 = \ldots = a_3$

Now $a_6 = t_5 - t_4 = 1$

Again $a_5 = 1 \Rightarrow t_4 - t_3 = 1 \Rightarrow 4 - t_3 = 1 \Rightarrow t_3 = 3$

Similarly $a_4 = 1 \Rightarrow t_3 - t_2 = 1 \Rightarrow 3 - t_2 = 1\ \therefore\ t_2 = 2$

Again $a_3 = 1 \Rightarrow t_2 - t_1 = 1 \Rightarrow 2 - t_1 = 1 \qquad \therefore\ t_1 = 1$

We get $t_n - t_{n-1} = 1, n \ge 3$

Also $t_1 = 1, t_2 = 2$

So, $\{t_n\}$ is an A.P. starting with $t_1 = 1$ and common difference being 1

$$\sum_{n=1}^{2009} t_n = 1 + 2 + \ldots + 2009 = \frac{2009 \times 2010}{2} = 2009 \times 1005 = 2019045$$

2. **(c) :** $n^2 = 1 + m + m^2$

If $m > 0$, then $m^2 < 1 + m + m^2 < 1 + 2m + m^2$

i.e. $m^2 < 1 + m + m^2 < (1+m)^2$. That is the number $1 + m + m^2$ lies between two consecutive square, and hence can't be a square.

If $m = 0$ then $n^2 = 1 \Rightarrow n = 1, -1$. Thus the ordered pair are $(0, 1), (0, -1)$.

Considering $m < -1$, we have $m^2 > m^2 + m + 1 > m^2 + 2m + 1$ and then there are no solution.

For $m = -1$, we have

$n^2 = 1 - 1 + 1 = 1 \quad \Rightarrow n = 1, -1$

The ordered pair solutions are $(-1, 1), (-1, -1)$

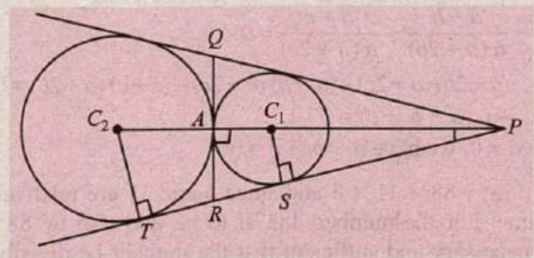
We summarize that the equation has exactly four integral ordered pair solution viz $(0, 1), (0, -1), (-1, 1)$ and $(-1, 1)$.

3. **(a) :** 8 points in a plane, no three of them being collinear, determine ${}^8C_3 = \dfrac{8 \cdot 7 \cdot 6}{6} = 56$ triangles. Assuming that all the 56 circumcircles corresponding to the triangles are different, we have the maximum possible number of circles as $M = 56$.

Again if these 8 points are themselves on a circle, then just one circle can be made to pass through them. This is the minimum possible number of circles. Thus $m = 1$, we then have $M - m = 56 - 1 = 55$.

4. **(b) :** Let C_1 and C_2 be the centre of the circle with radii 3 and 8 respectively.

Let the two circles touch each other externally at A and the three common tangents to these two circles intersect each other at P, Q and R as shown in the diagram.



We have $\Delta PC_1S \sim \Delta PC_2T$ (by A A similarity)

$\dfrac{PC_1}{PC_2} = \dfrac{C_1S}{C_2T} = 3/8\ \therefore\ PC_1 = \dfrac{3}{8}PC_2 = \dfrac{3}{8}(PC_1 + (11))$

$\Rightarrow\ 8PC_1 - 3PC_1 = 33\ \therefore\ PC_1 = 33/5$

$PA = PC_1 + AC_1 = \dfrac{33}{5} + 3 = 48/5$

$PS^2 = PC_1^2 - C_1S^2 = (33/5)^2 - 3^2 = \dfrac{33^2 - 15^2}{5^2} = \dfrac{48 \times 18}{25}$

$\Rightarrow\ PS = \dfrac{12\sqrt{6}}{5}$

Also $\Delta PSC_1 \sim \Delta PAR$ (by $A\,A$ similarity)

$\Rightarrow\ \dfrac{PS}{SC_1} = \dfrac{PA}{AR}\ \Rightarrow\ \dfrac{\frac{12\sqrt{6}}{5}}{3} = \dfrac{48/5}{AR}$

$\Rightarrow\ \dfrac{12\sqrt{6}}{15} = \dfrac{48}{5AR} = AR = 2\sqrt{6}\ \Rightarrow\ AR = 2\sqrt{6}$

area of $\Delta PQR = 2 \times$ area of $\Delta PAR = \dfrac{1}{2} \times 2 \cdot PA \cdot AR$

$= \dfrac{48}{5} \times 2\sqrt{6} = \dfrac{96\sqrt{6}}{5}\text{ cm}^2$

5. **(b) :** Let us use $f^k(n)$ to denote the composition f applied k times.

$f(14) = f(f(19)) = f^3(24) = \ldots f^{17}(94) = f^{18}(99)$

Note that $99 = 14 + (18-1)\cdot 5$, Again

$f^{19}(99) = f^{20}(104)$

Now let's unwind $f^{19}(104)$

$f^{19}(104) = f^{18}(f(104)) = f^{18}(101) = f^{17}(98)$

$f^{17}(98) = f^{18}(103)$

$f^{18}(103) = f^{17}(100) = f^{10}(97) = f^{17}(102) = f^{16}(99)$

Now $f^{18}(98) = f^{16}(99) = f^{14}(99) \ldots\ldots$

$= f^2(99) = f(f(98)) = f^3(103) = f(100) = 97$

6. **(b) :** From $\frac{1}{a}+\frac{1}{b}+\frac{1}{a+x}=0$ we have

$$\frac{1}{a+x}=-\frac{a+b}{ab} \Rightarrow a+x=-\frac{ab}{a+b}$$

$$\Rightarrow a+x=-\frac{ab}{a+b} \Rightarrow x=-a-\frac{ab}{a+b}=-\frac{a(a+2b)}{a+b}$$

Similarly from $\frac{1}{a}+\frac{1}{c}+\frac{1}{a+y}=0$ we have

$$y=-\frac{a(a+2c)}{a+c}$$

Substituting these values in

$\frac{1}{a}+\frac{1}{x}+\frac{1}{y}=0$, we have

$$\frac{1}{a}-\frac{a+b}{a(a+2b)}-\frac{a+c}{a(a+2c)}=0$$

$$\Rightarrow (a+2b)(a+2c)-(a+b)(a+2c)-(a+c)(a+2b)=0$$

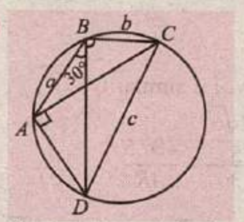
$$\Rightarrow -a(a+b+c)=0$$

As $a \neq 0$, we have $a+b+c=0$

7. **(a) :** $88 = 11 \times 8$ and since 8 and 11 are relatively prime. For the number $A837B$ to be divisible by 88 it is necessary and sufficient that the number be divisible by both 8 and 11. Divisibility by 8 implies that $37B$ is a multiple of 8 (which in term means that $5B$ is divisible by 8. Here $B = 6$). Again divisibility by 11 implies that $A+3+B$ are $8+7$ differ by a multiple of 11, that is, $9+A$ and is differ by a multiple of 11 or is zero. This mean $9+A-15=0 \Rightarrow A=6$

Thus A and B are both 6. The sum $A+B$ equals 12.

8. **(c) :**



We have from cosine rule in the triangle ABC,

$AC^2 = a^2 + b^2 - 2\,ab\cos 120° \Rightarrow a^2 + ab + b^2$

Also $\angle DAC = \angle DBC = 90°$, we have

$$c^2 = AC^2 \sec^2 30° = \frac{4}{3}(a^2+ab+b^2)$$

Again $c^2-(a+b)^2$

$$=\frac{4}{3}(a^2+ab+b^2)-(a^2+2ab+b^2)=\frac{(a-b)^2}{3} \geq 0$$

Thus $c \geq a+b$

Consider the product

$$p=(\alpha+\beta+\gamma)(\alpha-\beta-\gamma)(\alpha+\beta-\gamma)(\alpha-\beta+\gamma)$$

when $\alpha=\sqrt{c+a},\ \beta=\sqrt{c+b}, \gamma=\sqrt{c-a-b}$

Expanding the product

$$p=(c+a)^2+(c+b)^2+(c-a-b)^2-2(c+a)(c+b)$$
$$-2(c+a)(c-a-b)-2(c+b)(c-a-b)$$

Which on simplification reduce to

$$p=-3c^2+4a^2+4b^2+4ab=0$$

(Recall that $c^2=\frac{4}{3}(a^2+ab+b^2)$)

Thus at least one of the factors must be equal to zero. Since $\alpha+\beta+\gamma$ and $\alpha+\beta-\gamma$ are both positive, the product of remaining two factors is zero implying,

$\sqrt{c+a}-\sqrt{c+b}=\sqrt{c-a-b}$

or $\sqrt{c+b}-\sqrt{c+a}=\sqrt{c-a-b}$

Combining we have $\left|\sqrt{c+a}-\sqrt{c+b}\right|=\sqrt{c-a-b}$

9. **(a) :** Let $(\sqrt{n})=k$, then $k^2 < n < (k+1)^2$.

As k^3 divides n^2, we have k^2 divides n^2 and hence k divide n. The only possibilities for n are $n=k^2+k$ and $n=k^2+2k$.

CASE 1

Let $n=k^2+k$, then

$k^3 \mid n^2 \Rightarrow k^3 \mid (k^2+k)^2 = k^4+2k^3+k^2$

$\Rightarrow k^3 \mid k^2 \Rightarrow k=1$, Thus $n=2$

CASE 2

Let $n=k^2+2k$, then

$k^3 \mid n^2 \Rightarrow k^3 \mid (k^2+2k)^2 = k^4+4k^3+4k^2$

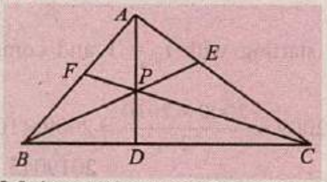
which implies that $k^3 \mid 4k^2$ or $k \mid 4$

Then $k = 1, 2$ or 4

Correspondingly $n = 3, 8, 24$

Hence $n = 2, 3, 8, 24$ are all positive integers satisfying the hypothesis of the problem.

10. **(b) :** 1[st] Solution :



Let $[x]$ denote the area of the figure x.

$$\text{Now } \frac{[ACF]}{[BCF]}=\frac{AF}{FB}=\frac{[APF]}{[BPF]}$$

$$\text{So } \frac{AF}{FB}=\frac{[ACF]-[APF]}{[BCF]-[BPF]}=\frac{[ACP]}{[BCP]} \quad ...(*)$$

$$\text{Again} \left[\frac{ABE}{CBE}\right]=\frac{AE}{EC}=\frac{[APE]}{[CPE]}$$

$$\Rightarrow \frac{AE}{EC}=\frac{[ABE]-[APE]}{[CBE]-[CPE]}=\frac{[ABP]}{[CBP]} \quad (**)$$

From (*) and (**) by addition

$$\frac{AF}{FB}+\frac{AE}{EC}=\frac{[ACP]+[ABP]}{[CBP]} \quad ...(1)$$

We now have

$$\frac{AP}{PD}=\frac{[ABP]}{[DBP]}=\frac{[CAP]}{[DCP]}$$

$$=\frac{[ABP]+[CAP]}{[DBP]+[DCP]}=\frac{[ABP]+[CAP]}{[CBP]} \quad ...(2)$$

from (1) and (2)

$$\frac{AF}{FB}+\frac{AE}{EC}=\frac{AP}{PD}$$

2nd Solution

Applying Ceva's theorem to the cevians AD, BE and CF which are concurrent at P, we have

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1$$

we have $\frac{AE}{EC}=\frac{AF}{FB}\cdot\frac{BD}{DC}$

Hence $\frac{AF}{FB}+\frac{AE}{AC}=\frac{AF}{FB}(1+\frac{BD}{DC})=\frac{AF}{FB}\cdot\frac{BC}{DC}$ ...(1)

Applying Menelau's theorem to triangle ABD, whose side are cut by the transversal FPC, we have

$$\frac{AF}{FB}\cdot\frac{BC}{DC}\cdot\frac{DP}{PA}=1$$

So we have $\frac{AF}{FB}\frac{BC}{PC}=\frac{AP}{PD}$ ...(2)

from (1) and (2), the result follows, viz

$$\frac{AF}{FB}+\frac{AE}{EC}=\frac{AP}{PD}$$

11. (b) : Let $p(x)=ax^3+bx^2+cx+d$

As $(x-1)$ divides $p(x)+2$ we have $p(1)+2=0$

$\Rightarrow\ a+b+c+d+2=0$

giving $d=-a-b-c-2$

Again $p(x)+2=a(x^3-1)+b(x^2-1)+c(x-1)$

$=(x-1)\{a(x^2+x+1)+b(x+1)+c\}$

Since $(x-1)^2$ divides $p(x)+2$, we find that

$(x-1)$ divides $a(x^2+x+1)+b(x+1)+c$

Thus implies $3a+2b+c=0$

Similarly $(x+1)^2$ divides $p(x)-2$, we get two more equations

$-a+b-c+d-2=0$ and $3a-2b+c=0$

Solving the system of equation obtained we have $b=d=0$ and $a=1$, $c=-3$.

Hence the requested polynomial is $p(x)=x^3-3x$, giving $p(3)=27-9=18$

12. (b) : Using modulo arithmetic, we have the last three digits of the number 7^n are given by finding 7^n modulo 1000.

Now $7^4\equiv 2401\equiv 40$ (modulo 1000)

$7^8\equiv 801$ (modulo 1000)

$7^{12}\equiv 201$ (mod 1000)

$7^{16}\equiv 601$ (mod 1000)

and $7^{20}\equiv 001$ (mod 1000)

This mean $7^{20k}=1$ (mod 1000)

Let $7^{9999}\equiv x$ (mod 1000)

Thus $7^{10000}\equiv(7^{20})^{500}\equiv 7x$ (modulo 1000)

mean $7x\equiv 1$ (modulo 1000)

$\Rightarrow\ 143\cdot 7x\equiv 143$ (mod 1000)

$\Rightarrow\ 1001\,x\equiv 143$ (mod 1000)

$\Rightarrow\ x\equiv 143$ (mod 1000)

Thus the last three digits of 7^{9999} are 143.

13. (c) : The problem amounts to finding m such that when $\underline{|100}$ is divided by 6^m, the quotient is not divisible by 6. In other words this is equivalent to finding the greatest m such that $\frac{\underline{|100}}{3m}$ is not divisible by 3.

The exponent of 3 in $\underline{|100}$ is

$$\left[\frac{100}{3}\right]+\left[\frac{100}{5}\right]+\left[\frac{100}{27}\right]+\left[\frac{100}{81}\right]$$
$$=33+11+3+1=48$$

Hence the number of zero's at the end of $\underline{|100}$ of decimal system expressed in base 6 will be 48.

14. (b) : Given any real number t, one can always find a real number $x\in(-\pi/2,\pi/2)$ such that $\tan x=t$. (Recall that the tangent function is strictly increasing on $(-\pi/2,\pi/2)$ and has range R)

We claim that the least value of n for which the assertion is true, is $n=5$.

Corresponding to given t_k, $b=1, 2, 5$ we can find $\tan x_k=t_k$, $b=1, 2, 5$ where $x_k\in(-\pi/2,\pi/2)$.

Divide the open interval $(-\pi/2,\pi/2)$ into four intervals $(-\pi/2,-\pi/4)$, $(-\pi/4,0)$, $(0,\pi/4)$, $(\pi/4,\pi/2)$

By pigeon hole principle, at least two of the number x_k must lie in the same interval. Suppose x_i and x_j lie in the same interval with $x_i>x_j$, thus

$0<x_i-x_j<\pi/4\ \Rightarrow\ 0<\tan(x_i-x_j)<\tan\pi/4$

$$=0<\frac{\tan x_i-\tan x_j}{1+\tan x_i\,\tan x_j}<1$$

Identifying $\tan x_i$ and $\tan x_j$ with a and b respectively we have

$$0<\frac{a-b}{1+ab}<1\ \Rightarrow\ 0<a-b<1+ab.$$

15. (b) : Let ϕ be the Euler-totient function,

$$\phi(92)=\phi(23\cdot 2^2)=23\cdot 2^2\left(1-\frac{1}{2}\right)\left(1-\frac{1}{2\cdot 3}\right)=44$$

from Euler's theorem

$19^{44}\equiv(1\bmod 92)$

Thus $19^{44}\equiv 1+92k$

Now $19^{92}=19^4\cdot 19^{88}=19^4(1+92k)^2$

Also $19^4\equiv 361^2=(-7)^2$ (mod 92) $\equiv 49$ (mod 92)

$\therefore\ \ 19^{92}\equiv 49$ (mod 92)

Thus the remainder when 19^{92} is divided by 92 is 49.

16. (a) : $f(15+x)=f(15-x)$

Changing x to $15-x$, we have

$f(30-x)=f(x)$

from the equation $f(30+x) = -f(30-x)$ we have on substitution

$f(x) = -f(30+x)$ ...(*)

changing x to $x + 30$ we have

$f(x+30) = -f(x+60)$ ...(**)

Thus (*) and (**) yield

$f(x) = f(x+60)$

Hence the function is periodic.

Again $f(30-x) = f(x)$ implies $f(30+x) = f(-x)$ ...(1)

(1) and (*) on comparison yield $f(-x) = -f(x)$

Thus f is odd.

Hence f is odd and periodic.

17. (c) : We have $r = 7$

and $R = 32.5 \Rightarrow 2R = 65$

Let a, b, c, be the side of the right angled triangle with $a^2 + b^2 = c^2$.

Now $\Delta = \frac{1}{2}ab$

Also $\dfrac{\frac{1}{2}ab}{\frac{a+b+c}{2}} = r \Rightarrow \dfrac{ab}{a+b+c} = r$

and $c = 2R$

The system is $\dfrac{ab}{a+b+2R} = r$ ...(*)

$a^2 + b^2 = 4R^2$

we have $(a+b)^2 - 2ab = 4R^2$

$\therefore\ a + b = \sqrt{4R^2 + 2ab}$ ...(**)

from (*) and (**) we have

$\dfrac{ab}{\sqrt{4R^2+2ab}+2R} = r \Rightarrow ab = 2Rr + r\sqrt{4R^2+2ab}$

$\Rightarrow\ ab = 2rR + r\sqrt{4R^2+2ab}$

$\Rightarrow\ (ab - 2rR)^2 = r^2(4R^2 + 2ab)$

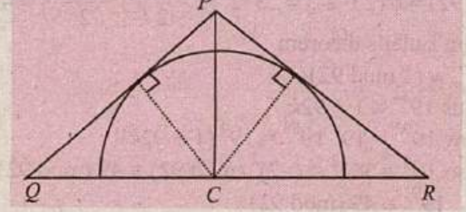
$\Rightarrow\ (a^2b^2 - 4rRab + 4r^2R^2 = 4r^2R^2 + 2abr^2$

$\Rightarrow\ ab - 4rR = 2r^2 \Rightarrow ab = 2r(r + 2R)$

$\therefore\ \frac{1}{2}ab = r(r+2R)$

Then area $= 7(7 + 65) = 7 \times 72 = 504$ cm^2

18. (a) :



Let C be the centre of the semicircle.

We have

$\frac{1}{2} \times r \times PR = [CPR]$ and $\frac{1}{2} \times r \times PQ = [CPQ]$

where $[x]$ denote the area of figure x.

Now adding we have

$\frac{1}{2} r(PQ + PR) = [PQR]$ ...(1)

$[PQR] = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{21 \cdot 12 \cdot 6 \cdot 3} = \sqrt{3 \cdot 7 \cdot 3 \cdot 4 \cdot 3 \cdot 2 \cdot 3} = 9 \times 2\sqrt{14}$

$= 18\sqrt{14}$

we have from (1)

$\frac{1}{2} \times r(15 + 9) = 18\sqrt{14} \Rightarrow 12r = 18\sqrt{14}$

$\therefore\ r = \frac{18}{12}\sqrt{14} = \frac{3}{2}\sqrt{14} = 3\sqrt{14/4} = 3\sqrt{7/2}$

19. (c) : Observe that $3^{3^{334}} + 1 = 3^{3^{333} \cdot 3} + 1$

$= \left(3^{3^{333}}\right)^3 + 1 = n^3 + 1$, where $n = 3^{3^{333}}$

Now $n^3 + 1 = (n+1)(n^2 - n + 1)$

Thus $3^{3^{334}} + 1 = \left(3^{3^{333}} + 1\right) \cdot k$

where k is a positive integer. Thus the g.c.d of

$3^{3^{333}} + 1$ and $3^{3^{334}} + 1$ is $3^{3^{333}} + 1$.

20. (a) : $n = 720 = 2^4 \cdot 3^2 \cdot 5$

The pure divisor of n are 2, 2^2, 2^3, 2^4, 3, 3^2, 5.

In the prime decomposition form of n

Only 5 occurs as a single power. Thus '5' is coprime with all the other pure divisors.

Thus $S_{\underline{|6}} = S_{720} = \{5\}$

21. (b) : Given $S_n = \{2, 3\}$

Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$

where $p_1, p_2, \ldots, p_k$ are distinct prime and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are non-negative integers.

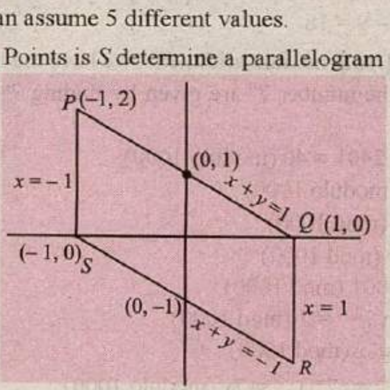
$S_n = \{2, 3\}$ if and only if the prime factorisation of n has the exponent of 2 and 3 as unity only and every other prime except 2 and 3 must have an exponent greater than 1.

Thus n can take the values (between 1 and 1000)

$2^1 3^1$ (= 6), $2^1 \cdot 3^1 \cdot 5^2$ (=150), $2^1 3^1 5^3$(= 750), $2^1 3^1 7^2$ (=294), $2^1 3^1 11^2$ (=726).

Here n can assume 5 different values.

22. (a) : Points is S determine a parallelogram



$|x + y| = 1$ is a set of parallel lines given by $x + y = 1$ and

$x+y=-1$

$|x|=1$ is a set of parallel lines given by $x=1$ and $x=-1$

Together these two pair of parallel line determine a parallelogram.

As the angle between adjacent side is not 90°, this is not a square.

The area of parallelogram = $[PQRS] = 2\cdot[PQS]$

$= 2\cdot\frac{1}{2}\times 2\cdot 2 = 4$ sq. units

23. (c) : Let x_1, x_2, x_3 be the angle of a triangle in T.

We have $x_1+x_2+x_3 = 180°$ ...(1)

$x_1, x_2, x_3 \geq 1$

Also $x_i's$ are integer, $(i = 1, 2, 3)$.

The number of unordered triplets (x_1, x_2, x_3) gives the number of elements in T.

The number of integral solutions, ordered of (1) is

$$^{177+3-1}C_{3-1} = {}^{179}C_2 = \frac{179.178}{2} = 15931$$

Let's count the number of unordered triplets $\{x_1, x_2, x_3\}$

There are three types of solution

1. No two of $x_i's$ are equal
2. Exactly two of $x_i's$ are equal, say $x_1 = x_2$
3. All three of $x_i's$ are equal *i.e.* $x_1 = x_2 = x_3$

Type of solutions	The number of solutions	The number of ordered solution
(x_1, x_2, x_3)	n	$6n$
(x_1, x_1, x_3)	$\left.\begin{matrix}(1,1,178)\\(2,2,176)\\ \vdots \\(89,89,1)\end{matrix}\right\}88$	3×88
(x_1, x_1, x_1)	(60, 60, 60) 1	$\frac{1}{6n+265}$

Note that (60, 60, 60) is not counted in (x_1, x_1, x_3).

Now $6n + 265 = 15931$

$$\Rightarrow\ n = \frac{15931-265}{6}\quad \therefore\ n = 2611$$

This is also the number of scalene triangles.

The total number of triangle = 2611 + 88 + 1 = 2700.

Note that of these 88 triangles are isosceles and 1 is equilateral. Thus there are 2700 triangles in T.

24. (c) : In problems of this type we try to construct the subset according to the given prescription.

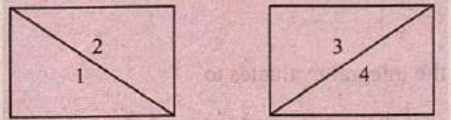
Since no element of A can be prime and every possible pair of elements are coprime, we choose an elements of A, p^2 where p is prime. Note that $31^2 = 961 < 1000$ and $37^2 = 1369 > 1000$. As 1 is coprime to any number, to get the largest subset 1 can be appended. Hence the desired subset A is

$A = \{1^2, 2^2, 3^2, 5^2, \ldots, 31^2\}$ which has 12 elements.

25. (c) :

The number of rectangle formed by the points belonging to the array is given by $^3C_2\cdot{}^3C_2 = 9$.

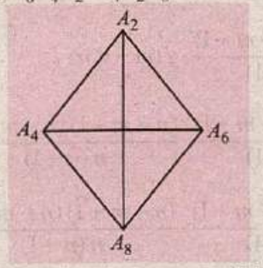
Each such rectangles given rise to 4 right triangle *i.e.*



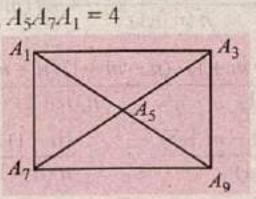
Thus the right triangle get from this type = 4 × 9 = 36

Again four more right triangle can be get from

$A_2A_6A_8, A_6A_8A_4, A_8A_4A_2, A_4A_2A_6 = 4$



Further four more triangles are possible, viz $A_5A_1A_3$, $A_5A_3A_9$, $A_5A_9A_7$, $A_5A_7A_1 = 4$



In total we have 36 + 4 + 4 = 44 right triangles.

26. (b) : $p(x)$ satisfies

$p(x^2) + 2x^2 + 10x = 2x\,p(x+1) + 3$

Let $p(x)$ be of degree n. As left side has degree $2n$ while right side has degree $(n+1)$. we have from the equality of polynomials $2n = n+1\ \therefore\ n = 1$

Set $p(x) = \lambda x + \mu$ to obtain

$\lambda x^2 + \mu + 2x^2 + 10x = 2x\,\{\lambda(x+1) + \mu\} + 3$

$\Rightarrow\ (\lambda+2)x^2 + 10x + \mu \equiv 2\lambda x^2 + 2(\lambda+\mu)x + 3$

Comparing the coeff. of x^2, x, x^0 we obtain $\lambda = 2$, $\mu = 3$. Thus $p(x)$ is the polynomial $2x+3$ and is of first degree. Hence there is unique polynomial satisfying the condition of the problem.

27. (d) : The integrand simplifies

$$\frac{\left(x+\frac{1}{x}\right)^6 - \left(x^6+\frac{1}{x^6}+2\right)}{\left(x+\frac{1}{x}\right)^3 + x^3 + \frac{1}{x^3}} = \frac{\left\{\left(x+\frac{1}{x}\right)^3\right\}^2 - \left(x^3+\frac{1}{x^3}\right)^2}{\left(x+\frac{1}{x}\right)^3 + \left(x^3+\frac{1}{x^3}\right)}$$

$$=\frac{\left\{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)\right\}\left\{\left(x+\frac{1}{x}\right)^3-\left(x^3+\frac{1}{x^3}\right)\right\}}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}$$

$$=\left(x+\frac{1}{x}\right)^3-\left(x^3+\frac{1}{x^3}\right)=x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)-\left(x^3+\frac{1}{x^3}\right)$$

$$=3\left(x+\frac{1}{x}\right)$$

Thus the integral evaluates to

$$\int 3\left(x+\frac{1}{x}\right)dx=3\int\left(x+\frac{1}{x}\right)dx=3\left(\frac{x^2}{2}+\ln|x|\right)+k$$

28. (c) : $f(m,n)=\left(1-\frac{m}{n}\right)\left(1-\frac{m}{n+1}\right)f(m-1,n)$

$$=\frac{(n-m)(n-m+1)}{n(n+1)}\cdot f(m-1,n)$$

$$=\frac{(n-m)(n-m+1)}{n(n+1)}\cdot\frac{(n-m+1)(n-m+2)}{n(n+1)}f(m-2,n)$$

$$=\frac{(n-m)(n-m+1)}{n(n+1)}\cdot\frac{(n-m+1)(n-m+2)}{n(n+1)}\cdot\frac{(n-m+2)(n-m+3)}{n(n+1)}f(m-3,n)$$

$$=\frac{(n-m)(n-m+1)}{n(n+1)}\cdot\frac{(n-m+1)(n-m+2)}{n(n+1)}\cdot\frac{(n-m+2)(n-m+3)}{n(n+1)}\ \cdots\cdots\ \frac{(n-1)n}{n(n+1)}f(0,n)$$

$$=\frac{(n-m)\{(n-m+1)(n-m+2)\ldots(n-1)\}^2\,n}{n^m(n+1)^m}\cdot 1$$

Setting $m=n-1$ we have

$$f(n-1,n)=\frac{1\cdot\{2.3\ldots.(n-1)\}^2\cdot n}{n^{n-1}(n+1)^{n-1}}=\frac{1\cdot(\lfloor n-1)^2\cdot}{n^{n-2}(n+1)^{n-1}}$$

Hence $f(2009,2010)=\dfrac{(\lfloor 2009)^2}{(2010)^{2008}\cdot(2011)^{2009}}$

29. (b) : $\underline{1^{st}\text{ Solution}}$

$S_n=a\left(\frac{1}{3}\right)^n+b$

We have

last term $=n^{th}$ term $=S_n-S_{n-1}, n\ge 2$

$$=\left\{a\left(\frac{1}{3}\right)^n+b\right\}-\left\{a\left(\frac{1}{3}\right)^{n-1}+b\right\}$$

$$=a\left\{\left(\frac{1}{3}\right)^n-\left(\frac{1}{3}\right)^{n-1}\right\}$$

$$=a\left(\frac{1}{3}\right)^{n-1}\left\{\frac{1}{3}-1\right\}=-\frac{2a}{3}\left(\frac{1}{3}\right)^{n-1}=-2a\left(\frac{1}{3}\right)^n$$

Thus $T_2=-\frac{2a}{9}$

Again common ratio $=\frac{T_n}{T_{n-1}}=\frac{1}{3}, n\ge 3$

As $\{T_n\}_{n\ge 1}$ is a geometric sequence we have

$T_2=\frac{1}{3}T_1 \Rightarrow -\frac{2a}{9}=\frac{1}{3}(S_1)$ (Recall $S_1=T_1$)

$=\frac{1}{3}\cdot\left(\frac{a}{3}+b\right) \Rightarrow -\frac{2a}{3}=\frac{a}{3}+b$

$\Rightarrow -\frac{2a}{3}-a/3=b \Rightarrow -a=b \therefore b=-a$

The sum to infinity

$$=\frac{\text{first term}}{1-\text{common ratio}}=\frac{\frac{a}{3}+b}{1-\frac{1}{3}}=1 \text{ (given)}$$

$\Rightarrow \frac{a}{3}+b=\frac{2}{3} \Rightarrow \frac{a}{3}-a=2/3$

$\Rightarrow -\frac{2a}{3}=2/3 \therefore a=-1$

which gives $b=1$.

$$S_{3^{33}}=1-\left(\frac{1}{3}\right)^{3^{33}}=1-\frac{1}{3^{3^{33}}}=1-\left(\frac{1}{3^{3^{32}}}\right)^3$$

$$=\left(1-\frac{1}{3^{3^{32}}}\right)\left[1+\frac{1}{3^{3^{32}}}+\left(\frac{1}{3^{3^{32}}}\right)^2\right]$$

$\underline{2^{nd}\text{ Solution}}$:

Recall that the formula for sum to n term of a *GP*

$$S_n=\frac{a(1-r^n)}{1-r}, n\ge 1$$

can be extended to include $n=0$ as well, for then $S_0=0$ which can be interpreted as the sum to zero term, *i.e.* no term, is zero.

So the given expression

$$S_n=a\left(\frac{1}{3}\right)^n+b, n\ge 1$$

can be extended to

$$S_n=a\left(\frac{1}{3}\right)^n+b, n\ge 0$$

with the understanding that $S_0=0$

we have $S_0=a+b=0$

giving $b=-a$

Again the sum to infinity $S_\infty=b$ only

Thus $b=1$ which yields $a=-1$.

Then the solution can be completed as in the first solution.

30. (d) : Let $p(t)=t-t^2+t^3-t^4+t^5$

then $p'(t)=1-2t+3t^2-4t^3+5t^4$

$=(1-t)^2+2t^2+2t^2(t^2-2t+1)+3t^4-2t^2$

$=(1-t)^2+2t^2(t^2-2t+1)+3t^4$

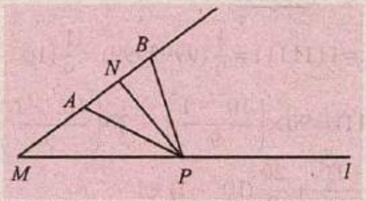
$= (1-t)^2 + 2t^2(1-t)^2 + 3t^4$
$= (1+2t^2)(1-t)^2 + 3t^4 > 0 \ \forall\ t \in R.$

This implies that $p(t)$ is strictly increasing for all t and therefore $g(x) = p(f(x))$ is increasing or decreasing according as $f(x)$ is increasing or decreasing.

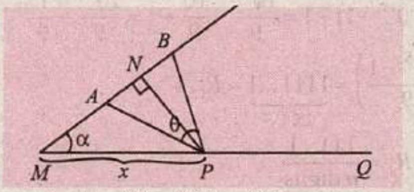
Observe that if we take $f(x) = x$. Then
$h(x) = x - x^2 + x^3 - x^4$
given $h(0) = 1 = h(1)$.

So a counter-example shows that h is a neither increasing nor decreasing.

31. Let AB (or AB produced) meet l at M.



For the moment, we assume that AB and l are not parallel.
Let P be any point on l as shown. PN the foot of perpendicular from P on AB.
Let $MA = a$ and $MB = b$, where a and b are known.
Let $MP = x$.



$MN = x\cos\alpha \quad \therefore \quad AN = MN - AM = x\cos\alpha - a$
Similarly $NB = MB - MN = b - x\cos\alpha$
Again $NP = x\sin\alpha$
$\tan\theta = \tan(\theta_1 + \theta_2)$ where $\theta_1 = \angle APN$ and $\theta_2 = \angle BPN$

$$\tan\theta_1 = \frac{AN}{NP} = \frac{x\cos\alpha - a}{x\sin\alpha}$$

$$\tan\theta_2 = \frac{NB}{NP} = \frac{b - x\cos\alpha}{x\sin\alpha}$$

$$\tan\theta = \frac{\dfrac{x\cos\alpha - a}{x\sin\alpha} + \dfrac{b - x\cos\alpha}{x\sin\alpha}}{1 - \dfrac{(x\cos\alpha - a)}{x\sin\alpha}\cdot\dfrac{(b - x\cos\alpha)}{x\sin\alpha}}$$

$$= \frac{(b-a)\,x\sin\alpha}{x^2\sin^2\alpha - \{-x^2\cos^2\alpha + (a+b)\,x\cos\alpha - ab\}}$$

$$= \frac{(b-a)\,x\sin\alpha}{x^2 - (a+b)\,x\cos\alpha + ab} = \frac{-(a-b)\sin\alpha}{x - (a+b)\cos\alpha + \dfrac{ab}{x}}$$

$$\tan\theta = \frac{-(a-b)\sin\alpha}{\left(x + \dfrac{ab}{x}\right) - (a+b)\cos\alpha}$$

for $\tan\theta$ to be greatest, the denominator should be minimum which occurs when $x + \dfrac{ab}{x}$ is a minimum, which by AM-GM inequality happen at $x = \dfrac{ab}{x} \Rightarrow x = \sqrt{ab}$.

The point P at which the segment AB subtends the greatest angle can be identified geometrically at the point T, the point of contact of l with the circle passing through A, B and touching l.

Note that this geometric interpretation still holds when the line segment AB is parallel to l, although the analysis would be different and can be simply done in the same fashion as outlined here.

32. Suppose that n can be written as a difference of two squares. Let $n = a^2 - b^2 = (a-b)(a+b)$.

Notice that $(a-b) + (a+b) = 2a$ = even, so $a - b$ and $(a+b)$ have the same parity *i.e.*, they are either both odd or both even.

Hence either n is odd or if n is even, it must be a multiple of 4.

These conditions are sufficient as well.

Write n as a product of cd, where c and d are both odd or both even and $c > d$.

Let $a = \dfrac{c+d}{2}, b = \dfrac{c-d}{2}$

We then have $n = a^2 - b^2 = \left(\dfrac{c+d}{2}\right)^2 - \left(\dfrac{c-d}{2}\right)^2 = cd$

For example $32 = 2\times 16 = \left(\dfrac{16+2}{2}\right)^2 - \left(\dfrac{16-2}{2}\right)^2 = 9^2 - 7^2$

The number of multiples of 4 in the set $S = \left(\dfrac{100}{4}\right) = 25$
The number odd number in the set $S = 50$

Hence there are 75 elements in S that can be written as a difference of two squares. The expression is not always unique.

33. Let $f(x) = \max\{\log((1+x^2), 1)\}$

$$\text{Thus } f(x) = \begin{cases} 1, & 0 \le x \le \sqrt{e-1} \\ \log(1+x^2), & \sqrt{e-1} \le x \le 2 \end{cases}$$

$$\text{Now } \int_0^2 \max\{\log((1+x^2), 1)\}\,dx$$

$$= \int_0^2 \max\{\log((1+x^2), 1)\}\,dx$$

$$\int_0^{\sqrt{e-1}} 1\,dx + \int_{\sqrt{e-1}}^2 \log(1+x^2)\,dx$$

$$= \sqrt{e-1} + [x\log(1+x^2)]_{\sqrt{e-1}}^2 - \int_{\sqrt{e-1}}^2 \frac{2x^2}{1+x^2}\,dx$$

$$= \sqrt{e-1} + 2\log 5 - \sqrt{e-1} - \int_{\sqrt{e-1}}^2 2dx + 2\int_{\sqrt{e-1}}^2 \frac{1}{1+x^2}\,dx$$

$$= \sqrt{e-1} + 2\log 5 - \sqrt{e-1} - 2\left(2 - \sqrt{e-1}\right) + 2\left[\tan^{-1}x\right]_{\sqrt{e-1}}^2$$

$$= 2\log 5 - 4 + 2\sqrt{e-1} + 2\tan^{-1}2 - 2\tan^{-1}\sqrt{e-1}$$

34. We have $f(x)+\frac{1}{x}>0$ from (b)

Changing x to $f(x)+\frac{1}{x}$ in the functional equation

$$f(x)\,f(f(x)+\frac{1}{x})=1 \qquad ...(1)$$

We have

$$f(f(x)+\frac{1}{x})+\left(f\left(f\left(f(x)+\frac{1}{x}\right)\right)+\frac{1}{f(x)+\frac{1}{x}}\right)=1$$

$$\Rightarrow \frac{1}{f(x)}\cdot f\left(\frac{1}{f(x)}+\frac{1}{f(x)+\frac{1}{x}}\right)=1 \quad \text{using (1)}$$

$$\Rightarrow f\left(\frac{1}{f(x)}+\frac{1}{f(x)+\frac{1}{x}}\right)=f(x)$$

As f is strictly increasing $\therefore\ f(a)=f(b)\Rightarrow a=b$

we thus have

$$\frac{1}{f(x)}+\frac{1}{f(x)+\frac{1}{x}}=x \Rightarrow \frac{2f+\frac{1}{x}}{f(f+\frac{1}{x})}=x$$

$$\Rightarrow 2f+\frac{1}{x}=xf^2+f \Rightarrow xf^2-f-\frac{1}{x}=0$$

$$\Rightarrow x^2f^2-xf-1=0$$

$$xf=\frac{1\pm\sqrt{1+4}}{2} \Rightarrow f(x)=\left(\frac{1\pm\sqrt{5}}{2}\right)\frac{1}{x}$$

As $f(x)$ is increasing, we have, $f(x)=\frac{1-\sqrt{5}}{2}\cdot\frac{1}{x}$

It is easy to check that the function indeed satisfies the hypothesis of the problem.

35. $a^5+b^5=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$

$=(a+b)\{a^4+a^2b^2+b^4-ab(a^2+b^2)\}$

$=(a+b)\{(a^2+ab+b^2)(a^2-ab+b^2)-ab(a^2+ab+b^2)+a^2b^2\}$

$=(a+b)\{(a^2+ab+b^2)(a^2-2ab+b^2)+a^2b^2\}$

$=(a+b)\{(a^2+ab+b^2)(a-b)^2+a^2b^2\}$

$\geq (a+b)a^2b^2 \quad \left[\because\ (a^2+ab+b^2)(a-b^2)\geq 0\right]$

Thus $a^5+b^5\geq(a+b)a^2b^2$

and equality holds when $a=b$.

$$\text{Thus}\quad \frac{ab}{a^5+b^5+ab}\leq\frac{ab}{a^2b^2(a+b)+ab}=\frac{1}{ab(a+b)+1}$$

$$=\frac{1}{ab(a+b)+abc}=\frac{1}{ab(a+b+c)} \qquad (\because\ abc=1)$$

$$=\frac{abc}{ab(a+b+c)}=\frac{c}{a+b+c} \qquad .....(i)$$

$$\text{Similarly}\quad \frac{bc}{b^5+c^5+bc}\leq\frac{a}{a+b+c} \qquad .....(ii)$$

$$\text{and}\quad \frac{ca}{c^5+a^5+ca}\leq\frac{b}{a+b+c} \qquad .....(iii)$$

Adding (i), (ii) and (iii) we get

$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca}$$

$$\leq\frac{a+b+c}{a+b+c}=1$$

And the equality holds when $a=b=c$, But $abc=1$, thus equality holds when $a=b=c=1$.

36. Let $R_7=\underbrace{1111111}_{7}$ is

$$\text{Now}\quad R_7=1111111=\frac{1}{9}(9999999)=\frac{1}{9}\left(10^7-1\right)$$

$$f(1111111)=90\times\left(\frac{10^7-1}{9}\right)^2+20\left(\frac{10^7-1}{9}\right)+1$$

$$=\frac{10(10^7-1)^2}{9}+\frac{20}{9}(10^7-1)+1$$

$$=\frac{10(10^7-1)}{9}\{10^7-1+2\}+1$$

$$=\frac{10}{9}(10^7-1)(10^7+1)+1$$

$$=\frac{10}{9}\left(10^{14}-1\right)+1=\frac{10^{15}}{9}-\frac{10}{9}+1=\frac{10^{15}}{9}-\frac{1}{9}$$

$$=\left(\frac{10^{15}-1}{9}\right)=\underbrace{1111...1}_{15\text{ 1's}}=R_{15}$$

$$\text{where } R_n=\frac{111..1}{n\text{ digits}}$$

The sum of digits of the number $f(1111111)=15$

37. At the start, when no chord have been drawn, there is just one region (the whole circle). At any intermediate stage, let us draw a new chord, and suppose it to be cut into k pieces by the chord already drawn. The number of new region created is $k+1$.

That is the number is 1 + number of intersections the chord has with the chords drawn earlier. Continuing in this manner, we establish that in the end the number of regions is one more than the sum of the number of chord and the number of points of intersection. The number of pieces, then is

$1+{}^nc_2+{}^nc_4$

$$\Rightarrow 1+\frac{n(n-1)}{2}+\frac{n(n-1)(n-2)(n-3)}{24}$$

$$=\frac{24+12n(n-1)+n(n-1)(n-2)(n-3)}{24}$$

$$=\frac{24+12n^2-12n+n^4-6n^3+11n^2-6n}{24}$$

$$=\frac{n^4-6n^3+23n^2-18n+24}{24}$$

38. If a, b, c are the sides of a triangle, there exists positive real x, y, z such that $a = y + z$, $b = z + x$, $c = x + y$

we have $s = \dfrac{(a+b+c)}{2} = (x+y+z)$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{xyz(x+y+z)}$$

$$r = \frac{\Delta}{s} = \frac{\sqrt{xyz(x+y+z)}}{x+y+z} = \sqrt{\frac{xyz}{x+y+z}}$$

$$R = \frac{abc}{4\Delta} = \frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}$$

We form $24Rr - 12r^2$

$$= 24\frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}\sqrt{\frac{xyz}{x+y+z}} - 12\frac{xyz}{x+y+z}$$

$$= \frac{6(x+y)(y+z)(z+x) - 12xyz}{x+y+z}$$

$$= \frac{6\{xy(x+y) + yz(y+z) + zx(z+x)\}}{(x+y+z)}$$

Again

$(x+y+z)(a^2+b^2+c^2) = (x+y+z)\{(x+y)^2 + (y+z)^2 + (z+x)^2\} = 2(x+y+z)(x^2+y^2+z^2+xy+yz+zx)$

$= 2(x^3+y^3+z^3+3xyz) + 4\{xy(x+y) + yz(y+z) + zx(z+x)\}$

$\geq 2\{xy(x+y) + yz(y+z) + zx(z+x)\} + 4\{xy(x+y) + yz(y+z) + zx(z+x)\}$

$= 6\{xy(x+y) + yz(y+z) + zx(z+x)\}$

Thus $a^2+b^2+c^2 \geq \dfrac{6\{xy(x+y) + yz(y+z) + zx(z+x)\}}{x+y+z}$

$\Rightarrow \; a^2+b^2+c^2 \geq 24Rr - 12r^2$

$\Rightarrow \; 24Rr - 12r^2 \leq a^2+b^2+c^2$

39. From Cauchy's mean value theorem between $F(x)$ and $G(x)$ we have

$$\frac{F(a)-F(b)}{G(a)-G(b)} = \frac{F'(c)}{G'(c)}$$

Applying Cauchy's *MVT* to $\dfrac{f(x)}{x}$ and $\dfrac{1}{x}$ we have a $c \in (a, b)$

such that

$$\frac{\left\{\dfrac{f(b)}{b} - \dfrac{f(a)}{a}\right\}}{\dfrac{1}{b} - \dfrac{1}{a}} = \frac{\dfrac{c\,f'(c) - f(c)}{c^2}}{-\dfrac{1}{c^2}} \qquad \ldots(1)$$

Note that $\left(\dfrac{f(x)}{x}\right) = \dfrac{xf' - f}{x^2}$

Also $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$

when rearranged (1), we get

$$\frac{af(b) - bf(a)}{a-b} = -\frac{cf'(c) - f(c)}{1}$$

Thus $\dfrac{af(b) - bf(a)}{a-b} = f(c) - cf'(c)$

40. Rewrite the given relation as

$$a_n = \frac{6a_{n-1}^2}{a_{n-2}} - \frac{8a_{n-1}a_{n-2}}{a_{n-3}}$$

We have $\dfrac{a_n}{a_{n-1}} = \dfrac{6a_{n-1}}{a_{n-2}} - \dfrac{8a_{n-2}}{a_{n-3}}$

Setting $b_n = \dfrac{a_n}{a_{n-1}}$ we have $b_2 = 2, b_3 = 12$

Also $b_n = 6b_{n-1} - 8b_{n-2}$

It can be shown that

$b_n = \dfrac{1}{4}4^n - \dfrac{1}{2}\cdot 2^n = 2^{n-1}\left(2^{n-1} - 1\right)$, for $n \geq 4$

Thus $\dfrac{a_2}{a_1}\cdot\dfrac{a_3}{a_2}\ldots\dfrac{a_n}{a_{n-1}} = b_2 \cdot b_3 \ldots b_n$

$$\Rightarrow \; a_n = 2^1\cdot 2^2 \ldots 2^{n-1}\prod_{k=1}^{n-1}(2^k - 1)$$

$$= 2^{n\frac{(n-1)}{2}}\prod_{k=1}^{n-1}(2^k - 1)$$

If $n = 2^l m$, where m is odd, then $l < n < \dfrac{n(n-1)}{2}$ and there exists $\phi(m) = k \leq m - 1$ such that m divides $2^k - 1$. This establishes that n divides a_n.

Form IV

1. Place of Publication	:	New Delhi
2. Periodicity of its publication	:	Monthly
3. Printer's and Publisher's Name	:	Mahabir Singh
Nationality	:	Indian
Address	:	Mathematics Today, 406, Taj Apartment, New Delhi - 110029.
4. Editor's Name	:	Mahabir Singh
Nationality	:	Indian
Address	:	Mathematics Today, 406, Taj Apartment, New Delhi - 110029.
5. Name and address of individuals who own the newspapers and partners or shareholders holding more than one percent of the total capital	:	Mahabir Singh 406, Taj Apartment New Delhi

I, Mahabir Singh, here by declare that particulars given above are true to the best of my knowledge and belief.
Mahabir Singh
Publisher

Mock Test PAPER 7

West Bengal JEE 2009

New Pattern

Exam on 19th April 2009

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SECTION - I

This section contains 80 multiple choice questions numbered 1 to 80. Each question has four choices of which one is correct. Each question carries +1 mark for correct answer and $-\frac{1}{3}$ mark for wrong answer.

1. If the roots of the equation $5x^2 - 7x + k = 0$ are reciprocal to each other, then the value of k is

   (a) 1 (b) $\frac{1}{5}$ (c) 5 (d) $-\frac{1}{7}$

2. The number of words that can be formed from the letters of the word ARTICLE so that the vowels occupy even places is

   (a) 360 (b) 574 (c) 300 (d) 144

3. Let $Y = \{1, 2, 3, 4, 5\}, A = \{1, 2\}, B = \{3, 4, 5\}$ and ϕ be the null set. If $A \times B$ denotes cartesian product of the sets A and B, then $(Y \times A) \cap (Y \times B)$ is equal to

   (a) Y (b) ϕ (c) A (d) B

4. $\frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + \ldots\ldots\ldots\ldots =$

   (a) $\frac{e^{-1}}{2}$ (b) e (c) $\frac{e}{4}$ (d) $\frac{e}{6}$

5. If $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \alpha \in R$, then $[F(\alpha)]^{-1}$ is equal to

   (a) $F(-\alpha)$ (b) $F(\alpha^{-1})$
   (c) $F(2\alpha)$ (d) none of these.

6. If $0 < x < 1$,

   then $\sqrt{1+x^2}\left[\left\{x\cos\left(\cot^{-1}x\right) + \sin^{-1}\left(\cot^{-1}x\right)\right\}^2 - 1\right]^{1/2}$ is equal to

   (a) $\frac{x}{\sqrt{1+x^2}}$ (b) x
   (c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

7. If $\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$, then the value of x is

   (a) 13 (b) ± 13 (c) $\pm\frac{13}{7}$ (d) $\frac{13}{7}$

8. If $\tan\alpha = (1+2^{-x})^{-1}$ and $\tan\beta = (1+2^{x+1})^{-1}$, then the value of $(\alpha+\beta)$ is

   (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

9. If a, b, c are the sides of the triangle ABC such that $a^4 + b^4 + c^4 = 2c^2(a^2 + b^2)$, then the angle opposite to the side c is

   (a) 45° or 135° (b) 30° or 120°
   (c) 60° or 150° (d) none of these.

10. The period of $2\sin\left(3\pi x + \frac{\pi}{4}\right) + 3|\cos 5\pi x|$ is

    (a) 1 (b) 2 (c) 3 (d) 5

11. If p^{th} term of an A.P. is q and q^{th} term is p, then 10^{th} term is

    (a) $p - q + 10$ (b) $p + q + 11$
    (c) $p + q - 9$ (d) $p + q - 10$

12. Out of 8 sailors on a boat, 3 can work on one side and 2 only on the other side. The number of ways the sailors can be arranged on the boat is

    (a) 1700 (b) 1720
    (c) 1728 (d) 1736

13. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, then $q =$

    (a) 49/4 (b) 4 (c) 3 (d) 12

14. If $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta - \frac{\pi}{3}\right) = K\tan 3\theta$, then K is equal to

    (a) 1 (b) 3
    (c) $\frac{1}{3}$ (d) none of these.

15. Let $R = \{(1, 3), (2, 4), (4, 2), (2, 3), (3, 1)\}$ be a relation on the set $A = \{1, 2, 3, 4\}$. Then the relation R on A is

By : Sankar Ghosh, HOD (Math), Takshyashila.

(a) $\left(\frac{3^{n+1}-3-2^n}{4}\right)$ (b) $\left(\frac{3^{n+1}+3-2^n}{4}\right)$

(c) $\left(\frac{3^{n+1}+3+2^n}{4}\right)$ (d) $\left(\frac{3^{n-1}-3+2^n}{4}\right)$

32. The eccentric angle of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ whose distance from the centre of the ellipse is 2, is

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{2}$ (c) $\frac{5\pi}{3}$ (d) $\frac{7\pi}{6}$

33. Two teams are to play a series of 5 matches between them. A match ends in a win or loss or draw for a team. A number of people forecast the result of each match and no two people make the same forecast for the series of matches. The smallest group of people in which one person forecasts correctly for all the matches, will contain 'n' people, where 'n' is

(a) 81 (b) 243

(c) 486 (d) none of these

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34. Odds 8 to 5 against a person who is 40 years old living till he is 70 and 4 to 3 against another person who is 50 years old will be living till he is 80. Probability that one of them will be alive next 30 years is

(a) 59/91 (b) 44/91

(c) 51/91 (d) 32/91

35. Solution to the differential equation $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$ where

(a) $\frac{y-x}{1+xy}=K$ (b) $xy=K$

(c) $x^2-y^2=K$ (d) none of these

36. If z is a non-real root of $\sqrt[7]{-1}$, then $z^{86}+z^{175}+z^{289}$ is equal to

(a) 0 (b) -1 (c) 3 (d) 1

37. The number of ways in which 6 different balls can be put in two boxes of different sizes so that no box remains empty is

(a) 62 (b) 64

(c) 36 (d) none of these

38. The number of even proper divisors of 1008 is

(a) 23 (b) 24

(c) 22 (d) none of these

39. The solution of the differential equation

$2x+\frac{dy}{dx}-y=3$; given $y(0)=-1$ represents

(a) straight line (b) circle

(c) parabola (d) ellipse

40. If the three consecutive coefficients in the expansion of $(1+x)^n$ are 28, 56 and 70, then 'n' equals

(a) 6 (b) 4 (c) 8 (d) 10

41. $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots\infty}}}$, then $\frac{dy}{dx}=$

(a) $\frac{2y-1}{\cos x}$ (b) $\frac{\cos x}{2y-1}$

(c) $\frac{2x-1}{\cos y}$ (d) $\frac{\cos y}{2y-1}$

42. The equation of the hyperbola referred to its axis as axis of co-ordinate and whose distance between the foci is 16 and eccentricity is $\sqrt{2}$, is

(a) $x^2-y^2=16$ (b) $x^2-y^2=32$

(c) $x^2-2y^2=16$ (d) $x^2-y^2=-16$

43. If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, then $\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=$

(a) $\frac{a_2}{a_2+a_3}$ (b) $\frac{a_2}{2(a_2+a_3)}$

(c) $\frac{2a_2}{a_2+a_3}$ (d) $\frac{2a_3}{a_2+a_3}$

44. $\frac{d}{dx}\left[\tan^{-1}(\log_{10}x)\right]=$

(a) $\frac{1}{[1+(\log_{10}x)]^2\cdot x\log_e 10}$

(b) $\frac{1}{1+(\log_{10}x)^2}$

(c) $\frac{1}{[1+(\log_{10}x)^2]\cdot x\log_{10}e}$

(d) none of these

45. If e, e' be the eccentricities of two conics $S=0$ and $S'=0$ and if $e^2+e'^2=3$ then both S and S' are

(a) hyperbolas (b) ellipses

(c) parabolas (d) none of these

SOLUTIONS

1. (d)	2. (d)	3. (b)	4. (c)	5. (c)
6. (d)	7. (b)	8. (b)	9. (b)	10. (a)
11. (a)	12. (b)	13. (d)	14. (a)	15. (c)
16. (b)	17. (b)	18. (a)	19. (a)	20. (b)
21. (b)	22. (b)	23. (a)	24. (a)	25. (b)
26. (c)	27. (a)	28. (b)	29. (a)	30. (c)
31. (a)	32. (a)	33. (b)	34. (b)	35. (a)
36. (b)	37. (a)	38. (a)	39. (a)	40. (c)
41. (b)	42. (b)	43. (c)	44. (a)	45. (a)

MOCK TEST PAPER

ISI 2009

Exam on 10th May 2009

Indian Statistical Institute

*Alok Kumar, BTech, IIT KANPUR

The Indian Statistical Institute (ISI), Kolkata, is considered as one of the foremost centres in the world for training and research in statistics and the related sciences. The B.Stat (Hons) degree program, the flagship programme of the institute, offers comprehensive instruction in the theory, method and application of statistics, in addition to several areas of Mathematics and some basic areas of computer science.

Each candidate applying for admission to this programme has to take a selection test comprising Objective type and Short-answer type questions in mathematics at the Higher Secondary level (10 + 2 year's programme).

The selection tests consist of

(1) A multiple choice type test having about 30 questions, and

(2) A short-answer type test having about 10 questions.

Questions will be set on the following and related topics.

Algebra : Sets, operations on sets, prime numbers, factorization of integers and divisibility, rational and irrational numbers, permutations and combinations, binomial theorem, logarithms, theory of quadratic equations, polynomial and remainder theorem, arithmetic and geometric progressions, inequalities involving A.M., G.M., and H.M., complex numbers.

Geometry : Plane geometry of class X level. Geometry of 2 dimensions with cartesian and polar co-ordinates. Concept of a locus, equation of a line, angle between two lines, distance from a point to a line. Areas of a triangle, equations of a circle, parabola, ellipse and hyperbola and equations of their tangents and normals, mensuration.

Trigonometry : Measures of angles, trigonometric and inverse trigonometric functions, trigonometric identities including addition formulae, solutions of trigonometric equations. Properties of triangles, heights and distances.

Calculus : Functions, one-one functions, onto functions, limits and continuity, derivatives and methods of differentiation, slope and curve, tangents and normals, maxima and minima, use of calculus in sketching graph of functions, methods of integration, definite and indefinite integrals, evaluation of area using integrals.

Logical Reasoning : Consistency of statements.

In response to growing demand from students preparing for the ISI, we bring to you the second Mock ISI paper, which closely simulates the real exam.

Multiple Choice Type Questions

1. Let $R = \dfrac{48^{52} - 46^{52}}{96^{26} + 92^{26}}$. Then R satisfies

(a) $R < 1$ (b) $23^{26} < R < 24^{26}$
(c) $1 < R < 23^{26}$ (d) $R > 24^{26}$

2. The number of positive integers $n \le 99$ such that $1 + x^{100} + x^{200} + ... + x^{100n}$ is divisible by $1 + x + x^2 + ... + x^n$ is

(a) 38 (b) 40 (c) 39 (d) 41

3. The number of pairs of integers (m, n) satisfying $m^2 + mn + n^2 = 1$ is

(a) 8 (b) 6 (c) 4 (d) 2

4. From a group of 20 persons, belonging to an association, a president, a secretary and three members are to be elected for the executive committee. The number of ways this can be done, is

(a) 310080 (b) 155040
(c) 620160 (d) 77520

5. The side of a triangle ABC satisfy $a^2 = b(b + c)$. Also the degree measure of an angle B is 75°. The degree measure of an angle $\angle BAC$ is

(a) 90° (b) 120° (c) 150° (d) 100°

6. The sum of digits of the number $100^{13} - 26$, written in decimal notation is

(a) 227 (b) 218 (c) 228 (d) 219

7. Let $S = \{(x_1, x_2, x_3) | 0 \le x_i \le 9$ and $x_1 + x_2 + x_3$ is

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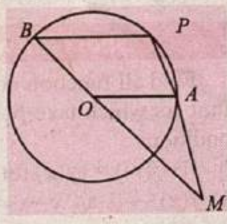
divisible by 3}. Then the number of elements in S is
(a) 334 (b) 333 (c) 327 (d) 336

8. A 3-digit number in base 9, when expressed in base 11 has its digits reversed. The number of such numbers is
(a) 0 (b) 2 (c) 3 (d) 5

9. From a point P on the circle shown with centre O, the chord $PA = 8$ cm is drawn. The radius of the circle is 24 cm. Let PB be drawn parallel to OA. Suppose BO extended meet PA extended at M. The length of MA equals (in cm)



(a) 16 (b) 8 (c) 10 (d) 9

10. Let x and y be positive reals with $x < y$. Also $0 < b < a < 1$. Define $E = \log_a\left(\frac{y}{x}\right) + \log_b\left(\frac{x}{y}\right)$. Then E can't take the value
(a) -2 (b) -1
(c) $-\sqrt{2}$ (d) 2

11. Let $S = \{(x, y) | (x-1)^2 + (y+2)^2 \le 25 \text{ and } x, y \in Z\}$. The number of elements in S is
(a) 81 (b) 80 (c) 85 (d) 102

12. The greatest common divisor (gcd) of $2^{2^{22}} + 1$ and $2^{2^{222}} + 1$ is
(a) 1 (b) $2^{2^{22}} + 1$
(c) $2^{2^{11}} - 1$ (d) $2^{2^{21}} - 1$

13. A function f is said to be odd if $f(-x) = -f(x)$ for all x. Which of the following is not odd ?
(a) A function f such that $f(x+y) = f(x) + f(y)$ for all x, y
(b) $f(x) = \dfrac{xe^{x/2}}{1+e^x}$
(c) $f(x) = x - [x]$, where $[x]$ denotes the greatest integer which is less than or equal to x.
(d) $f(x) = x^2 \sin x + x^3 \cos x$

14. The $\lim\limits_{x\to 0} \dfrac{\cos x - \sec x}{x^2(1+x)}$ is
(a) -1 (b) 1
(c) 0 (d) doesn't exist

15. Let S be the set of all 3-digit numbers such that
(i) the digits in each number are all from the set $\{1, 2, 3, ..., 9\}$
(ii) exactly one digit in each number is even
The sum of all numbers in S is
(a) 96100 (b) 133200
(c) 66600 (d) 99800

16. Let $y = \dfrac{x}{x^2+1}$. Then $y^4(1)$, where $y^n(x)$ denotes the n^{th} derivative of x equals
(a) 4 (b) -3 (c) 3 (d) -4

17. The number of real roots of the equation
$1 + \dfrac{x}{1} + \dfrac{x^2}{2} + \dfrac{x^3}{3} + ... + \dfrac{x^7}{7} = 0$ (without factorial) is
(a) 7 (b) 5 (c) 3 (d) 1

18. Let $f(m, n) = 36^m - 5^n$, with m and n being positive integers. The smallest value of $|f(m, n)|$ is
(a) 1 (b) 11 (c) 5 (d) 9

19. Which of the following is false ?
(a) The square of any integer when divided by 3 leaves no remainder or a remainder of 1
(b) The square of any integer, when divided by 5 leaves no remainder or a remainder of 1 or a remainder of 4
(c) The square of an odd integer, when divided by 8 leaves a remainder of 1
(d) p and $p^2 + 8$ are primes for infinitely many primes p.

20. Let z be a complex number satisfying $z - 1/z = i$. Then a possible value of $z^{2009} + \dfrac{1}{z^{2009}}$ is
(a) i (b) $-\sqrt{3}$ (c) $2\sqrt{3}$ (d) $-i$

21. OA is a radius of a circle with centre at O. R is a point on OA through which a chord CD perpendicular to OA is drawn. Let a chord through A meet the chord CD at M and the circle at B. Also S is the perpendicular from O on chord AB. The radius of circle is 18 cm. R is the midpoint of AO and $AM/MB = 1/2$. The length of OS (in cm) is
(a) $12\sqrt{3}$ (b) 9 (c) $9\sqrt{3}$ (d) 12

22. Let $a = \dfrac{\dfrac{(\log_{1000} 100)(\log_3(\log_{27} 3))}{(\log_{36} \log_3(729))}}{\log_3 9 + \log_9 3}$

Then a equals to
(a) $-2/15$ (b) $2/15$
(c) $4/15$ (d) $-4/15$

23. The number of ways in which a player can be dealt a hand of 13 cards, if all four suits are to be represented in the deal is
(a) ${}^{52}C_{13} - 3 \cdot {}^{39}C_{13} + 4 \cdot {}^{26}C_{13} - 3$
(b) $\sum\limits_{\substack{a,b,c,d \\ a+b+c+d=13}} \binom{13}{a}\binom{13}{b}\binom{13}{c}\binom{13}{d}$ where $0 \le a, b, c, d < 13$,
(c) ${}^{52}C_{13} - 4 \cdot {}^{39}C_{13} + 6 \cdot {}^{26}C_{13} - 4$
(d) $\sum\limits_{a=1}^{10} \sum\limits_{b=1}^{11-a} \sum\limits_{b=1}^{13-a-b} \binom{13}{a}\binom{13}{b}\binom{13}{c}\binom{13}{13-a-b-c}$

24. Let $R_n = \underbrace{1111...1}_{n}$. Then the remainder when R_{813} is divided by 29 is

(a) 24 (b) 11 (c) 1 (d) 9

25. Let x, y, z be different from 1 satisfying $x+y+z=2007, xy+yz+zx=4011$.

Then the value of $\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}$ is

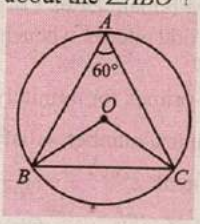
(a) 0 (b) 1

(c) 2008 (d) $\frac{1}{2008}$

26. The number $a7389b$, a, b are digits, is divisible by 72, then $a+b$ equals

(a) 10 (b) 9 (c) 11 (d) 12

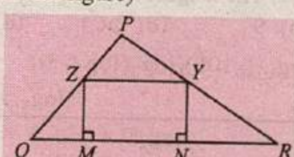
27. Let O be the centre of the circle and ABC be an inscribe triangle ($\angle BAC = 60°$). Which of the following is necessarily true about the $\angle ABO$?



(a) $60° < \angle ABO < 90°$ (b) $0 < \angle ABO < 60°$

(c) $\angle ABO = 30°$ (d) $0 < \angle ABO < 30°$

28. Let QPR be a right angle triangle, right angled at P. Let $\frac{PZ}{ZY}=\frac{PY}{YR}$. $PQ = 6$ cm, $PR = 8$ cm. If the ratio of area of ΔPZY and ΔPQR is 2 : 5, then the area of rectangle $ZMNY$ is (see in the figure)



(a) $48\sqrt{5}\{2-\sqrt{2}\}$ cm² (b) $48\left\{\sqrt{\frac{2}{5}}-\frac{2}{5}\right\}$cm²

(c) $24\left\{\sqrt{\frac{2}{5}}-\frac{2}{5}\right\}$ cm² (d) $24\sqrt{5}\{2-\sqrt{2}\}$cm²

29. In a special version of chess, a rook moves either horizontally or vertically on the chessboard. The number of ways to place 8 rooks of different colours on a 8×8 chessboard such that no rook lies on the path of the other rook at the start of the game is

(a) $8 \times \underline{|8}$ (b) $\underline{|8} \times \underline{|8}$

(c) $2^8 \times \underline{|8}$ (d) $2^8 \times {}^{64}C_8$

30. The co-efficient of x^{100} in the expansion of $(1+x)(1-2x)(1+3x)(1-4x)\ldots\ldots(1+101x)$ is

(a) $\underline{|101}\left\{\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{100}\right)-\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{101}\right)\right\}$

(b) $\underline{|101}\left\{\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{101}\right)-\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{100}\right)\right\}$

(c) $\left\{\left(1+\frac{1}{3}+\ldots+\frac{1}{101}\right)-\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{100}\right)\right\}$

(d) $\left\{\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{100}\right)-\left(1+\frac{1}{3}+\ldots+\frac{1}{101}\right)\right\}$

Short Answer Type Questions

31. Find all function f defined on the set of positive real numbers which take positive real values and satisfy the conditions :

(i) $f(xf(y)) = yf(x)$ for all positive x, y ;

(ii) $f(x) \to 0$ as $x \to \infty$.

32. In a triangle ABC, prove that

$$a(s-a)+b(s-b)+c(s-c) < 9Rr.$$

where a, b, c are the sides of the triangle, s the semi-perimeter and R, r respectively the circum-radius and in-radius.

33. Define $N = 2^n(2^{n+1}-1), n \in N$ such that $2^{n+1}-1$ is a prime number. Prove that

(i) the sum of the divisors of N is $2N$

(ii) the sum of the reciprocals of the divisors of N is 2

34. Find all real numbers satisfying

$$6^x + 2^{2x} - 36^x + 24^x - 16^x = 1$$

35. (i) Do there exist function $f : R \to R$ and $g : R \to R$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^3$ for all $x \in R$?

(ii) Do there exist functions $f : R \to R$ and $g : R \to R$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^4$ for all $x \in R$?

36. Let P_n be the number of permutation of n symbols $(1, 2, 3, \ldots, n-1, n)$ in which no element occupies a place that is its own. Such a permutation is known as a 'Derangement'. Let us agree to set $P_0 = 1$. Prove that

$$\underline{|n} = {}^nC_0 \cdot P_n + {}^nC_1 \cdot P_{n-1} + \ldots + {}^nC_{n-1}P_n + {}^nC_n \cdot P_0$$

37. Two boxes contain between them 6 balls of several different sizes. Each ball is white, black, red or yellow. If you take any five balls of the same colour, at least two of them will always be of the same size (radius). Prove that there are at least three balls which lie in the same box, have the same colour and are of the same size.

38. The diagonals AC and CE of a regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that $\frac{AM}{AC}=\frac{CN}{CE}=k$.

Determine k if B, M, N are collinear.

39. Find all continuous function $f : (0, \infty) \to (0, \infty)$ such that $f(1) = 1$ and $\frac{1}{2}\int_0^x (f(t))^2 dt = \frac{1}{x}\left(\int_0^x f(t)dt\right)^2$.

40. Prove that a triangle has maximum area given the length of a side and the angle opposite it, when it is isosceles.

SOLUTIONS

MULTIPLE CHOICE TYPE QUESTIONS

1. **(b)** : $R=\dfrac{48^{52}-46^{52}}{96^{26}+92^{26}}$

When $R=\dfrac{(2\cdot 24)^{52}-(2\cdot 23)^{52}}{(4\cdot 24)^{26}+(4\cdot 23)^{26}}=\dfrac{2^{52}(24^{52}-23^{52})}{4^{26}(24^{26}+23^{26})}$

$$=\frac{2^{52}}{2^{52}}\cdot\frac{(24^{26})^2-(23^{26})^2}{24^{26}+23^{26}}$$

$$=\frac{(24^{26}+23^{26})(24^{26}-23^{26})}{24^{26}+23^{26}}=24^{26}-23^{26}$$

As $R=24^{26}-23^{26}<24^{26}$

$\therefore\quad R<24^{26}$ ...(1)

Also $R=24^{26}-23^{26}=(1+23)^{26}-23^{26}$

$=23^{26}+{}^{26}C_1\cdot 23^{25}+{}^{26}C_2\cdot 23^{24}+...+1-23^{26}$

$=26\times 23^{25}+{}^{26}C_2\times 23^{24}+...+1$

$>26\cdot 23^{25}>23\cdot 23^{25}=23^{26}$

thus $R>23^{26}$ ...(2)

combining (1) and (2) we get

$23^{26}<R<24^{26}$

2. **(c)** : We have

$$1+x^{100}+x^{200}+x^{300}+...+x^{100n}=\frac{(x^{100})^{n+1}-1}{x^{100}-1}$$

Also $1+x+x^2+...+x^n=\dfrac{x^{n+1}-1}{x-1}$

$$\therefore\quad \frac{1+x^{100}+x^{200}+...+x^{100n}}{1+x+x^2+...+x^n}=\frac{(x^{100})^{n+1}-1}{x^{100}-1}\cdot\frac{x-1}{x^{n+1}-1}$$

$$=\frac{x^{100(n+1)}-1}{x^{100}-1}\cdot\frac{x-1}{x^{n+1}-1} \quad \text{........ (A)}$$

The key ideas is to apply a result regarding the roots of unity. If k and l are relatively prime (positive integers) then x^k-1 and x^l-1 has exactly one common, *viz* $x-1$.

Thus from (A) the given quotient to be polynomial, it's necessary that $n+1$ and 100 should be relatively prime. This condition is sufficient as well.

Note that $x^{100}-1\,|\,(x^{100})^{n+1}-1$ and also $x^{n+1}-1\,|\,(x^{n+1})^{100}-1$.

Hence the number of positive integers $n\le 99$, such that $n+1$ is relatively prime to 100 is the answer to our problem.

As $100=2^2\cdot 5^2$

$$\phi(100)=100\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right)=100\cdot\frac{1}{2}\cdot\frac{4}{5}=40$$

Of these 40 values '1' is a value which can't be taken by $n+1$. Thus $(n+1)$ can take 39 values and since corresponding to every value of $n+1$ there is a value of n and vice-versa a bijection, n can take 39 distinct values.

3. **(b)** : $m^2+mn+n^2=1$

As the equation is symmetric in m and n, we make the substitution $u=m+n$ and $v=m-n$

so that $u^2+v^2=2(m^2+n^2)$ and $u^2-v^2=4mn$

Multiplying the given equation by 4

$4m^2+4mn+4n^2=4$

$\Rightarrow\quad 4(m^2+n^2)+4mn=4 \qquad \Rightarrow\quad 2(u^2+v^2)+u^2-v^2=4$

$\Rightarrow\quad 3u^2+v^2=4$

Set $u^2=r$, $v^2=s$ with $r, s\ge 0$, we have $3r+s=4$

The ordered pairs (r, s) satisfying the above equation in integers are $(0, 4)$ and $(1, 1)$

We then have

$u^2=0$ and $u^2=1$

$v^2=4$ and $v^2=1$

i.e. $u=0, v=2$; $\quad u=0, v=-2$; $u=1,\ v=1$;

$u=1, v=-1$; $\quad u=-1, v=1$ and $u=-1, v=-1$

giving 6 ordered pair solutions (m, n) *viz*

$(1,-1), (-1, 1), (1, 0), (0, 1), (0,-1), (-1, 0)$

4. **(a)** : **1st solution** : The president can be elected in ${}^{20}C_1$ ways, following which the secretary can be elected in ${}^{19}C_1$, and then the three members in ${}^{18}C_3$ ways.

The number of ways $={}^{20}C_1\cdot{}^{19}C_1\cdot{}^{18}C_3=20\cdot 19\cdot\dfrac{18\cdot 17\cdot 16}{6}$

$=20\cdot 19\cdot 3\cdot 17\cdot 16$

$=310080$

2nd solution : The number of ways of dividing 20 persons into three groups, two groups consisting of 1 person each, one group consisting of 3 persons and yet another 1 group consisting of 15 persons is

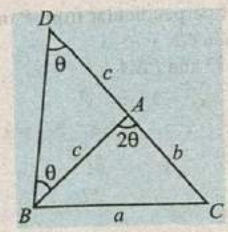
$$=\frac{\underline{|20}}{\underline{|1}\,\underline{|1}\,\underline{|3}\,\underline{|15}}\cdot\frac{1}{\underline{|2}}$$

Now the 3-member group can serve as the three member, 5 to be elected for the executive committee of the two 1-member group, one can serve as president and the other as secretary. Then the number of ways

$$=\left\{\frac{\underline{|20}}{\underline{|1}\,\underline{|1}\,\underline{|3}\,\underline{|15}}\cdot\frac{1}{\underline{|2}}\right\}\cdot\underline{|2}$$

$$=\frac{\underline{|20}}{\underline{|1}\,\underline{|1}\,\underline{|3}\,\underline{|15}}$$

5. **(c)** :



Produce CA to D such that $AD=AB$

$\therefore\quad \angle ABD=\angle ADB$

$\angle BAC=\angle ABD+\angle ADB=2\theta$

from ΔACB and ΔBCD (as $a^2=b(b+c)$)

$$\frac{a}{b+c}=\frac{b}{a}\quad\Rightarrow\quad\frac{CB}{CD}=\frac{AC}{BC}$$

and $\angle C$ is common

So the triangles are similar.

$\angle CBA=\angle CDB=\angle B$ Also $\angle ADB=\angle ABD$

$\angle BAC=\angle ADB+\angle ABD$ (Exterior angle)

$\angle BAC=2\angle ADB=2\angle CDB$

$=2\angle B$

given $\angle B=75°\ \therefore\ \angle BAC=150°$

6. **(a)** : $100^{13}-26=10^{26}-26=1\underbrace{000000}_{26\text{ zeroes}}-26$

$=\underbrace{99999.....9}_{24\text{ 9's}}74$

The sum of digits $=24\times 9+7+4$

$=216+11=227$

7. **(a)** : With each (x_1, x_2, x_3) identify a three digit code, where reading zeroes are allowed. We have a bijection between S and the set of all non-negative integers less than or equal to 999 divisible by 3.

The number of numbers between 1 and 999, inclusive, divisible by 3 is $\left(\frac{999}{3}\right) = 333$

Also '0' is divisible by 3.

Hence the number of element in S = 333 + 1 = 334

8. **(a)** : Let abc be the number in base 9 we have $(abc)_9 = (cba)_{11}$

$\Rightarrow\ 81a + 9b + c = 121c + 11b + a$

$\Rightarrow\ 120c + 2b - 80a = 0 \Rightarrow\ 60c + b - 40a = 0$

$\Rightarrow\ 40a - 60c = b \ \Rightarrow\ 20(2a - 3c) = b$

But $0 \le b < 9$

So $b = 0$, implying $2a - 3c = 0$

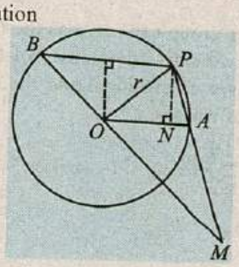
Again $0 \le a, c \le 9$ yield the solution

$a = 3, a = 6$

$c = 2, c = 4$

The two possible solutions are $(302)_9$ and $(609)_9$.

9. **(d)** : 1st Solution



Let $PA = \lambda r, 0 < \lambda < 1$

Let N be the foot of perpendicular from P on AO.

Suppose $AN = x$, then $ON = r - x$

In right triangle PNO and PNA

$$PN^2 = r^2 - (r - x)^2 = \lambda^2 r^2 - x^2$$

$$\Rightarrow\ 2rx - x^2 = \lambda^2 r^2 - x^2 \quad \Rightarrow\ 2rx = \lambda^2 r^2$$

$$\therefore\ x = \frac{\lambda^2 r}{2}$$

$OA \parallel BP$,

$$\Rightarrow\ \frac{MA}{MP} = \frac{AO}{PB} \quad \Rightarrow\ \frac{MA}{MA + AP} = \frac{AO}{PB}$$

$$\Rightarrow\ 1 + \frac{AP}{MA} = \frac{PB}{AO} \quad \Rightarrow\ \frac{AP}{MA} = \frac{PB}{AO} - 1$$

$$\Rightarrow\ MA = \frac{AP}{\frac{PB}{AO} - 1} = \frac{\lambda r}{\frac{2(r-x)}{r} - 1}$$

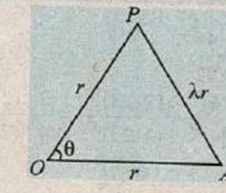
$$= \frac{\lambda r}{1 - \frac{2x}{r}} = \frac{\lambda r}{1 - \frac{2\lambda^2 r}{2 \cdot r}} = \frac{\lambda r}{1 - \lambda^2}$$

Now $r = 24$ cm, $AP = 8$ cm, we have

$$MA = \frac{\frac{1}{3} \cdot 24}{1 - \frac{1}{9}} = \frac{3}{8} \cdot 24 = 9 \text{ cm}$$

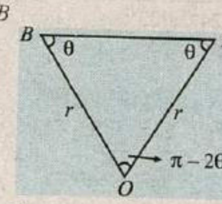
2nd solution

From the cosine rule in triangle AOP



$$\lambda r = 2r \sin\frac{\theta}{2} \ \Rightarrow\ \lambda = 2\sin\frac{\theta}{2}$$

Again from ΔPOB



$$PB = 2r\sin\left(\frac{\pi - 2\theta}{2}\right) = 2r\sin\left(\frac{\pi}{2} - \theta\right) = 2r\cos\theta$$

$$= 2r\left\{1 - 2\sin^2\frac{\theta}{2}\right\} = 2r\left\{1 - 2 \cdot \frac{\lambda^2}{4}\right\}$$

$$= 2r\left(1 - \frac{\lambda^2}{2}\right) = r(2 - \lambda^2)$$

Again as $PB \| OA$, we have

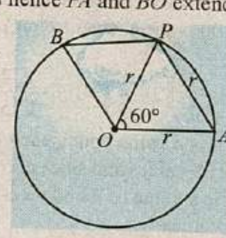
$$\Rightarrow\ \frac{MP}{MA} = \frac{PB}{OA}$$

$$\Rightarrow\ \frac{MA}{OA} = \frac{MP}{PB} = \frac{MP - MA}{PB - OA} = \frac{PA}{PB - OA}$$

$$\Rightarrow\ MA = \frac{OA}{PB - OA} \cdot PA = \frac{r}{r(2 - \lambda^2) - r} \cdot \lambda r$$

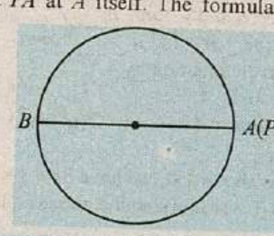
$$= \frac{\lambda r^2}{(1 - \lambda^2)r} = \frac{\lambda r}{1 - \lambda^2}$$

Remarks : When $\lambda = 1$, *i.e.* when $PA = r$, we have $OAPB$ as a parallelogram and hence PA and BO extended don't meet.



The formula for $\lambda = 1$ yields $MA \to \infty$.

Also when $\lambda = 0$, *i.e.* when A and P become coincident, BO extended meet PA at A itself. The formula for $\lambda = 0$ yields $MA = 0$



10. **(d)** : $E = \log_a\left(\frac{y}{x}\right) + \log_b\left(\frac{x}{y}\right) = \log_a\left(\frac{y}{x}\right) + \log_b\left(\frac{y}{x}\right)^{-1}$

$$= \log_a \frac{y}{x} - \log_b \frac{y}{x} = \frac{\log \frac{y}{x}}{\log a} - \frac{\log \frac{y}{x}}{\log b}$$

$$= \log\left(\frac{y}{x}\right)\left\{\frac{1}{\log a} - \frac{1}{\log b}\right\} = \log\left(\frac{y}{x}\right)\left\{\frac{\log b - \log a}{(\log a)(\log b)}\right\}$$

$$= \log\left(\frac{y}{x}\right)\frac{\log\left(\frac{b}{a}\right)}{(\log a)(\log b)} = -\log\left(\frac{y}{x}\right)\cdot\frac{\log\left(\frac{a}{b}\right)}{(\log a)(\log b)}$$

Now $0 < a < 1$, $0 < b < 1$ $\therefore$ $\log a$ and $\log b$ are both –ve.

Also $\frac{y}{x} > 1$ and $\frac{a}{b} > 1$. Thus $\log\left(\frac{y}{x}\right)$ and $\log\left(\frac{a}{b}\right)$ are both positive. Finally E turns out to be a negative quantity. Thus E can't take the value '2'.

11. **(a)** : The number of elements in $S = \{(x, y) | (x-1)^2 + (y+2)^2 \le 25$ and $x, y > \in z\}$ is same as the number of lattice points, the points having both its co-ordinates integers satisfying $x^2 + y^2 \le 25$

Let $T = \{(x, y) | x^2 + y^2 \le 25, x, y \in z\}$. Then we have $|T| = (S)$ we must do some systematic counting.

In general if (x, y) is a solution of $x^2 + y^2 \le 25$, then so are $(x, -y)$, $(-x, y)$, $(-x, -y)$, (y, x), $(y, -x)$, $(-y, x)$ and $(-y, -x)$. So any solution, in this way can generate a maximum of 8 solutions. Considering the change of sign and an interchange of x and y.

Let's count our solutions

With $a, b > 0$ $(a < b)$

any solution (a, b) indeed realizes all possible eight solution, *viz*, $(a, -b)$, $(-a, b)$, $(-a, -b)$, (b, a), $(b, -a)$, $(-b, a)$, $(-b, -a)$.

$$\left.\begin{matrix}(1, 2)\\(1, 3)\\(1, 4)\\(2, 3)\\(2, 4)\\(3, 4)\end{matrix}\right\} \rightarrow 6 \times 8 = 48 \text{ solutions}$$

With $a, b > 0$, $a = b$; any solution (a, a) gives rise to (a, a), $(a, -a)$, $(-a, a)$, $(-a, -a)$ 4 solutions

$$\left.\begin{matrix}(1, 1)\\(2, 2)\\(3, 3)\end{matrix}\right\} \rightarrow 3 \times 4 = 12 \text{ solutions}$$

With $b > 0$ any solution $(0, b)$ gives rise to $(0, -b)$, $(b, 0)$ and $(-b, 0)$, *i.e* in all 4 solutions

$$\left.\begin{matrix}(0, 1)\\(0, 2)\\(0, 3)\\(0, 4)\\(0, 5)\end{matrix}\right\} \rightarrow 5 \times 4 = 20 \text{ solutions}$$

Also $(0, 0)$ doesn't produce any other solution

$(0, 0)\} \rightarrow 1$ solution

The number of solutions $= 48 + 12 + 20 + 1 = 81$

12. **(a)** : Let $F_n = 2^{2^n} + 1$, with $m < n$

$$F_n - 2 = 2^{2^n} + 1 - 2$$

$$= 2^{2^n} - 1 = (2^{2^{n-1}})^2 - 1 = (2^{2^{n-1}} + 1)(2^{2^{n-1}} - 1)$$

$$= (2^{2^{n-2}} + 1)(2^{2^{n-2}} - 1)(2^{2^{n-1}} - 1)$$

$$= (2^{2^m} + 1)(2^{2^m} - 1)\ldots(2^{2^{n-1}} - 1) = \lambda F_m$$

Now $F_n - \lambda F_m = 2$

let $d | F_n$ and $d | F_m$ then $d | 2$. Then $d = 1$ or 2. But F_m and F_n are both odd. Hence $gcd = 1$.

13. **(d)** : $f(x + y) = f(x) + f(y)$ for all x, y

Let $x = y = 0$

$\Rightarrow$ $f(0 + 0) = f(0) + f(0)$ $\quad\Rightarrow$ $f(0) = f(0) + f(0)$

$\therefore$ $f(0) = 0$

Again replacing y with $-x$, we have

$f(x - x) = f(x) + f(-x)$

$\Rightarrow$ $f(0) = f(x) + f(-x)$

$\Rightarrow$ $f(x) + f(-x) = 0$

$\therefore$ $f(-x) = -f(x)$. Thus f is odd

Again for $f(x) = \frac{xe^{x/2}}{1+e^x}$, we have

$$f(-x) = \frac{(-x)(e^{-x/2})}{1+e^{-x}} = \frac{(-x)(e^{-x/2})\cdot e^x}{1+e^x}$$

$$= -\frac{xe^{x/2}}{1+e^x} = -f(x)$$

establishing that f is odd

for $f(x) = x^2\sin x + x^3\cos x$

$f(-x) = (-x)^2\sin(-x) + (-x)^3\cos(-x)$

$= -x^2\sin x - x^3\cos x = -f(x)$, implying that f is odd

But $f(x) = x - [x]$ is not odd. To prove it we take a counter-example.

$f(-2.3) = -2.3 - [-2.3]$

$= -2.3 - (-3) = 3 - 2.3 = 0.7$

$f(2.3) = 2.3 - [2.3] = 2.3 - 2 = 0.3$

$f(-2.3) \ne -f(2.3)$

thus f is not odd.

14. **(a)** :

$$\lim_{x\to 0}\frac{\cos x - \sec x}{x^2(x+1)} = \lim_{x\to 0}\frac{\cos x - \frac{1}{\cos x}}{x^2(x+1)} = \lim_{x\to 0}\frac{\cos^2 x - 1}{\cos x \cdot x^2(x+1)}$$

$$= \lim_{x\to 0}\frac{-\sin^2 x}{\cos x\cdot(x^2)(x+1)} = -\lim_{x\to 0}\frac{1}{\cos x}\cdot\left(\frac{\sin x}{x}\right)^2\cdot\frac{1}{x+1}$$

$$= -1\cdot 1\cdot 1 = -1$$

15. **(b)** : **1st solution**

Let abc be a number in S.

Then all possible arrangments of x, y and z will also be numbers in S.

Sum of all possible arrangements of a, b and c is

$$(abc)_{10} + (acb)_{10} + (bca)_{10} + (bac)_{10} + (cab)_{10} + (cba)_{10}$$
$$= 200(a + b + c) + 20(a + b + c) + 2(a + b + c)$$
$$= 222(a + b + c)$$

All the digits in S contain exactly 1 even digit. It can be one of 2, 4, 6 or S.

The sum of the other two digits can be

$1 + 3 = 4$, $1 + 5 = 6$, $1 + 7 = 8$, $1 + 9 = 10$

$3 + 5 = 8$, $3 + 7 = 10$, $3 + 9 = 12$

$5 + 7 = 12$, $5 + 9 = 14$, $7 + 9 = 16$

Let 'a' denote the even digit, the sum of all elements of S for a given value of a is

$222(a+4)+222(a+6)+222(a+8)+222(a+10)$
$+222(a+8)+222(a+10)+222(a+12)+222(a+12)$
$+222(a+14)+222(a+16)$
$=222(10a+100)=2220(a+10)$

Now the sum of all number in $S = \sum_a 2220(a+10)$

$=2220\{(2+4+6+8)+10\cdot 4\}$
$=2220\{60\}=133200$

2nd solution

The sum of digits in unit place of all the numbers in S will be same as the sum in tens or hundreds place.

The only even digit can have any of the three positions, *i.e.* ${}^3C_1$, ways.

And the digit itself has 4 choices (2, 4, 6 or 8). The other two digits can be filled in $5\cdot 4=20$ ways.

Then the number of numbers in $S=240$

Number of numbers containing the even digit in units place = $4\times 5\times 4=80$

So the other 160 numbers have digits 1, 3, 5, 7 or 9 in units place, with each digit appearing $\frac{160}{5}=32$ times

Sum in units place $=32(1+3+5+7+9)+20(2+4+6+8)$

$=32\cdot 5^2+20\cdot 2\cdot\frac{4\cdot 5}{2}$

$=32\cdot 25+20\cdot 20=1200$

The sum of all numbers $=1200(1+10+10^2)$
$=1200\cdot 111=133200$

16. (b) : Simply differentiating would be tedious. We take advantage of 'i' the square root of '-1'

$$y=\frac{x}{x^2+1}=\frac{1}{2}\left\{\frac{1}{x-i}+\frac{1}{x+i}\right\}$$

$$\frac{d^4y}{dx^4}=\frac{1}{2}\left\{\frac{\underline{|4}}{(x-i)^5}+\frac{\underline{|4}}{(x+i)^5}\right\}$$

Note that $\frac{d^n}{dx^n}\left\{\frac{1}{x+a}\right\}=\frac{(-1)^n\underline{|n}}{(x+a)^{n+1}}$

So, $y^4(x)=\frac{\underline{|4}}{2}\left\{\frac{1}{(x-i)^5}+\frac{1}{(x+i)^5}\right\}$

$$y^4(1)=12\left\{\frac{1}{(1-i)^5}+\frac{1}{(1+i)^5}\right\}=12\left\{\frac{1-i}{(-2i)^3}+\frac{1+i}{(2i)^3}\right\}$$

$$=12\left\{\frac{1-i}{8i}+\frac{1+i}{-8i}\right\}=12\left(-\frac{1}{8}-\frac{1}{8}\right)=-3$$

17. (d) : First we will prove a lemma

the polynomial $f(x)=1+\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\frac{x^6}{6}$ has no real zeroes.

Let f has a minimum at $x=x_0$, then $f'(x_0)=0$

$\Rightarrow\ 1+x_0+x_0^2+x_0^3+x_0^4+x_0^5=0$

$\Rightarrow\ \frac{x_0^6-1}{x_0-1}=0 \quad\Rightarrow\quad \frac{(x_0^3-1)(x_0^3+1)}{(x_0-1)}=0$

$\Rightarrow\ (x_0^2+x_0+1)(x_0^2-x_0+1)(x_0+1)=0$

which has a real root $x_0=-1$

But $f(-1)=1-1+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\frac{1}{6}>0$

Then $f(x)>0$ and hence f has no real zeroes

Let $g(x)=1+\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+...+\frac{x^7}{7}$

An odd degree polynomial has at least one real root. If our polynomial g has more than one zero say x_1 and x_2 then by Rolle's theorem in (x_1, x_2) we have a 'x_3' such that

$g'(x_3)=0$

$\Rightarrow\ 1+x_3+x_3^2+...+x_3^6=0$

But this has no real zeroes.

Hence the given polynomial $1+\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+...+\frac{x^7}{7}$ has exactly one real zero.

18. (b) : The unit digit of 5^n is 5 for all positive integers n and the unit digit of 36^m is 6.

The unit digit of $|f(m,n)|$ is 1 or 9 depending upon which of 5^n and 36^m is greater.

The least possible value of $|f(m,n)|$ can be 1.

Note that $|f(m,n)|=1 \Rightarrow 36^m-5^n=\pm 1$

Modulo '5' the above congruence reads

$1=\pm 1 \pmod 5$ *i.e.* $2=0 \pmod 5$ [impossible]

The next possible value can be 9.

$36^m-5^n=\pm 9 \Rightarrow 36^m\mp 9=5^n$

which reads modulo '3' [impossible]

$0\equiv(-1)^n \pmod 3$

Continuing in this fashion, the next possible value of $|36^m-5^n|$ would be 11. We show that 11 is realized also so it is indeed the least value that $|f(m,n)|$ can take $|f(1,2)|=36-25=11$.

19. (d) : We will prove the result in the language of congruence.

Any integer with respect to modulo 3 has three form

$x=3m+1, x=3m, x=3m-1$

In all cases $x^2\equiv 1 \pmod 3$ or $x^2=0 \pmod 3$

Again an integer x with respect to modulo 5 has 5 form

$x=5m, x=5m+1, x=5m+2, x=5m-2, x=5m-1$

$x^2\equiv -1, 0, 1 \pmod 5$

Lastly an odd integer is of the form $2k+1$

we have $(2k+1)^2=4k^2+4k+1\equiv\frac{8k(k+1)}{2}+1\equiv 1 \pmod 8$

Note that $\frac{k(k+1)}{2}$ is an integer for all $k\in Z$.

Statement (4) is false. $s+p\neq 3$, then $3+p$ and $p^2\equiv 1 \pmod 3$ thus $3/p^2+8$. Hence $p=3$ is the only prime.

20. (b) : $z-\frac{1}{z}=i \quad\Rightarrow z^2-iz-1=0$

$\Rightarrow\ -(iz)^2-(iz)-1=0$

$\Rightarrow\ (iz)^2+(iz)+1=0$

Identifying this with the relation $1+\omega+\omega^2=0$, ω being a non-real cube roots of unity, we have

$iz=\omega, \omega^2$

Taking z to be '$-i\omega$' $\therefore z=-i\omega, -i\omega^2$

$$z^{2009}+\frac{1}{z^{2009}}=(-i\omega)^{2009}+\frac{1}{(-i\omega)^{2009}}$$

$$=(-i)^{2009}\cdot\omega^{2009}+\frac{1}{(-i)^{2009}}\cdot\frac{1}{(\omega)^{2009}}$$

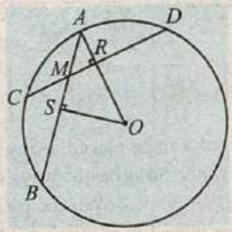
$$(-i)^{2008+1}\cdot(\omega)^{2007+2}+\frac{1}{(-i)^{2008+1}}\cdot\frac{1}{(\omega)^{2007+2}}$$

$$= (-i)(\omega^2) + \frac{1}{(-i)}\cdot\frac{1}{\omega^2} = -i\omega^2 + i\omega$$
$$= i(\omega - \omega^2) = i(i\sqrt{3}) = -\sqrt{3}$$

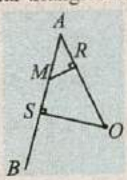
Similarly taking $z = -i\omega^2$, we have

$$z^{2009} + \frac{1}{z^{2009}} = i(\omega^2 - \omega) = \sqrt{3}$$

21. (b) :



The idea is the spot similar triangles in this mess of a diagram.

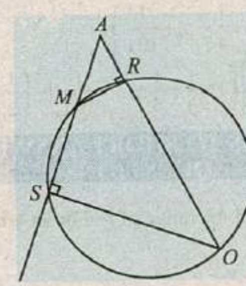


The triangles ARM and ASO are similar triangles. Thus

$$\frac{AM}{AO} = \frac{AR}{AS} \quad \Rightarrow AR\cdot AO = AM\cdot AS$$

The above result follows rather beautifully if one realizes that $ORMS$ is a cyclic quadrilateral.

We have the following figure and then $AM\cdot AS = AR\cdot AO$ follows



$$\frac{AM}{MB} = \frac{1}{2} \quad \Rightarrow \quad \frac{AM}{1} = \frac{MB}{2} = \frac{AM+MB}{3} = \frac{AB}{3}$$

$$\therefore \quad AM = \frac{1}{3}AB$$

Also $AR = 9$ cm, $AO = 18$ cm

We have $AR\cdot AO = AM\cdot AS$

$$\Rightarrow \quad 9.18 = \frac{1}{3}AB\cdot\frac{1}{2}AB$$

$$\Rightarrow \quad AB^2 = 3\cdot 2\cdot 9\cdot 18 \qquad \therefore \quad AB = 18\sqrt{3}$$

Now $AS = \frac{1}{2}AB = 9\sqrt{3}$

We have $OS^2 = AO^2 - AS^2 = 18^2 - (9\sqrt{3})^2$
$= 18^2 - 9^2\cdot 3 = 9^2\{2^2 - 3\} = 9^2$

$\therefore \quad OS = 9$ cm

22. (a) : $a = \dfrac{(\log_{1000}100)(\log_3(\log_{27}3))(\log_{36}(\log_3(729)))}{\log_3 9 + \log_9 3}$

$$= \frac{(\log_{10^3}10^2)(\log_3(\log_{3^3}3))(\log_{36}(\log_3 3^6))}{\log_3 3^2 + \log_{3^2}3}$$

$$= \frac{\frac{2}{3}\cdot\left(\log_3\frac{1}{3}\right)(\log_{36}6)}{2+\frac{1}{2}} = \frac{\frac{2}{3}\cdot(-1)\log_{6^2}6}{\frac{5}{2}}$$

$$= \frac{2}{3}\cdot\frac{2}{5}(-1)\cdot\frac{1}{2} = -\frac{2}{15}$$

23. (d) : This is an application of principle of inclusion and exclusion (PIE).

Let A = set of all cards having no Clubs
B = " " " Spades
C = " " " Hearts
D = " " " Diamonds

We have

$n(A) = n(B) = n(C) = n(D) = {}^{39}C_{13}$
$n(A\cap B) = n(A\cap C) = n(A\cap D) = n(B\cap C) = n(B\cap A)$
$= n(C\cap D) = {}^{26}C_{13}$
$n(A\cap B\cap C) = n(A\cap B\cap D) = n(A\cap C\cap D) = n(B\cap C\cap D)$
$= 1$
$n(A\cap B\cap C\cap D) = 0$

Applying PIE we have

$n(A\cup B\cup C\cup D)$
$= \Sigma n(A) - \Sigma n(A\cap B) + \Sigma n(A\cap B\cap C) - n(A\cap B\cap C\cap D)$
$= 4\cdot{}^{39}C_{13} - 6\cdot{}^{26}C_{13} + 4 - 0$
$= 4\cdot{}^{39}C_{13} - 6\cdot{}^{26}C_{13} + 4$

We want $n(A'\cap B'\cap C'\cap D') = n(S) - n(A\cup B\cup C\cup D)$
$= {}^{52}C_{13} - 4\cdot{}^{39}C_{13} + 6\cdot{}^{26}C_{13} - 4$

Another way to look at the problem is to consider this. Let a, b, c, d be the number of cards of the suits clubs, spades, hearts and diamonds respectively. The number of deals then is

$$\binom{13}{a}\cdot\binom{13}{b}\binom{13}{c}\binom{13}{d}$$

Also we must have $a, b, c, d \in N$ and $1 \le a, b, c, d \le 13$, $a+b+c+d = 13$ as a, b, c, d are all positive integer, we must have $a \le 10$, $a+b \le 11$ and $a+b+c \le 12$. Suppose we fix 'a', then b can be no more than $11 - a$. Suppose we fix b with $1 \le b \le 1 + a$. Then c can be no more than $12 - a - b$. Suppose we fix c, with $1 \le c \le 12 - a - b$, then d gets fixed, $d = 13 - a - b - c$.

So the expression is

$$\sum_{a=1}^{10}\sum_{b=1}^{11-a}\sum_{c=1}^{12-a-b}\binom{13}{a}\binom{13}{b}\binom{13}{c}\binom{13}{13-a-b-c}$$

24. (c) : We first prove a result, if $p \ge 7$ is a prime, then $R_{p-1} = \underbrace{1111...1}_{(p-1)\text{ 1's}}$ is divisible by p.

we have $R_{p-1} = \dfrac{10^{p-1}-1}{9}$

As p and 10 are coprime to each other, Fermat's little theorem applies $\therefore$ p is a divisor of $10^{p-1} - 1$.

we have $R_{p-1} = \dfrac{p\cdot\alpha}{9}$

where $p\alpha = 10^{p-1} - 1$

As 9 divides $10^{p-1} - 1$, so $9/p\alpha$. But 9 and p are coprime, hence $9/\alpha$.

Thus $R_{p-1} = kp$

Thus we have proved that $p(\geq z$, a prime, divides $R_{p-1})$.

Thus R_{28} is divisible by 29. As $813 = 28 \times 29 + 1$, the remainder. When R_{813} is divided by 29, the remainder is 1.

25. (a) :

$$\frac{1}{(1-x)}+\frac{1}{(1-y)}+\frac{1}{(1-z)}=\frac{(1-y)(1-z)+(1-x)(1-z)+(1-x)(1-y)}{(1-x)(1-y)(1-z)}$$

$$=\frac{\Sigma 1-(y+z)+yz}{(1-x)(1-y)(1-z)}=\frac{3-2(x+y+z)+(xy+yz+zx)}{(1-x)(1-y)(1-z)}$$

$$=\frac{3-2\cdot 2007+2011}{(1-x)(1-y)(1-z)}=0$$

26. (b) : $72 = 8 \times 9$, and 8 and 9 are coprime. As the number $a7389b$ is divisible by 72, it is divisible by both 9 and 8. For divisibility by 8, the last three digits must be divisible by 8, *i.e.* $800 + 80 + 10 + b$ should be divisible by 8, so $b + 2$ should be divisible by $8 \Rightarrow b = 6$ only.

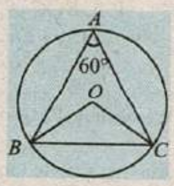
For divisibility by 9, the sum of digits, $a + 7 + 3 + 8 + 9 + b$ should be divisible by 9 *i.e.* $a + 7 + 3 + 8 + 6 \equiv 0 \pmod 9$

$\Rightarrow \quad a + 6 \equiv 0 \pmod 9 \qquad \Rightarrow \quad a \equiv -6 \pmod 9$

$\Rightarrow \quad a \equiv 3 \pmod 9 \quad \therefore \quad a = 3$ only

Hence $a + b$ equals 9.

27. (b) :



Let $\angle ABO = \theta$, now

$\angle BOC = 2 \times 60° = 120°$.

Then $\angle OBC = \angle OCB = 30°$

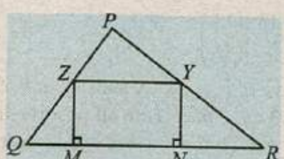
In triangle ABC, we have

$\Rightarrow \quad (30 + \theta)° + 30° + \angle ABO = 120°$

$\Rightarrow \quad \theta + \angle ABO = 60°$

$\Rightarrow \quad 0 < \theta < 60°$

28. (b) :



In triangle PZY and PQR, we have

$\dfrac{PZ}{PQ}=\dfrac{PY}{PR}$ and $\angle QPR$ is common

Hence $\Delta PZY \sim \Delta PQR$. (Also $ZY \parallel QR$)

$$\frac{ar(\Delta PZY)}{ar(\Delta PQR)}=\frac{(ZY)^2}{(QR)^2}$$

As $ZY \parallel QR$ and $\angle ZMN = 90° = \angle YMN$, $ZMNY$ is a rectangle.

The ratio of heights of ΔPZY and ΔPQR is $\dfrac{ZY}{QR} = k$(say)

Area of rectangle $ZMNY = (ZY)(YN)$

$= (kQR)\cdot(h_1 - h_2)$,

h_1 = height PQR, h_2 = height of PZY

$$= kQR\cdot\left\{h_1 - h_1\cdot\frac{ZY}{QR}\right\} = kh_1QR\left\{1-\frac{ZY}{QR}\right\},$$

Given $PQ = 6$, $PR = 8$, $\dfrac{ar(\Delta PZY)}{ar(\Delta PQR)}=\dfrac{2}{5}$

We have

$$\frac{ZY^2}{QR^2}=\frac{2}{5} \quad \therefore \quad k=\sqrt{\frac{2}{5}}$$

Also $QR=\sqrt{6^2+8^2}=10$ $(\because \angle QPR = \pi/2)$

$$\therefore \text{ Area of rectangle } ZMNY = \sqrt{\frac{2}{5}}\cdot\frac{\frac{1}{2}\times 6\times 8}{\frac{1}{2}\times 5}\cdot 10\left\{1-\frac{\sqrt{2}}{\sqrt{5}}\right\}$$

$$=48\sqrt{\frac{2}{5}}\left\{1-\sqrt{\frac{2}{5}}\right\}=48\left\{\sqrt{\frac{2}{5}}-\frac{2}{5}\right\}\text{ cm}^2$$

29. (b) : The first rook can be placed in any row in 8 ways and in any column in 8 ways. So it has 8^2 ways to be disposed off. Since no other rook can be placed in the path of the first rook, a second rook can be placed in 7^2 ways for there now remains only 7 rows and 7 column. Counting in this manner the number of ways $= 8^2 \cdot 7^2 \cdot 6^2 \ldots\ldots 1^2 = (\underline{|8})^2$.

30. (b) : $(1 + x)(1 - 2x)(1 + 3x)(1 - 4x) \ldots (1 + 101x)$

$$=1\cdot 2\cdot 3\cdot 4\cdot\ldots 101\left((1+x)\left(\frac{1}{2}-x\right)\left(\frac{1}{3}+x\right)\left(\frac{1}{4}-x\right)\ldots\left(\frac{1}{101}+x\right)\right)$$

$$=\underline{|101}(x+1)\left(x-\frac{1}{2}\right)\left(x+\frac{1}{3}\right)\left(x-\frac{1}{4}\right)\ldots\left(x-\frac{1}{100}\right)\left(x+\frac{1}{101}\right)\cdot(-1)^{50}$$

(the number of even numbers in $\{1, 2, 3, \ldots, 101\}$ is 50

$$=\underline{|101}(x+1)\left(x-\frac{1}{2}\right)\left(x+\frac{1}{3}\right)\left(x-\frac{1}{4}\right)\ldots\left(x-\frac{1}{100}\right)\left(x+\frac{1}{101}\right)$$

$\therefore$ The coefficient of x^{100} is

$$\underline{|101}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{101}\right)$$

$$=\underline{|101}\left\{\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{101}\right)-\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{100}\right)\right\}$$

SOLUTIONS TO SHORT ANSWER TYPE QUESTIONS

31. Note that 1 is in the range of f. For an arbitrary $x_0 > 0$, let

$$y_0=\frac{1}{f(x_0)}$$

The given equation then reads $f(x_0 f(y_0)) = y_0 f(x_0)$

$\Rightarrow \quad f(x_0 f(y_0)) = 1$

Thus 1 is in the range of f. The same argument can be extended to show that any positive real number is in the range of f.

Now $\exists$ y such that $f(y) = 1$.

Setting $x = 1$, $f(y) = 1$ is our equation

$f(1 - 1) = f(1) = yf(1)$

since $f(1) > 0$ we have $y = 1$, and so $f(1) = 1$.

But $y = x$ in the equation to obtain

$f(xf(x)) = xf(x)$ for all $x > 0$

Hence $xf(x)$ is a fixed point of f. Let a and b fixed points of f, then $f(a) = a$, $f(b) = b$

setting $x = a$, $y = b$ gives

$f(ab) = ab$

which mean that ab is also a fixed point of f. Thus the set of fixed points of f is closed under multiplication.

In particular, if a is a fixed point, all non-negative integral powers a^n of a are fixed points.
As $f(x) \to 0$ when $x \to \infty$ there can be any fixed point > 1.
Also $xf(x)$ is a fixed point, it follows that $xf(x) \le 1$

$$\Rightarrow \quad f(x) \le \frac{1}{x} \qquad \text{...(A)}$$

Let $a = tf(t)$, so $f(a) = a$

set $x = \frac{1}{a}$, $y = a$ in our equation

$$f\left(\frac{1}{a} f(a_1)\right) = f(1) = 1 = a\, f\left(\frac{1}{a}\right)$$

$$\text{or} \quad f\left(\frac{1}{a}\right) = \frac{1}{a} \quad \text{or} \quad f\left(\frac{1}{tf(t)}\right) = \frac{1}{tf(t)}$$

This shows that $\frac{1}{xf(x)}$ is also a fixed point of f for all $x > 0$

$$\text{therefore } f(x) \ge \frac{1}{x} \qquad \text{...(B)}$$

(A) and (B) together imply

$$f(x) = \frac{1}{x}$$

The function given above is the only solution satisfying the given conditions of the problem.

32. Let $a = y + z$, $b = z + x$, $c = x + y$ where x, y and z are positive reals

$$s = \frac{a+b+c}{2} = \frac{y+z+z+x+x+y}{2} = x+y+z$$

$$R = \frac{abc}{4\Delta} = \frac{(x+y)(y+z)(z+x)}{\Delta\sqrt{xyz(x+y+z)}}$$

$$r = \frac{\Delta}{s} = \frac{\sqrt{xyz(x+y+z)}}{x+y+z} = \sqrt{\frac{xyz}{x+y+z}}$$

we have

$$9Rr = \frac{9(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}} \cdot \sqrt{\frac{xyz}{x+y+z}}$$
$$= \frac{9(x+y)(y+z)(z+x)}{4(x+y+z)}$$

$$a(s-a) + b(s-b) + c(s-c) = (y+z)x + y(z+x) + z(x+y)$$
$$= 2(xy + yz + zx)$$

The inequality to be proved below

$$2(xy + yz + zx) \le \frac{9(x+y)(y+z)(z+x)}{4(x+y+z)}$$

i.e. $8(x+y+z)(xy+yz+zx) \le 9(x+y)(y+z)(z+x)$ ......(i)

Now $(x+y+z)(xy+yz+zx)$

$$= xy(x+y) + yz(y+z) + zx(z+x) + 3xyz$$
$$= \{xy(x+y) + yz(y+z) + zx(z+x) + 2xyz\} + xyz$$
$$= (x+y)(y+z)(z+x) + xyz \qquad \text{...(A)}$$

Now $x + y \ge 2\sqrt{xy}$

$y + z \ge 2\sqrt{yz}$

$z + x \ge 2\sqrt{zx}$

On multiplying we get $(x+y)(y+z)(z+x) \ge 8xyz$ ...(B)
using (B) in (A)
$(x+y+z)(xy+yz+zx)$

$$\le (x+y)(y+z)(z+x) + \frac{1}{8}(x+y)(y+z)(z+x)$$
$$= \frac{9}{8}(x+y)(y+z)(z+x)$$

$$\Rightarrow \quad 8(x+y+z)(xy+yz+zx) \le 9(x+y)(y+z)(z+x)$$

Thus the inequality (i) is proved.

33. (i) Let $p = 2^{n+1} - 1$, p being prime
Now $N = 2^n p$.
The divisors of N are $1, 2, 2^2, ..., 2^n, p, 2p, 2^2p, ... 2^np$
The sum of the divisors of N is

$$\sigma(N) = (1 + 2 + 2^2 + ... + 2^n) + p(1 + 2 + 2^2 + ... + 2^n)$$
$$= (1+p)(1 + 2 + 2^2 + ... + 2^n)$$
$$= (1+p) \cdot \frac{2^{n+1}-1}{2-1}$$
$$= (1+p)(2^{n+1} - 1) = (1 + 2^{n+1} - 1)(2^{n+1} - 1)$$
$$= 2^{n+1}(2^{n+1} - 1) = 2 \cdot 2^n(2^{n+1} - 1) = 2N$$

The above can also be done by using the formula for

$$\sigma(N) = \frac{(p_1^{k_1+1}-1)}{p_1-1} \cdot \frac{(p_2^{k_2+1}-1)}{p_2-1} \cdots \frac{(p_n^{k_n+1}-1)}{p_n-1}$$

whose $N = p_1^{k_1}, p_2^{k_2} ... p_n^{k_n}$ is the prime decompsition of N.
We have $N = 2^n p$

$$\text{So } \sigma(N) = \frac{(2^{n+1}-1)}{2-1} \cdot \frac{p^2-1}{p-1} = (2^{n+1}-1)(p+1) \text{ as before}$$

(ii) The sum of reciprocals of the divisors of N

$$= \left(\frac{1}{1} + \frac{1}{2} + ... + \frac{1}{2^n}\right) + \frac{1}{p}\left(1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^n}\right)$$
$$= \left(1 + \frac{1}{2} + ... + \frac{1}{2^n}\right)\left(1 + \frac{1}{p}\right)$$
$$= \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} \cdot \frac{1+p}{p} = \frac{2\{2^{n+1}-1\}}{2^{n+1}} \cdot \left(\frac{1+p}{p}\right)$$
$$= \frac{2^{n+1}-1}{2^n} \cdot \left(\frac{1+p}{p}\right) = \frac{2^{n+1}-1}{2^n} \cdot \frac{2^{n+1}}{2^{n+1}-1} = 2$$

34. 1st solution
Rewrite the given relation as

$$6^x + 4^x - 36^x + 24^x - 16^x = 1$$

Let $6^x = a$, $4^x = b$, we have

$$a + b - a^2 + ab - b^2 = 1$$
$$\Rightarrow \quad a^2 - ab + b^2 - a - b + 1 = 0 \qquad \text{...(1)}$$

Rearranging as a quadratic in a

$$a^2 - a(b+1) + b^2 - b + 1 = 0$$

The discriminant must be non-negative, which gives

$$(b+1)^2 - 4(b^2 - b + 1) \ge 0$$
$$\Rightarrow \quad b^2 + 2b + 1 - 4b^2 + 4b - 4 \ge 0 \quad \Rightarrow \quad -3b^2 + 6b - 3 \ge 0$$
$$\Rightarrow \quad b^2 - 2b + 1 \le 0 \qquad \Rightarrow \quad (b-1)^2 \le 0$$

which gives $(b-1)^2 = 0, \Rightarrow b = 1$
As the equation (1) is symmetric in a and b, we have $a = 1$
So the equation (1) is satisfies only for $a = 1$, $b = 1$
$\Rightarrow \quad 6^x = 1$ and $4^x = 1$, giving $x = 0$ only

2nd solution

Equation (1) when multiplied by 2 becomes

$$2a^2 - 2ab + 2b^2 - 2a - 2b + 2 = 0$$
$$\Rightarrow \quad (a^2 - 2ab + b^2) + (a^2 - 2a + 1) + (b^2 - 2b + 1) = 0$$
$$\Rightarrow \quad (a-b)^2 + (a-1)^2 + (b-1)^2 = 0$$

which holds only when $a = b$, $a = 1$ and $b = 1$. All of these are

satisfies at $x = 0$.

35. (i) $f(x^3) = f(g f(x)) = \{f(x)\}^2$

Now $x \in \{-1, 0\} \Rightarrow x^3 = x \Rightarrow f(x) = \{f(x)\}^2$

$\Rightarrow f(x) \in \{0, 1\}$

Hence there exist different $a, b \in \{-1, 0, 1\}$ such that $f(a) = f(b)$

But then $a^3 = g(f(a)) = g(f(b)) = b^3$, a contradiction.

Thus the function f and g satisfying the given conditions don't exist.

(ii) we give a conductive proof.

Define

$$g(x) = \begin{cases} |x|^{\ln|x|} & \text{if } |x| \ge 1 \\ |x|^{-\ln|x|} & \text{if } 0 < |x| < 1 \\ 0 & \text{if } x = 0 \end{cases}$$

g is even and $|a| = |b|$ whenever $g(a) = g(b)$. So we can define f as an even function such that

$f(x) = y^2$, where y satisfies $g(\pm y) = x$

$f(g(x)) = x^2$ is clearly verified by the definition of f.

$$g(f(x)) = g(y^2) = \begin{cases} (y^2)^{\ln(y^2)} = (y^{\ln y})^4 & , \quad y \ge 1 \\ (y^2)^{-\ln(y^n)} = (y^{-\ln y})^4, & 0 < y < 1 \end{cases}$$

are $g(f(x)) = 0$ if $y = 0$

thus $g(f(x)) = g(y^2) = (g(y))^4 = x^4$.

36. Let's classify the n permutation in terms of how many fixed points they have. A fixed point is any element that occupies its right place. For example, for $n = 5$, the permutation (2, 3, 1, 4, 5) has two fixed points, viz 4 and 5.

Suppose the number of fixed points is i. Since there are n elements, i fixed points may be picked in ${}^nC_2$ ways. The remaining $(n - 2)$ elements must form an arrangement, as the permutation has EXACTLY 'i' fixed point. This mean that they can be arranged in D_{n-i} ways. So the number of permutation units exactly 'i' fixed points is

$$\binom{n}{i} D_{n-i}$$

As i can take values from 0 to n, the total number of permutation is $\underline{|n} = \sum_{i=0}^{n} \binom{n}{i} D_{n-i}$

$$\therefore \quad \underline{|n} = {}^nC_0 D_n + {}^nC_1 \cdot D_{n-1} + \dots + {}^nC_{n-1} D_1 + {}^nC_n D_0$$

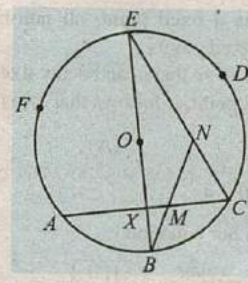
37. We will make repeated use of pigon-hole-principle (*PHP*)). As there are 65 balls and 2 boxes, one of the boxes must contain at least $\left[\frac{65}{2}\right] + 1 = 33$ balls. Consider that box, now we have four colours (white, black, red, yellow) and hence there must be at least $\left(\frac{33}{4}\right) + 1 = 9$ balls of the same colour.

There can be at most 4 different sizes available for these 9 balls of the same colour. For if there were 5(or more) different sizes, then collection of 5 balls, all of different sizes, would not satisfy the given property.

Thus of these 9 balls (of the same colour and in the same box) there must be at least 3 balls of the same size.

38. 1st solution

Let each side of hexagon have length '1'



let X denote the intersection of AC and BE. Now N is on CE, B on EX and M on XC, we apply Menelaus theorem to triangle,

$$\frac{CN}{NE} \cdot \frac{EB}{BX} \cdot \frac{XM}{MC} = -1$$

CE is the side opposite to the 120° angle in a isosceles triangle with two sides of lengths '1'

we have $CE = \sqrt{3}, CN = \sqrt{3}r, EB = 2, BX = -\frac{1}{2}$

$$MC = AC - AM = \sqrt{3}(1 - r)$$

$$XM = \frac{\sqrt{3}}{2} - MC = \frac{\sqrt{3}}{2} - \sqrt{3} + r\sqrt{3} = \sqrt{3}\left(r - \frac{1}{2}\right)$$

Substituting into the obtained result

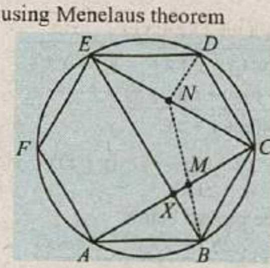
$$\frac{\sqrt{3}r}{\sqrt{3}(1-r)} \cdot \frac{2}{-1/2} \cdot \frac{\sqrt{3}\left(r - \frac{1}{2}\right)}{\sqrt{3}(1-r)} = -1$$

$$\Rightarrow \quad \frac{4r\left(r - \frac{1}{2}\right)}{(1-r)^2} = 1 \Rightarrow 4r^2 - 2r = 1 - 2r + r^2$$

$$\Rightarrow \quad 3r^2 = 1 \qquad \therefore \; r = \frac{1}{\sqrt{3}}.$$

2nd solution

This is without using Menelaus theorem



Since $AC = EC$, we have from $\dfrac{AM}{AC} = \dfrac{CN}{CE}$ that $AM = CN$

And so $CM = EN$. Also $\Delta BMC \cong \Delta DNE$

$\therefore \quad \angle XBC = \angle EDN$

Also $\angle BND = \angle BNC + \angle CND$

$= (90° - \angle NBC) + \angle CED + \angle NDE$

$= 90° - \angle NBC + 30° + \angle NBC = 120°$

Then BD subtends an angle of 120° from N and also from the centre O of the circumcircle of the hexagon. It is now clear that N lies on a circle with centre C and redius $CD = CB = CN$.

From right triangle BCE, $\angle EBC = 60°$ giving

$$R = \frac{CN}{CE} = \frac{CB}{CE} = \frac{1}{\sqrt{3}}$$

39. Define $F(x)=\int_0^x f(t)dt$ and $G(x)=\int_0^x (f(t))^2\,dt$

since $f:(0,\infty)\to(0,\infty)$ we have $F(x)>0$ for all $x>0$

Also $\frac{1}{2}G(x)=\frac{1}{x}\{F(x)\}^2$, from the given condition

on differentiation we have

$$\frac{1}{2}G'(x)=\frac{1}{x}\cdot 2F(x)\cdot F'(x)-\frac{1}{x^2}(F(x))^2$$

This mean that

$$\frac{1}{2}(F(x))^2=\frac{2}{x}F(x)F'(x)-\frac{1}{x^2(F(x))^2}$$

or $$\frac{1}{2}\left(\frac{xF'(x)}{F(x)}\right)^2=2\frac{xF'(x)}{F(x)}-1$$

Solving this equation as a quadratic in $\frac{xF'(x)}{F(x)}$ we have

$$\frac{xF'(x)}{F(x)}=2\pm\sqrt{2}=k(\text{say})$$

on integration, we obtain

$$\int\frac{dF(x)}{F(x)}=k\int\frac{dx}{x}$$

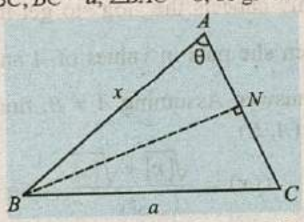
$\Rightarrow\ \ln F(x)=k\ln x+\ln\lambda$

$\therefore\ F(x)=\lambda x^k$

Hence $F(x)=\lambda k x^{k-1}$

Given $F(1)=1$ we have $\lambda k=1$, thus $f(x)=x^k=x^{1+\sqrt{2}}$ or $x^{1-\sqrt{2}}$.

40. Let in ΔABC, $BC=a$, $\angle BAC=\theta$, be given



let BN be the altitude through B, let $AB=x$, $AC=y$.

Then $AN=x\cos\theta$, $BN=x\sin\theta$, $CN=\sqrt{a^2-x^2\sin^2\theta}$

$$y=AN+CN=x\cos\theta+\sqrt{a^2-x^2\sin^2\theta}$$

area of $\Delta ABC=A=\frac{1}{2}AC\cdot BN$

$$=\frac{1}{2}\cdot x\sin\theta(x\cos\theta+\sqrt{a^2-x^2\sin^2\theta})$$

$$=\frac{1}{2}\sin\theta\{x^2\cos\theta+x\sqrt{a^2-x^2\sin^2\theta}\}$$

As 'θ' is fixed, so we have to maximize

$$u(\theta)=x^2\cos\theta+x\sqrt{a^2-x^2\sin^2\theta}$$

$\Rightarrow\ (u-x^2\cos\theta)^2=x^2(a^2-x^2\sin^2\theta)$

$\Rightarrow\ x^4-(a^2+2u\cos\theta)x^2+u^2=0$

on completing the square

$$\left(x^2-\frac{2u\cos\theta+a^2}{2}\right)^2-\left(\frac{2u\cos\theta+a^2}{2}\right)^2+u^2=0$$

$$\Rightarrow\ u^2=\left(\frac{2u\cos\theta+a^2}{2}\right)^2-\left(x^2-\frac{2u\cos\theta+a^2}{2}\right)^2$$

so u is maximum when the second term on R.H.S. is zero. This gives

$$u^2=\left(\frac{2u\cos\theta+a^2}{2}\right)^2$$

$$\Rightarrow\ u=\frac{2u\cos\theta+a^2}{2}$$

$\Rightarrow\ 2u=2u\cos\theta+a^2 \qquad \Rightarrow 2u(1-\cos\theta)=a^2$

$\Rightarrow\ 4u\sin^2\frac{\theta}{2}=a^2$

Also $$x^2=\frac{2u\cos\theta+a^2}{2}=\frac{\frac{a^2}{2\sin^2\frac{\theta}{2}}\cos\theta+a^2}{2}$$

$$=\frac{a^2\left(\cos\theta+2\sin^2\frac{\theta}{2}\right)}{4\sin^2\frac{\theta}{2}}=\frac{a^2}{4\sin^2\frac{\theta}{2}}$$

This mean that the median from A is the altitude from A. So ABC is an isosceles triangle with $AB=AC$.

EXAM DATES (2009)

Karnataka CET	: 6th and 7th May
CBSE-PMT (Mains)	: 10th May
Indian Statistical Institute	: 10th May
Uttarakhand (UKSEE)	: 10th May
Maharashtra CET	: 12th May (Revised)
DUMET	: 17th May
AMU (Engg.)	: 20th May
Orissa JEE	: 24th May
Kerala PET	: 25 & 26th May
Kerala PMT	: 27 & 28th May
AMU (Medical)	: 28th May
DCE	: 30th May
AIIMS	: 1st June
JIPMER	: 7th June
BHU (Screening)	: 10th June (Revised)
BHU (Mains)	: 30th June (Revised)

MOCK TEST PAPER

ISI 2009

Exam on 10th May 2009

Indian Statistical Institute

OBJECTIVE TYPE

1. If the quadratic equation $x^2+ax+b+1=0$ has non-zero integer solutions, then
(a) a^2+b^2 is a prime number
(b) ab is a prime number
(c) both (a) and (b) (d) neither (a) nor (b).

2. Let $u=(\sqrt{5}-2)^{1/3}-(\sqrt{5}+2)^{1/3}$ and $v=(\sqrt{189}-8)^{1/3}-(\sqrt{189}+8)^{1/3}$, then for each positive integer n, $u^n+v^{n+1}=$
(a) -1 (b) 0 (c) 1 (d) 2.

3. The number of real values of x satisfying the equation $x\cdot 2^{1/x}+\frac{1}{x}\cdot 2^x = 4$ is/are
(a) one (b) two (c) three (d) four.

4. If $a+b+c=3$, $a^2+b^2+c^2=1$ and $a^3+b^3+c^3=3$, then $abc=$
(a) 1 (b) 2 (c) 3 (d) 4

5. Let a, b, c be real numbers satisfying the equations, $ab-a=b+119$, $bc-b=c+59$ and $ca-c=a+71$, then the number of possible values of $a+b+c$ is/are
(a) 2 (b) 4 (c) 8 (d) 16

6. The number of distinct real roots of the equation $x^4+8x^2+16=4x^2-12x+9$ is
(a) one (b) two (c) three (d) four.

7. Let $f(x)$ and $g(x)$ be functions, which take integers as arguments. Let $f(x+y)=f(x)+g(y)+8$ for all integers x and y. Let $f(x)=x$ for all negative numbers x and let $g(8)=17$, then $f(0)=$
(a) 8 (b) 9 (c) 17 (d) 72.

8. Let $x=\left[\dfrac{2007\cdot 2006\cdot 2004\cdot 2003}{\frac{1}{3}\times(2005)^4}\right]$, where $[x]$ denotes greatest integer less than or equal to x. Then $\dfrac{((x^2+1)\cdot x^2)+1}{(x^2+1)}$ is
(a) 80 (b) 80.2 (c) 80.5 (d) 81.

9. A graph is defined in polar co-ordinates by
$$r(\theta)=\cos\theta+\frac{1}{2}$$
The smallest x-co-ordinate of any point on this graph is
(a) 1/16 (b) $-1/16$ (c) 1/8 (d) $-1/8$

10. A monic polynomial is one in which the coefficient of the highest order term is 1. The monic polynomial $P(x)$ (with integer coefficient) of least degree that satisfies $p(\sqrt{2}+\sqrt{5})=0$ is
(a) $x^4-x^3-14x^2+9=0$ (b) $x^4-14x^2+9=0$
(c) $x^4+x^3-14x^2+9=0$ (d) $x^4+14x^2-9=0$.

SUBJECTIVE TYPE

1. X, a weak math student, sees the expression $\frac{\log A}{\log B}$. She mistakenly cancels the "log" to get the expression $\frac{A}{B}$. But, when she puts in values of A and B, she gets the correct answer. Assuming $A\neq B$, find all possible ordered pairs (A, B).

2. Let $x\geq 1$, $f(x)=\dfrac{\sqrt{[x]}+\sqrt{\{x\}}}{\sqrt{x}}$ where $[\cdot]$ denotes G.I.F. and $\{\ \}$ denotes fractional part. Determine the smallest number k such that $f(x)\leq k$ for each $x\geq 1$.

3. Solve the equation $(\sqrt{2+\sqrt{2}})^x+(\sqrt{2-\sqrt{2}})^x = 2^x$.

4. Let A, B, C be three pairwise orthogonal faces of a tetrahedron meeting at one of its vertices and having respective areas a, b, c. Let the face D opposite this vertex have area d. Prove that $a^2+b^2+c^2=d^2$.

5. Let $f(x)$ be a polynomial with real coefficient for which the equation $f(x)=x$ has no real solution. Prove that the equation $f(f(x))=x$, has no real solution, either.

6. Let $a\in[0, 4]$. Prove that the area bounded by the curves $y=1-|x-1|$ and $y=|2x-a|$ cannot exceed 1/3.

7. Determine a value of the parameter θ so that $f(x)=\cos^2 x+\cos^2(x+\theta)-\cos x\cos(x+\theta)$ is a constant function of x.

By : **Er. Tapas Kr. Yogi**, Director, Pillar IIT Academy, Bhubaneshwar

8. Is it true that any pair of triangles sharing a common angle, inradius and circumradius must be congruent?

SOLUTIONS

OBJECTIVE TYPE

1. **(d)** : $\alpha+\beta=-a,\ \alpha\beta=b+1$

so, $a^2+b^2=(\alpha+\beta)^2+(\alpha\beta-1)^2=(\alpha^2+1)(\beta^2+1)$

2. **(b)** : $u^3=(\sqrt{5}-2)-(\sqrt{5}+2)$
$$-3\cdot(\sqrt{5}-2)^{1/3}(\sqrt{5}+2)^{1/3}\cdot(u)$$

i.e. $u^3=-4-3u \Rightarrow (u-1)(u^2-u+4)=0$

u^2-u+4 is always +ve. So, $u=1$

Similarly $v^3+15v+16=0$

$=(v+1)(v^2-v+16) \quad \Rightarrow \quad v=-1$

So, for each n, $u^n+v^{n+1}=0$.

3. **(a)** : If $x<0$, LHS = –ve but RHS = +ve

If $x=0$, LHS = not defined

If $x>0$, applying AM $\geq$ GM

$$x\cdot 2^{1/x}+\frac{1}{x}\cdot 2^x \geq 2\sqrt{2^{\frac{1}{x}+x}} \geq 2\cdot\sqrt{2^2}=4$$

$$\Rightarrow x\cdot 2^{1/x}=\frac{1}{x}\cdot 2^x. \text{ So, } x=1.$$

4. **(d)** : Use $a^3+b^3+c^3-3abc$

$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

5. **(a)** : $ab-a-b=119 \Rightarrow (a-1)(b-1)=120$ etc.

6. **(a)** : $(x^2+4)^2=(2x-3)^2 \Rightarrow x^2+4=\pm(2x-3)$

7. **(c)** : Put $x=-8$, $y=8$ in the given functional equation.

8. **(b)** : $x=\left[3\cdot\frac{2007}{2005}\cdot\frac{2006}{2005}\cdot\frac{2004}{2005}\cdot\frac{2003}{2005}\right]$

$$=\left[3\cdot\left(1+\frac{2}{2005}\right)\left(1+\frac{1}{2005}\right)\left(1-\frac{1}{2005}\right)\left(1-\frac{2}{2005}\right)\right]$$

$$x=\left[3\left(1-\frac{4}{(2005)^2}\right)\left(1-\frac{1}{(2005)^2}\right)\right] \Rightarrow x=2.$$

9. **(b)** : $x=r\cos\theta=\cos^2\theta+\frac{1}{2}\cos\theta$

$$=\left(\cos\theta+\frac{1}{4}\right)^2-\frac{1}{16}$$

10. **(b)** : Let $x=\sqrt{2}+\sqrt{5}$.

Squaring, $x^2=7+2\sqrt{10}$.

$\Rightarrow \quad x^2-7=2\sqrt{10}$

squaring again $x^4-14x^2+9=0$.

SUBJECTIVE TYPE

1. $\frac{\log A}{\log B}=\frac{A}{B} \Rightarrow B\log A=A\log B$

or $A^B=B^A \Rightarrow A=2$ or 4

2. Let $x=a+b$ where $a=[x]$, $b=\{x\}$

$$f(x)=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}}$$

$$(f(x))^2=\frac{a+b+2\sqrt{ab}}{a+b}=1+\frac{2\sqrt{ab}}{a+b}$$

using (AM $\geq$ GM) $\leq 1+1 \quad \Rightarrow f(x)\leq\sqrt{2}$.

3. Note that $1+\frac{\sqrt{2}}{2}=1+\cos\frac{\pi}{4}=2\cos^2\frac{\pi}{8}$

$$\left(\frac{2+\sqrt{2}}{4}\right)^{x/2}+\left(\frac{2-\sqrt{2}}{4}\right)^{x/2}=\left(\cos\frac{\pi}{8}\right)^x+\left(\sin\frac{\pi}{8}\right)^x$$

$\Rightarrow x=2$.

4. Let the three planes be the three co-ordinate planes and the 4th plane be $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$, where α,β,γ are +ve.

The vertices of the tetrahedron are $(0,0,0)$, $(\alpha,0,0)$, $(0,\beta,0)$ and $(0,0,\gamma)$. Then volume of the tetrahedron is

$$V=\frac{1}{3}a\alpha=\frac{1}{3}b\beta=\frac{1}{3}c\gamma=\frac{1}{3}dk$$

where k = distance of the plane from origin

$$=\frac{|0+0+0-1|}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}} \Rightarrow \frac{1}{k^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$$

i.e. $d^2=a^2+b^2+c^2$.

5. Suppose, if possible that $f(f(a))=a$, let $b=f(a)$ then $f(b)=a$. By hypothesis $b\neq a$. Assume that $a<b$ then $f(a)-a>0$ and $f(b)-b<0$. So, by Intermediate value theorem $f(x)-x=0$ should have a root between (a,b). But this contradicts our assumption. Hence, $f(f(x))$ can have no real solution.

6. When $a\in[0,1]$, the area is a triangle formed by $(0,0)$, $(1/2,0)$ and $(1,1)$ with area equals 1/4. When $a\in[1,3]$, the area is a quadrilateral with vertices at $\left(\frac{a}{3},\frac{a}{3}\right)$, $\left(\frac{a}{2},0\right)$, $\left(\frac{a+2}{3},\frac{4-a}{3}\right)$ and $(1,1)$. So, the net area is $\frac{1}{3}-\frac{(a-2)^2}{6}$ which also does not exceed 1/3.

When $a\in[3,4]$, area is same as when $a\in[0,1]$.

7. $f(x)=\sin^2\theta+(2\cos\theta-1)(\cos^2x\cos\theta-\sin 2x\sin\theta)$.

The function $f(x)$ is constant when $2\cos\theta-1=0$ *i.e.* at $\theta=\pi/3$ and the constant value is 3/4.

8. Let angle A, inradius r and circumradius R are fixed.

Then $a=2R\sin A$ = fixed

$b+c-a=2(s-a)=r\cot(A/2)$

$\Rightarrow b+c=a+r\cot(A/2)$ = fixed

and $bc=\frac{2\Delta}{\sin A}$ = fixed

so, b and c are also fixed *i.e.* triangles are congruent. ■

Self Assessment Test

This assessment series is based on the full syllabus of IIT-JEE. It's very helpful for different Engineering Entrance Examinations. Students can assess themselves by solving the problems given in assessment series.

Give yourself 1 mark for each correct answer. Assess your performance as per the table given here.

41-50	Genius
31-40	Very Good
21-30	Good
11-20	Satisfactory
less than 10	Average

OBJECTIVE TYPE

1. If $a = \log_2 3$, $b = \log_3 5$, $c = \log_7 2$, then $\log_{140} 63$ in terms of a, b, c is

(a) $\dfrac{2ac+1}{2a+abc+1}$ (b) $\dfrac{2ac+1}{2b+abc+1}$

(c) $\dfrac{2ac+1}{2c+abc+1}$ (d) none of these.

2. The mean square deviation of n observations $x_1, x_2, \ldots, x_n$, about -2 and 2 are 18 and 10 respectively. Then, S.D. of the given set is

(a) 1 (b) 2 (c) 3 (d) 4

3. The image of the point $(1, 2, -1)$ in the plane $\vec{r}\cdot(3\hat{i}-5\hat{j}+4\hat{k})=5$, is

(a) $\left(-\dfrac{73}{25},\dfrac{6}{5},\dfrac{39}{25}\right)$ (b) $\left(\dfrac{73}{25},\dfrac{-6}{5},\dfrac{39}{25}\right)$

(c) $\left(\dfrac{73}{25},\dfrac{6}{5},\dfrac{-39}{25}\right)$ (d) none of these.

4. A unit vector in the xy-plane that makes an angle of 45° with the vector $\hat{i}+\hat{j}$ and an angle of 60° with the vector $3\hat{i}-4\hat{j}$, is

(a) $\hat{i}$ (b) $\dfrac{\hat{i}+\hat{j}}{2}$

(c) $\dfrac{\hat{i}-\hat{j}}{2}$ (d) none of these.

5. A letter is taken out at random from 'ASSISTANT' and another taken out from 'STATISTICS'. The probability that they are the same letters, is

(a) 1/45 (b) 13/90

(c) 19/90 (d) none of these.

6. The slope of tangent at (x, y) to a curve passing through $\left(1, \dfrac{\pi}{4}\right)$ is given by $\dfrac{y}{x}-\cos^2\dfrac{y}{x}$, then equation of curve is

(a) $y=\tan^{-1}\left(\log\left(\dfrac{e}{x}\right)\right)$ (b) $y=x\tan^{-1}\left(\log\dfrac{e}{x}\right)$

(c) $y=x\tan^{-1}\left(\log\left(\dfrac{e}{x}\right)\right)$ (d) none of these.

7. Area of the region bounded by the curve $y=\begin{cases}x^2, & x<0\\ x & x\ge 0\end{cases}$ and the line $y=4$, is

(a) 10/3 (b) 20/3

(c) 40/3 (d) none of these.

8. If f, g, h are continuous functions on $[0, a]$ such that $f(x)=f(a-x)$, $g(x)=-g(a-x)$ and $4h(x)-3h(a-x)=7$, then $\int_0^a f(x)\,g(x)\,h(x)dx=$

(a) 0 (b) 1

(c) 5/4 (d) none of these.

9. If $\int f(x)dx=(ax^2-a^2)^5+C$, then $f(x)$ is

(a) $5\,(ax^2-a^2)^4$ (b) $2ax\,(ax^2-a^2)^4$

(c) $5\,(ax^2-a^2)^4.\,2ax$ (d) $5(ax^2-a^2)^4\,(2ax-2a)$.

10. All the points on the curve $y^2=4a\left(x+a\sin\dfrac{x}{a}\right)$ at which the tangents are parallel to the axis of x, lie on a

(a) circle (b) parabola

(c) line (d) none of these.

11. The set of all values of 'a' for which

$$f(x)=(a^2-3a+2)\left(\cos^2\frac{x}{4}-\sin^2\frac{x}{4}\right)+(a-1)x+\sin 1$$

does not possess critical points is

(a) $[1, \infty)$ (b) $(-2, 4)$

(c) $(1, 3)\cup(3, 5)$ (d) $(0, 1)\cup(1, 4)$.

12. If $f(x) = (\cos x + i\sin x)(\cos 3x + i\sin 3x)\ldots\ldots(\cos(2n-1)x + i\sin(2n-1)x)$, then $f''(x)$ is

(a) $n^2 f(x)$ (b) $-n^4 f(x)$

8. Is it true that any pair of triangles sharing a common angle, inradius and circumradius must be congruent?

SOLUTIONS

OBJECTIVE TYPE

1. **(d)** : $\alpha+\beta=-a, \alpha\beta=b+1$

so, $a^2+b^2=(\alpha+\beta)^2+(\alpha\beta-1)^2=(\alpha^2+1)(\beta^2+1)$

2. **(b)** : $u^3=(\sqrt{5}-2)-(\sqrt{5}+2)$
$$-3\cdot(\sqrt{5}-2)^{1/3}(\sqrt{5}+2)^{1/3}\cdot(u)$$

i.e. $u^3=-4-3u \Rightarrow (u-1)(u^2-u+4)=0$

u^2-u+4 is always +ve. So, $u=1$

Similarly $v^3+15v+16=0$

$=(v+1)(v^2-v+16) \Rightarrow v=-1$

So, for each n, $u^n+v^{n+1}=0$.

3. **(a)** : If $x<0$, LHS = –ve but RHS = +ve

If $x=0$, LHS = not defined

If $x>0$, applying AM ≥ GM

$$x\cdot 2^{1/x}+\frac{1}{x}\cdot 2^x \geq 2\sqrt{2^{\frac{1}{x}+x}} \geq 2\cdot\sqrt{2^2}=4$$

$$\Rightarrow x\cdot 2^{1/x}=\frac{1}{x}\cdot 2^x. \text{ So, } x=1.$$

4. **(d)** : Use $a^3+b^3+c^3-3abc$

$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

5. **(a)** : $ab-a-b=119 \Rightarrow (a-1)(b-1)=120$ etc.

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8. **(b)** : $x=\left[3\cdot\frac{2007}{2005}\cdot\frac{2006}{2005}\cdot\frac{2004}{2005}\cdot\frac{2003}{2005}\right]$

$$=\left[3\cdot\left(1+\frac{2}{2005}\right)\left(1+\frac{1}{2005}\right)\left(1-\frac{1}{2005}\right)\left(1-\frac{2}{2005}\right)\right]$$

$$x=\left[3\left(1-\frac{4}{(2005)^2}\right)\left(1-\frac{1}{(2005)^2}\right)\right] \Rightarrow x=2.$$

9. **(b)** : $x=r\cos\theta=\cos^2\theta+\frac{1}{2}\cos\theta$

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$\Rightarrow x^2-7=2\sqrt{10}$

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1. $\frac{\log A}{\log B}=\frac{A}{B} \Rightarrow B\log A=A\log B$

or $A^B=B^A \Rightarrow A=2$ or 4

2. Let $x=a+b$ where $a=[x]$, $b=\{x\}$

$$f(x)=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}}$$

$$(f(x))^2=\frac{a+b+2\sqrt{ab}}{a+b}=1+\frac{2\sqrt{ab}}{a+b}$$

using (AM ≥ GM) $\leq 1+1 \Rightarrow f(x)\leq\sqrt{2}$.

3. Note that $1+\frac{\sqrt{2}}{2}=1+\cos\frac{\pi}{4}=2\cos^2\frac{\pi}{8}$

$$\left(\frac{2+\sqrt{2}}{4}\right)^{x/2}+\left(\frac{2-\sqrt{2}}{4}\right)^{x/2}=\left(\cos\frac{\pi}{8}\right)^x+\left(\sin\frac{\pi}{8}\right)^x$$

$\Rightarrow x=2$.

4. Let the three planes be the three co-ordinate planes and the 4[th] plane be $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$, where α, β, γ are +ve.

The vertices of the tetrahedron are $(0,0,0)$, $(\alpha,0,0)$, $(0,\beta,0)$ and $(0,0,\gamma)$. Then volume of the tetrahedron is

$$V=\frac{1}{3}a\alpha=\frac{1}{3}b\beta=\frac{1}{3}c\gamma=\frac{1}{3}dk$$

where k = distance of the plane from origin

$$=\frac{|0+0+0-1|}{\sqrt{\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}}} \Rightarrow \frac{1}{k^2}=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$$

i.e. $d^2=a^2+b^2+c^2$.

5. Suppose, if possible that $f(f(a))=a$, let $b=f(a)$ then $f(b)=a$. By hypothesis $b\neq a$. Assume that $a<b$ then $f(a)-a>0$ and $f(b)-b<0$. So, by Intermediate value theorem $f(x)-x=0$ should have a root between (a,b). But this contradicts our assumption. Hence, $f(f(x))$ can have no real solution.

6. When $a\in[0,1]$, the area is a triangle formed by $(0,0)$, $(1/2,0)$ and $(1,1)$ with area equals 1/4. When $a\in[1,3]$, the area is a quadrilateral with vertices at $\left(\frac{a}{3},\frac{a}{3}\right)$, $\left(\frac{a}{2},0\right)$, $\left(\frac{a+2}{3},\frac{4-a}{3}\right)$ and $(1,1)$. So, the net area is $\frac{1}{3}-\frac{(a-2)^2}{6}$ which also does not exceed 1/3.

When $a\in[3,4]$, area is same as when $a\in[0,1]$.

7. $f(x)=\sin^2\theta+(2\cos\theta-1)(\cos^2 x\cos\theta-\sin 2x\sin\theta)$.

The function $f(x)$ is constant when $2\cos\theta-1=0$ *i.e.* at $\theta=\pi/3$ and the constant value is 3/4.

8. Let angle A, inradius r and circumradius R are fixed.

Then $a=2R\sin A$ = fixed

$b+c-a=2(s-a)=r\cot(A/2)$

$\Rightarrow b+c=a+r\cot(A/2)$ = fixed

and $bc=\frac{2\Delta}{\sin A}$ = fixed

so, b and c are also fixed *i.e.* triangles are congruent.

SECTION - IV

Matrix-Match Type

This section contains 3 questions numbered 20 to 22. Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D) in Column I have to be matched with statements (p, q, r, s) in Column II.

The answers to these questions have to be appropriately bubbled as illustrated in the following example. If the correct matches are A-p, A-s, B-q, B-r, C-p, C-q and D-s, then the correctly bubbled 4 × 4 matrix should be as follows : [For each correct answer 6 marks no negative marking other cases 0 mark].

	p	q	r	s
A	●	○	○	●
B	○	●	●	○
C	●	●	○	○
D	○	○	○	●

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20. Match the following, if a, b, c are in H.P., then

	Column I		Column II
(A)	$\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$	(p)	H.P.
(B)	$\frac{1}{b-a}, \frac{1}{b}, \frac{1}{b-c}$	(q)	G.P.
(C)	$a-\frac{b}{2}, \frac{b}{2}, c-\frac{b}{2}$	(r)	A.P.
(D)	$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$	(s)	none of these

21. Match the statements of Column I with the value in Column – II.

	Column I		Column II
(A)	Point $A(0, 0)$, $B(2, 0)$, $C(2, 1)$ and $D(0, 1)$ are given. The minimum possible value of $PA + PB + PC + PD$ for all positions of P is	(p)	–1
(B)	The two tangents from $P(39, 52)$ meet the circle defined by $x^2 + y^2 = 625$ at Q and R. The length QR is	(q)	4
(C)	Let P denote the perimeter of a right angled triangle and Q the sum of the squares of its three sides. If the area of the triangle is expressed as, $xP^2 + yP\sqrt{Q} + zQ^2$, then the value of $x + y + z$ is	(r)	1
(D)	A rhombus of side length x has the property that there is a point on its longer diagonal such that the distance from that point to the vertices are 1, 1, 1 and x. The value of x is	(s)	46

22.

	Column I		Column II
(A)	$\int_{-\pi/4}^{\pi/4} \ln\{e^a\sqrt{1+\sin 2x}\}.dx$	(p)	$\frac{\pi}{4}(2a - \ln 2)$
(B)	$\int_0^a \sqrt{a^2-x^2}\left(\cos^{-1}\frac{x}{a}\right)^2.dx$	(q)	$\frac{\pi}{48}a^2(\pi^2+6)$
(C)	$\int_0^{2\pi} e^{\sin^2 ax}\tan axdx (a \in 1)$	(r)	$\frac{a\pi^3 \ln 2}{192}$
(D)	$\int_0^{\pi/4}(\pi ax - 4ax^2)\ln(1+\tan x)dx$	(s)	none of these

SOLUTIONS

Paper I

1. (b)	2. (b)	3. (b)	4. (b)	5. (d)
6. (d)	7. (a, d)	8. (b, c)	9. (b, c)	
10. (a, c)	11. (c)	12. (c)	13. (a)	14. (a)
15. (d)	16. (d)	17. (c)	18. (c)	19. (b)
20. (a)	21. (b)	22. (a)	23. (d)	

Paper II

1. (b)	2. (d)	3. (d)	4. (a)	5. (b)
6. (c)	7. (d)	8. (c)	9. (d)	10. (a)
11. (a)	12. (d)	13. (c)	14. (c)	
15. (b)	16. (b)	17. (b)	18. (b)	19. (d)

20. (A) → (p), (B) → (r), (C) → (q), (D) → (p)
21. (A) → (q), (B) → (s), (C) → (r), (D) → (r)
22. (A) → (p), (B) → (q), (C) → (s), (D) → (r)

For detailed solution, visit www.vidyalankar.org

Solution Sender of Maths Musing

SET-84

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3. Anil. B. Murthy, Bangalore, Karnataka
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MOCK TEST PAPER

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Indian Statistical Institute

SUBJECTIVE TYPE

1. Let f be a continuous function on the interval $[0, 1] \to R$ such that $f(0) = f(1)$. Prove that there exists a point c in $[0, 1/2]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$

2. Let f be a differentiable function on $[0, 1]$. Suppose $f(0) = f(1) = 0$ and $f\left(\frac{1}{4}\right) = f\left(\frac{3}{4}\right) = 1$. Show that there are three distinct elements $c_1, c_2, c_3 \in (0, 1)$ such that $f'(c_1) + f'(c_2) + f'(c_3) = 0$.

3. Let $f : [a, b] \to R$ be a continuous function on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = a$ and $f(b) = b$. Show that there is an element $c \in (a, b)$ such that $f'(c) = 1$. Further show that there are distinct $c_1, c_2 \in (a, b)$ such that $f'(c_1) + f'(c_2) = 2$.

4. Let f be a continuous real valued function on the interval $[a, b]$ and let n_1, n_2 be real numbers such that $n_1 n_2 > 0$. Prove that the equation $f(x) = \frac{n_1}{a-x} + \frac{n_2}{b-x}$ has atleast one solution in the interval (a, b).

5. Let $a < b$ be positive real numbers. Prove that the equation $\left(\frac{a+b}{2}\right)^{x+y} = a^x b^y$ has atleast one solution in (a, b).

6. Suppose f is continuous on $[a, b]$, differentiable on (a, b) and satisfies $f^2(a) - f^2(b) = a^2 - b^2$. Then show that the equation $f(x)f'(x) = x$ has atleast one root in (a, b).

7. Let $f : [0, 1] \to R$ be a continuous function. Show that there exists $x_0 \in [0, 1]$ such that

$$f(x_0) = \frac{1}{3}\left[f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right]$$

8. Let f be twice differentiable on $[0, 2]$. Show that if $f(0) = 0, f(1) = 2$ and $f(2) = 4$ then there is some $x_0 \in [0, 2]$ such that $f''(x_0) = 0$.

9. Let $n > 1$ be an integer and let $f : [0, 1] \to R$ be a continuous function such that $\int_0^1 f(x)dx = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$. Prove that there is a real number $x_0 \in (0, 1)$ such that $f(x_0) = \frac{1 - x_0^n}{1 - x_0}$.

10. Let $f : [a, b] \to R$ be a continuous function such that $\int_a^b f(x)dx \neq 0$. Prove that there are numbers $a < \alpha < \beta < b$ such that $\int_a^\alpha f(x)dx = (b - \alpha)f(\beta)$.

11. Let $f : [0, 1] \to R$ be differentiable, $f\left(\frac{1}{2}\right) = \frac{1}{2}$ and $0 < \alpha < 1$ for some α. Suppose $|f'(x)| \leq \alpha$ for all $x \in [0, 1]$. Show that $|f(x)| < 1$ for all $x \in [0, 1]$.

12. Let f and g be continuous on $[a, b]$, differentiable on (a, b) and let $f(a) = f(b) = 0$. Prove that there is a point $c \in (a, b)$ such that $f'(c) + g'(c)f(c) = 0$.

13. Let $a > 0$ and $f : [-a, a] \to R$ be continuous. Suppose $f'(x)$ exists and $f'(x) \geq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$ then show that $f(x) = x$ for every $x \in (-a, a)$.

14. Consider the continuous function, $f, g : [a, b] \to R$. Prove that the equation $f(x)\int_a^x g(t)dt = g(x)\int_x^b f(t)\,dt$ has atleast one solution in the interval (a, b).

15. Prove that if the differentiable functions f and g satisfy $f'(x)g(x) = g'(x)f(x)\ \forall x \in R$, then between any two roots of $f(x) = 0$ there is a root of $g(x) = 0$.

16. Suppose that $f : [0, 1] \to R$ is differentiable, $f(0) = 0$ and $f(x) > 0$ for $x \in (0, 1)$. Prove that there is a number c in $(0, 1)$ such that $\frac{2f'(c)}{f(c)} = \frac{f'(1-c)}{f(1-c)}$.

SOLUTIONS

1. Consider the function $g(x) = f(x) - f\left(x + \frac{1}{2}\right)$

$$g(0) = f(0) - f\left(\frac{1}{2}\right)$$

$$g\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(0) \qquad \text{[ATQ]}$$

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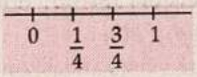
$\Rightarrow$ $g(0)$ and $g\left(\frac{1}{2}\right)$ are of opposite sign.

Hence, there exists atleast one $c \in \left(0, \frac{1}{2}\right)$ such that

$$g(c) = 0 \text{ or } = f(c) - f\left(c + \frac{1}{2}\right) = 0 \Rightarrow f(c) = f\left(c + \frac{1}{2}\right)$$

2. Applying MVT on $\left[0, \frac{1}{4}\right]$ gives

$$f'(c_1) = \frac{f\left(\frac{1}{4}\right) - f(0)}{\frac{1}{4} - 0}$$



$\Rightarrow$ $f'(c_1) = 4$

Applying MVT on $\left[\frac{1}{4}, \frac{3}{4}\right]$ gives

$$f'(c_2) = \frac{f\left(\frac{3}{4}\right) - f\left(\frac{1}{4}\right)}{\frac{3}{4} - \frac{1}{4}}$$

$\Rightarrow$ $f'(c_2) = 0$

Applying MVT on $\left[\frac{3}{4}, 1\right]$ gives

$$f'(c_3) = \frac{f(1) - f\left(\frac{3}{4}\right)}{1 - \frac{3}{4}} = -4$$

So, $f'(c_1) + f'(c_2) + f'(c_3) = 4 + 0 + (-4) = 0$.

3. Consider the function $g(x) = f(x) - x$

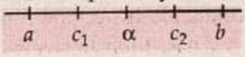
$g(a) = 0 = g(b)$ (ATQ)

Hence, there exists atleast one $c \in (a, b)$ such that

$g'(c) = 0$

(RT), *i.e.*, $f'(c) - 1 = 0$ or $f'(c) = 1$

Consider a quantity $\alpha \in (c_1, c_2)$ such that $f(\alpha) = \alpha$.



Applying RT on $[a, \alpha]$ gives $f'(c_1) = 1$ and applying RT on $[\alpha, b]$ gives $f'(c_2) = 1$.

Hence $f'(c_1) + f'(c_2) = 2$

4. Consider the function

$g(x) = (x - a)(x - b)f(x) + n_1(x - b) + n_2(x - a)$

Now, g is a continuous function and

$g(a) \cdot g(b) = -n_1 n_2 (a - b)^2 < 0$ (ATQ)

Hence, there exist a point $c \in (a, b)$ such that

$$g(c) = 0 \text{ i.e., } f(c) = \frac{n_1}{a - c} + \frac{n_2}{b - c}.$$

5. Applying MVT for the function $f(u) = \log u$ on $\left[a, \frac{a+b}{2}\right]$ gives

$$\frac{\log\left(\frac{a+b}{2}\right) - \log a}{\frac{a+b}{2} - a} = \frac{1}{x}, x \in \left(a, \frac{a+b}{2}\right)$$

$$\text{i.e., } \log\left(\frac{a+b}{2a}\right)^x = \frac{b-a}{2} \quad \text{...(i)}$$

Similarly, applying MVT on $\left[\frac{a+b}{2}, b\right]$ gives

$$\frac{\log b - \log\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} = \frac{1}{y}, y \in \left(\frac{a+b}{2}, b\right)$$

i.e., on rearranging, $\log\left(\frac{2b}{a+b}\right)^y = \frac{b-a}{2}$ ...(ii)

Hence from equations (i) & (ii), we have

$$\log\left(\frac{a+b}{2a}\right)^x = \log\left(\frac{2b}{a+b}\right)^y$$

$$\text{i.e., } \left(\frac{a+b}{2}\right)^{x+y} = a^x . b^y.$$

6. Consider the function

$$g(x) = \frac{1}{2}[(f(x))^2 - x^2]$$

$$g(a) = \frac{1}{2}[f^2(a) - a^2] \text{ and } g(b) = \frac{1}{2}[f^2(b) - b^2]$$

$\Rightarrow$ $g(a) = g(b)$ (ATQ)

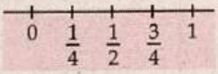
So, there exists atleast one $x \in (a, b)$ such that

$g'(x) = 0$ (RT)

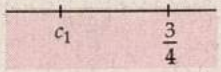
i.e., $1/2[2f(x)f'(x) - 2x] = 0$ or, $f(x)f'(x) = x$.

7. Using IVT,

$$f(c_1) = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right)\right]$$



$$\frac{1}{4} < c_1 < \frac{1}{2}$$



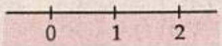
$$f(x_0) = \frac{2f(c_1) + f\left(\frac{3}{4}\right)}{3}$$

$$c_1 < x_0 < \frac{3}{4}.$$

$$\text{so, } f(x_0) = \frac{1}{3}\left[f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right]$$

8. Applying MVT on $[0, 1]$

$$f'(x_1) = \frac{f(1) - f(0)}{1 - 0} = 2$$



Applying MVT on $[1, 2]$

$$f'(x_2) = \frac{f(2) - f(1)}{2 - 1} = 2$$

$\Rightarrow$ $f'(x_1) = f'(x_2)$

So, there exists atleast one $x_0 \in (x_1, x_2)$ such that

$f''(x_0) = 0$ (RT)

9. Consider the function $g : [0, 1] \to R$,
$g(x) = f(x) - (1 + x + x^2 + ... + x^{n-1})$

Now, g is continuous and

$$\int_0^1 g(x)\,dx = \int_0^1 f(x)\,dx - \int_0^1 (1 + x + x^2 + ... + x^{n-1})\,dx$$

$= 0$ (ATQ).

So, from IVT, there exists atleast one $x_0 \in (0, 1)$ such that

$$i.e., f(x_0) = 1 + x_0 + x_0^2 + ... + x_0^{n-1}$$

$$= \frac{1 - x_0^n}{1 - x_0} \quad \text{[sum of } n \text{ G.P. terms]}$$

10. Consider the function $g : [a, b] \to R$

$$g(t) = \int_a^t f(x)\,dx - \int_t^b f(x)\,dx$$

Now, g is continuous and $g(a).g(b)$ = Negative

So, there exists atleast one $\alpha \in (a, b)$, such that $g(\alpha) = 0$

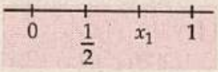
$$i.e., \int_a^\alpha f(x)\,dx = \int_\alpha^b f(x)\,dx \quad ...(i)$$

Applying MVT, there is a $\beta \in (\alpha, b)$ such that

$$\int_\alpha^b f(x)\,dx = (b - \alpha) f(\beta) \quad \text{[Average value]}$$

Hence, $\int_a^\alpha f(x)\,dx = (b - \alpha) f(\beta)$. [Using eqn. (i)]

11. $|f'(x)| \le \alpha$



$$\Rightarrow \left|\frac{f(x_1) - f\left(\frac{1}{2}\right)}{x_1 - \frac{1}{2}}\right| \le \alpha \quad \text{[MVT]}$$

$$\Rightarrow \left|f(x_1) - f\left(\frac{1}{2}\right)\right| \le \alpha \left|x_1 - \frac{1}{2}\right|$$

$$\text{or, } -\alpha\left(x_1 - \frac{1}{2}\right) \le f(x_1) - \frac{1}{2} \le \alpha\left(x_1 - \frac{1}{2}\right)$$

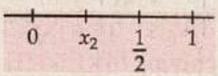
$$\text{or, } \frac{1}{2} - \alpha\left(x_1 - \frac{1}{2}\right) \le f(x_1) \le \alpha\left(x_1 - \frac{1}{2}\right) + \frac{1}{2}$$

So, Max. $f(x_1) = \frac{1}{2} + \alpha\left(x_1 - \frac{1}{2}\right) < 1$

and min. $f(x_1) = \frac{1}{2} - \alpha\left(x_1 - \frac{1}{2}\right) > -1$

Since, $\alpha\left(x_1 - \frac{1}{2}\right) < 1$

and $|f'(x)| \le \alpha$



$$\Rightarrow \left|\frac{f\left(\frac{1}{2}\right) - f(x_2)}{\frac{1}{2} - x_2}\right| \le \alpha \quad \text{[MVT]}$$

$$i.e., \frac{1}{2} - \alpha\left(\frac{1}{2} - x_2\right) \le f(x_2) \le \frac{1}{2} + \alpha\left(\frac{1}{2} - x_2\right)$$

$$\text{Max.} f(x_2) = \frac{1}{2} + \alpha\left(\frac{1}{2} - x_2\right) < \frac{1}{2} + \frac{1}{2} = 1$$

and Min. $f(x_2) = \frac{1}{2} - \alpha\left(\frac{1}{2} - x_2\right) > -1$

So, in any case (x_1) or (x_2)

$|f(x)| < 1$.

12. Consider the function $h(x) = f(x).e^{g(x)}$

Now, $h(a) = h(b) = 0$ (ATQ)

So, there exists atleast one $c \in (a, b)$ such that

$h'(c) = 0$ (RT)

i.e., $f'(c) + f(c)\,g'(c) = 0$.

13. Consider the function $g(x) = f(x) - x$

Now, $g(a) = 0$, $g(-a) = 0$ (ATQ)

and $g'(x) = f'(x) - 1 \ge 0$ (ATQ)

i.e., $g(x)$ is an increasing function

and hence, $g(x) = 0$ for all $x \in (-a, a)$

14. Consider the function $h : [a, b] \to R$

$$h(x) = \int_a^x g(t)\,dt \cdot \int_x^b f(t)\,dt$$

Now, h is differentiable and $h(a) = 0 = h(b)$

So, applying RT on $[a, b]$, there exists atleast one $c \in (a, b)$ such that $h'(c) = 0$

$$i.e., f(c)\int_a^c g(t)\,dt = g(c) \cdot \int_c^b f(t)\,dt$$

15. Let 'a' and 'b' be two roots of $f(x) = 0$, $a < b$. The condition implies that neither 'a' nor 'b' are roots of $g(x) = 0$. Suppose that g has no zeros between 'a' and 'b'. Then, by IVT, the sign of $g(x)$ is always same for all $x \in [a, b]$. Now, consider the function $h(x) = \frac{f(x)}{g(x)}$.

Hence, by RT, $h'(c) = 0$ for $c \in (a, b)$.

$$i.e., \frac{g(c)f'(c) - g'(c)f(c)}{g^2(c)} = 0,$$

but this contradicts the given condition. So, by contradiction $g(x)$ must have a root between 'a' and 'b'.

16. Consider the function $g(x) = f^2(x) f(1 - x)$ and apply RT on [0, 1].

Is there a number d in (0, 1) such that

$$\frac{3f'(d)}{f(d)} = \frac{f'(1-d)}{f(1-d)}?$$

Abbreviations used

ATQ – According to question
RT – Rolle's Theorem
MVT – Lagrange'sMean Value Theorem
IVT – Intermediate Value Theorem

OLYMPIAD CORNER

Challenging problems for Olympiads, IIT-JEE and other contests.

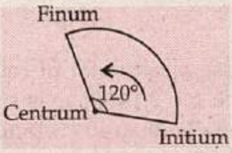
1. D and E are points on sides AB and AC of a triangle ABC such that $DE \| BC$, and P is an interior point of ΔADE, PB and PC meet DE at F and G respectively. Let O_1 and O_2 be the circumcentres of ΔPDG and ΔPFE respectively. Prove that $AP \perp O_1O_2$.

2. Show that for all natural numbers $n > 1$ the inequality $\left(\frac{1+(n+1)^{n+1}}{n+2}\right)^{n-1} > \left(\frac{1+n^n}{n+1}\right)^n$ is valid.

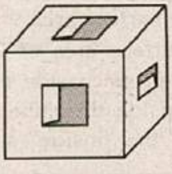
3. Find the exact value of
$$\cos\left(\frac{2\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\ldots\cos\left(\frac{16\pi}{17}\right)$$

4. Determine the area of a triangle of sides a, b, c and semiperimeters if
$$(s-b)(s-c)=\frac{a}{h},(s-c)(s-a)=\frac{b}{k},\ (s-a)(s-b)=\frac{c}{l},$$
where h, k, l are consistent given constants.

5. Two routes lead from Initium to Finum in a certain country. The Circular Route follows the arc of a circle of radius 100 km centred at Centrum and passing through Initium and Finum. The Radial Route follows the radius from Initium to Centrum and then the radius from Centrum to Finum. The arc of the circular route measures 120°, and the speed limit on the Circular Route is 105 km/hr while that on the Radial Route is 100 km/hr. Which is the quicker route to take?



6. A wooden cube 9 cm × 9 cm × 9 cm has three square holes drilled through it, each of which forms a 3 cm × 3 cm × 9 cm tunnel through the centre of opposite faces. What is the total surface area of the exposed wood?



7. Prove that for each positive integer n
$(2n^2 + 3n + 1)^n \geq 6^n(n!)^2$.

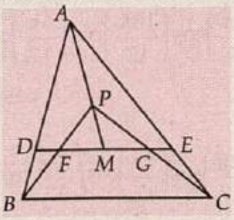
8. Sketch the graphs of $2x^2 + y^2 = 3$ and $xy = 1$ using the same axes for both graphs. Determine the co-ordinates of the points of intersection of the two curves.

9. Let A, B, C be vectors from the circumcentre of a triangle ABC to the respective vertices. Prove that
$$\frac{(B+C)|B-C|}{|B+C|}+\frac{(C+A)|C-A|}{|C+A|}+\frac{(A+B)|A-B|}{|A+B|}=0$$

10. Prove that
$3(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2) \geq xyz(x+y+z)^2$, where $x, y, z \geq 0$.

SOLUTIONS

1. In order to prove that $AP \perp O_1O_2$, it suffices to show that AP is the radical axis of the circles (O_1) and (O_2), because in this case AP would be perpendicular to the line of centres O_1O_2. Suppose AP intersects DE at the point M. The power of M with respect to (O_1) is $MD \cdot MG$, and with respect to (O_2) is $MF \cdot ME$. Therefore we must show $MD \cdot MG = MF \cdot ME$, or



$$\frac{MD}{MF}=\frac{ME}{MG} \quad \text{...(1)}$$

By the Menelaus theorem applied to triangle AME with the line CGP,
$$\frac{CE}{EA}\cdot\frac{AP}{PM}\cdot\frac{MG}{GE}=1; \quad \text{...(2)}$$
and the same theorem applied to triangle AMD with the line BFP gives us
$$\frac{BD}{DA}\cdot\frac{AP}{PM}\cdot\frac{MF}{FD}=1 \quad \text{...(3)}$$
Moreover, $\frac{AE}{EC}=\frac{AD}{BD}$, and therefore from (2) and (3)

we obtain $\frac{MG}{GE}=\frac{MF}{FD}$. By a property of proportions, this means $\frac{MG}{ME}=\frac{MF}{MD}$, and (1) follows.

2. By the power means inequality, we have

$$\sqrt[n]{\frac{1+(n+1)^{n+1}}{n+2}}=\sqrt[n]{\frac{1+(n+1)^{n}+...+(n+1)^{n+1}}{n+2}}$$
$$>\sqrt[n-1]{\frac{1+(n+1)^{n-1}+...+(n+1)^{n-1}}{n+2}}$$

If we take away one of the $(n+1)^{n-1}$ terms in this average, the average will obviously decrease, so

$$\sqrt[n]{\frac{1+(n+1)^{n+1}}{n+2}}>\sqrt[n-1]{\frac{1+(n+1)^{n-1}+...+(n+1)^{n-1}}{n+1}}$$
$$>\sqrt[n-1]{\frac{1+n^{n-1}+...+n^{n-1}}{n+1}}$$
$$=\sqrt[n-1]{\frac{1+n\cdot n^{n-1}}{n+1}}=\sqrt[n-1]{\frac{1+n^{n}}{n+1}}$$

and the proof is complete.

3. Let $I=\cos\left(\frac{2\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)...\cos\left(\frac{16\pi}{17}\right)$; then

$$I\sin\left(\frac{2\pi}{17}\right)=\frac{1}{2}\sin\left(\frac{4\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)...\cos\left(\frac{16\pi}{17}\right)$$
$$=\frac{1}{4}\sin\left(\frac{8\pi}{17}\right)\cos\left(\frac{8\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\cos\left(\frac{10\pi}{17}\right)$$
$$\cos\left(\frac{12\pi}{17}\right)\cos\left(\frac{14\pi}{17}\right)\cos\left(\frac{16\pi}{17}\right)$$
$$=\frac{1}{8}\sin\left(\frac{16\pi}{17}\right)\cos\left(\frac{16\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\cos\left(\frac{10\pi}{17}\right)$$
$$\cos\left(\frac{12\pi}{17}\right)\cos\left(\frac{14\pi}{17}\right)$$
$$=-\frac{1}{16}\sin\left(\frac{2\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\cos\left(\frac{10\pi}{17}\right)\cos\left(\frac{12\pi}{17}\right)\cos\left(\frac{14\pi}{17}\right)$$

Thus

$$I\sin\left(\frac{6\pi}{17}\right)=-\frac{1}{16}\sin\left(\frac{6\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\cos\left(\frac{10\pi}{17}\right)$$
$$\cos\left(\frac{12\pi}{17}\right)\cos\left(\frac{14\pi}{17}\right)$$
$$=-\frac{1}{32}\sin\left(\frac{12\pi}{17}\right)\cos\left(\frac{12\pi}{17}\right)\cos\left(\frac{10\pi}{17}\right)\cos\left(\frac{14\pi}{17}\right)$$
$$=\frac{1}{64}\sin\frac{10\pi}{17}\cos\frac{10\pi}{17}\cos\frac{14\pi}{17}$$
$$=-\frac{1}{128}\sin\left(\frac{14\pi}{17}\right)\cos\left(\frac{14\pi}{17}\right)$$
$$=-\frac{1}{256}\sin\left(\frac{28\pi}{17}\right)=\frac{1}{256}\sin\left(\frac{6\pi}{17}\right)$$

Therefore, $I=\frac{1}{256}$

4. $h=\frac{a}{(s-b)(s-c)}=\frac{1}{(s-b)}+\frac{1}{(s-c)}$,

$$k=\frac{1}{(s-c)}+\frac{1}{(s-a)},\ l=\frac{1}{(s-a)}+\frac{1}{(s-b)}$$

Hence, h, k, l must satisfy the triangle inequality. Letting $2s'=h+k+l$, it follows by addition that

$$s'=\frac{1}{(s-a)}+\frac{1}{(s-b)}+\frac{1}{(s-c)}$$

and then

$$s-a=\frac{1}{(s'-h)},\ s-b=\frac{1}{(s'-k)},\ s-c=\frac{1}{(s'-l)}$$

Adding the latter three equations, we get

$$s=\frac{1}{(s'-h)}+\frac{1}{(s'-k)}+\frac{1}{(s'-l)}$$

Finally, the area of the triangle is given by

$$\Delta=\{s(s-a)(s-b)(s-c)\}^{1/2}$$
$$=\left\{\frac{\frac{1}{(s'-h)}+\frac{1}{(s'-k)}+\frac{1}{(s'-l)}}{(s'-h)(s'-k)(s'-l)}\right\}^{1/2}$$

5. The length of the Radial Route is 100 + 100 = 200 km, and the top speed there is 100 km/hr, so you could drive it in 2 hours. For the Circular Route, since 120° is one-third of a complete circle, the distance of the Circular Route is

$$\frac{1}{3}\times2\times\pi\times100=\frac{200\pi}{3},$$

and the top speed there is 105 km/hr, so you could drive it in

$$\frac{200\pi}{3\times105}=\frac{40\pi}{3\times21}=\frac{40\pi}{63}\text{ hours.}$$

Now the question is, which is bigger,

$$2\text{ or }\frac{40\pi}{63}?$$

We could just use a calculator of course, but let's do it without. Since π = 3.14159 ... is smaller than 3.15, $40\pi/63$ will be smaller than $40\times3.15/63$ which is 126/63 which is exactly 2; thus the time required to drive the Circular Route is less, so the Circular Route is quicker.

6. The outside of the cube has six faces, each of which is a 9×9 square with a 3×3 square missing, so each will have surface area $9^2-3^2=72$ cm^2, for a total outside surface area of $72\times6=432$ cm^2. As well, drilled into each side is a $3\times3\times3$ hole with four sides inside the cube, each side of area $3\times3=9$ cm^2, so the "inside" surface area will be

$9 \times 4 \times 6 = 216$ cm^2. Thus the total exposed surface area is $432 + 216 = 648$ cm^2. (Note that the central $3 \times 3 \times 3$ "hole" in the cube has no "sides" to count.)

7. By the Arithmetic Mean-Geometric Mean Inequality we have

$$\frac{n(n+1)(2n+1)}{6n} = \frac{1^2+2^2+...+n^2}{n} \geq (1^2 2^2 .... n^2)^{1/n}$$
$$= (n!)^{2/n}$$

or $(n+1)(2n+1) \geq 6(n!)^{2/n}$
or $(2n^2+3n+1)^n \geq 6^n(n!)^2$
with equality if and only if $n = 1$.
Next we give Widhagen's version.
For $1 \leq k \leq n$ we have $k(n+1-k) \leq ((n+1)/2)^2$,

hence $(n!)^2 = \prod_{k=1}^{n} k(n+1-k) \leq \left(\frac{n+1}{2}\right)^{2n}$.

Therefore it suffices to show that

$$(2n+1)^n(n+1)^n \geq 6^n \cdot \left(\frac{n+1}{2}\right)^{2n}$$

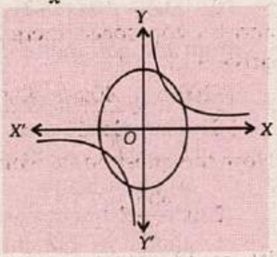
or $(2n+1)(n+1) \geq 6\left(\frac{n+1}{2}\right)^2$

or $2n+1 \geq \frac{3}{2}(n+1)$.

The last inequality holds trivially (and is strict unless $n = 1$).

8. From $y = \frac{1}{x}$ we get $2x^2 + \frac{1}{x^2} = 3$

so $2x^4 - 3x^2 + 1 = 0$
$\Rightarrow (2x^2-1)(x^2-1) = 0$
This gives $x^2 - 1 = 0$
or $2x^2 - 1 = 0$
so $x = \pm 1, \pm\frac{1}{\sqrt{2}}$.



The points of intersection are $(1, 1)$, $(-1, -1)$,

$\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right), \left(-\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$

9. The lengths of

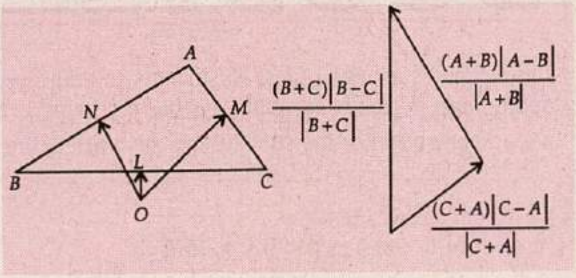
$$\frac{B+C}{|B+C|}|B-C|, \frac{C+A}{|C+A|}|C-A|, \frac{A+B}{|A+B|}|A-B|$$

are $|B-C| = a$, $|C-A| = b$, $|A-B| = c$,

so a triangle with sides equal to these lengths must be congruent to triangle ABC. With circumcentre as origin the directions of these vectors are perpendicular to BC, CA, AB respectively. So the given equation (in question) is simply a mapping of the relation

$(B-C) + (C-A) + (A-B) = 0$

under rotation by 90º. This is true always provided O is internal to triangle ABC. The figure below illustrates the situation for O an external point, where evidently a sign has to be adjusted appropriately.



OL, OM, ON are in directions
$B+C$, $C+A$, $A+B$

It can be seen that for $\angle A$ obtuse one has

$$\frac{(C+A)|C-A|}{|C+A|} + \frac{(A+B)|A-B|}{|A+B|} = \frac{(B+C)|B-C|}{|B+C|}$$

(When $\angle A = 90°$, $B + C$ vanishes and the relation degenerates and requires further interpretation).
When O is internal the situation is satisfactory, as the diagrams below show.

Now realtion (1) holds.

10. $(x^2y + y^2z + z^2x)(zx^2 + xy^2 + yz^2) \geq$

$$\left(x^2\sqrt{yz} + y^2\sqrt{zx} + z^2\sqrt{xy}\right)^2$$

Hence, it suffices to show that

$$\left\{\frac{(x^{3/2}+y^{3/2}+z^{3/2})}{3}\right\}^2 \geq \left\{\frac{(x+y+z)}{3}\right\}^3$$

But this follows immediately from the power mean inequality. There is equality iff $x = y = z$.

concept BOOSTERS

Class XI

Trigonometry - I

– MTG Editorial Board

This column is aimed at Class XI students so that they can prepare for competitive exams such as IIT, AIEEE, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult, the easy and the challenging.

Trigonometric Identities

1. (i) Let $0 < \alpha < \pi$, $0 < \beta < \pi$, and
$$\cos\alpha + \cos\beta - \cos(\alpha+\beta) = \frac{3}{2}.$$
Prove that $\alpha = \beta = \dfrac{\pi}{3}$

(ii) Let $\cos\theta + \cos\varphi = a$, $\sin\theta + \sin\varphi = b$ compute $\cos(\theta + \varphi)$ and $\sin(\theta + \varphi)$.

(iii) If $\cos^2\theta = \dfrac{\cos\alpha}{\cos\beta}, \cos^2\varphi = \dfrac{\cos\gamma}{\cos\beta}, \dfrac{\tan\theta}{\tan\varphi} = \dfrac{\tan\alpha}{\tan\gamma}$, then prove that $\tan^2\left(\dfrac{\alpha}{2}\right).\tan^2\left(\dfrac{\gamma}{2}\right) = \tan^2\left(\dfrac{\beta}{2}\right)$.

(iv) Prove that if $\dfrac{\cos x - \cos\alpha}{\cos x - \cos\beta} = \dfrac{\sin^2\alpha\cos\beta}{\sin^2\beta\cos\alpha}$ then one of the value of $\tan\left(\dfrac{x}{2}\right)$ is, $\tan\left(\dfrac{\alpha}{2}\right).\tan\left(\dfrac{\beta}{2}\right)$.

(v) Prove that if $\cos\theta = \cos\alpha\cos\beta$, $\cos\varphi = \cos\alpha_1\cos\beta$, $\tan\left(\dfrac{\theta}{2}\right)\tan\left(\dfrac{\varphi}{2}\right) = \tan\left(\dfrac{\beta}{2}\right)$, then $\sin^2\beta = \left(\dfrac{1}{\cos\alpha} - 1\right)\left(\dfrac{1}{\cos\alpha_1} - 1\right)$

2. (i) Let $\dfrac{\sin(\theta-\alpha)}{\sin(\theta-\beta)} = \dfrac{a}{b}, \dfrac{\cos(\theta-\alpha)}{\cos(\theta-\beta)} = \dfrac{c}{d}$ prove that $\cos(\alpha-\beta) = \dfrac{ac+bd}{ad+bc}$

(ii) Given $\dfrac{e^2-1}{1+2e\cos\alpha+e^2} = \dfrac{1+2e\cos\beta+e^2}{e^2-1}$.

Prove that

(a) $\dfrac{e^2-1}{1+2e\cos\alpha+e^2} = \dfrac{e+\cos\beta}{e+\cos\alpha}$
$$= \pm\frac{\sin\beta}{\sin\alpha} = -\frac{1+e\cos\beta}{1+e\cos\alpha}$$

(b) $\tan\left(\dfrac{\alpha}{2}\right).\tan\left(\dfrac{\beta}{2}\right) = \pm\dfrac{1+e}{1-e}$

(iii) Given that α and β are different solutions of the equation $a\cos x + b\sin x = c$. Prove that
$$\cos^2\left(\frac{\alpha-\beta}{2}\right) = \frac{c^2}{a^2+b^2}$$

(iv) Show that if $\dfrac{\cos x}{a} = \dfrac{\cos(x+\theta)}{b} = \dfrac{\cos(x+2\theta)}{c} = \dfrac{\cos(x+3\theta)}{d}$, then $\dfrac{a+c}{b} = \dfrac{b+d}{c}$

(v) Let $x\cos(\alpha + \beta) + \cos(\alpha - \beta) = x\cos(\beta + \gamma) + \cos(\beta - \gamma) = x\cos(\gamma + \alpha) + \cos(\gamma - \alpha)$. Prove that
$$\frac{\tan\alpha}{\tan\frac{1}{2}(\beta+\gamma)} = \frac{\tan\beta}{\tan\frac{1}{2}(\alpha+\gamma)} = \frac{\tan\gamma}{\tan\frac{1}{2}(\alpha+\beta)}$$

3. (i) Show that the result of elimination of θ and φ from the equations $\cos\theta = \dfrac{\sin\beta}{\sin\alpha}, \cos\varphi = \dfrac{\sin\gamma}{\sin\alpha}$, $\cos(\theta - \varphi) = \sin\beta\sin\gamma$ is $\tan^2\alpha = \tan^2\beta + \tan^2\gamma$

(ii) Eliminate θ and φ from the equations $a\sin^2\theta + b\cos^2\theta = a\cos^2\varphi + b\sin^2\varphi = 1$, $a\tan\theta = b\tan\varphi$.

(iii) Prove that if $\cos(\theta - \alpha) = a$, $\sin(\theta - \beta) = b$, then $a^2 - 2ab\sin(\alpha - \beta) + b^2 = \cos^2(\alpha - \beta)$.

(iv) Prove that from the equalities $x\cos\theta + y\sin\theta = x\cos\varphi + y\sin\varphi = 2a$ and $2\sin\left(\dfrac{\theta}{2}\right)\sin\left(\dfrac{\varphi}{2}\right) = 1$ follows $y^2 = 4a(a - x)$.

(v) Let $\cos\alpha = \cos\beta\cos\varphi = \cos\gamma\cos\theta$, $\sin\alpha = 2\sin\left(\dfrac{\varphi}{2}\right)\sin\left(\dfrac{\theta}{2}\right)$. Prove that
$$\tan^2\left(\frac{\alpha}{2}\right) = \tan^2\left(\frac{\beta}{2}\right).\tan^2\left(\frac{\gamma}{2}\right).$$

4. Solve the following equations

(i) $\cos 3x\cos^3 x + \sin 3x\sin^3 x = 0$.

(ii) $\sin 2x + \cos 2x + \sin x + \cos x + 1 = 0$.

(iii) $\tan^2 x = \dfrac{1-\cos x}{1-\sin x}$

(iv) $32\cos^6 x - \cos 6x = 1$

(v) $\sin^4 x + \cos^4 x - 2\sin 2x + \dfrac{3}{4}\sin^2 2x = 0$

5. Solve the following equations :

(i) $\sin 6x + \sin 4x = 0$

(ii) $\sin x = \cos 2x$

(iii) $\sin 2x + \cos 2x = -1$

(iv) $12\cos x - 5\sin x = -13$

(v) $\sin 3x + \cos 2x = 1$

Trigonometric Equations with a Parameter

6. (i) Solve and analyze the equation $\sin 3x + \sin 2x = m\sin x$.

(ii) Solve the equation

$$(1+k)\frac{\cos x\cos(2x-\alpha)}{\cos(x-\alpha)} = 1 + k\cos 2x$$

(iii) Find $\cot x$ from the equation $\cos^2(\alpha + x) + \cos^2(\alpha - x) = a$, where $0 < a < 2$. For what α the problem is solvable?

(iv) For what a the equation $\sin^2 x - \sin x\cos x - 2\cos^2 x = a$ is solvable? Find the solutions.

(v) Determine all the values of a for which the equation $\sin^4 x - 2\cos^2 x + a^2 = 0$ is solvable. Find the solutions.

(vi) Determine the range of the values of the parameter λ for which the equation $\sec x + \operatorname{cosec} x = \lambda$, possesses a root x satisfying the inequality $0 < x < \dfrac{\pi}{2}$.

Miscellaneous

7. (i) Find the least and greatest values of the function $f(x) = \sin(\cos(\sin x))$ on the closed interval $\left[\dfrac{\pi}{2}, \pi\right]$.

(ii) Prove that $\sin\alpha+\sin\beta+\sin\gamma - \sin(\alpha + \beta + \gamma)$

$$= 4\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta+\gamma}{2}\right)\sin\left(\frac{\gamma+\alpha}{2}\right)$$

(iii) Solve the equation $\sin^2 4x + \cos^2 x = 2\sin 4x\cos^4 x$.

(iv) (a) For $\alpha + \beta + \gamma = \pi$ prove the identity

$$\sin 2n\alpha + \sin 2n\beta + \sin 2n\gamma = (-1)^{n+1} 4\sin n\alpha\sin n\beta\sin n\gamma$$

where n is an integer.

(b) Prove that if $\sin\alpha = A\sin(\alpha + \beta)$, then $\tan(\alpha+\beta) = \dfrac{\sin\beta}{\cos\beta - A}$ for all permissible values of α and β.

(c) Prove that if the angles α and β satisfy the relation $\dfrac{\sin\beta}{\sin(2\alpha+\beta)} = \dfrac{n}{m}(|m|>|n|)$, then

$$\frac{1+\dfrac{\tan\beta}{\tan\alpha}}{m+n} = \frac{1-\tan\alpha.\tan\beta}{m-n}$$

(d) Prove that if $\cos x.\cos y.\cos z \neq 0$, the formula $\cos(x + y + z) = \cos x\cos y\cos z(1 - \tan x\tan y - \tan y\tan z - \tan z\tan x)$ holds true.

(e) Rewrite as a product the expression $\cot^2 2x - \tan^2 2x - 8\cos 4x\cot 4x$.

8. (i) Show that if $(x-a)\cos\theta + y\sin\theta = (x-a)\cos\theta_1 + y\sin\theta_1 = a$ and $\tan\left(\dfrac{\theta}{2}\right) - \tan\left(\dfrac{\theta_1}{2}\right) = 2l$, then $y^2 = 2ax - (1 - l^2)x^2$.

(ii) Let $\cos\theta = \cos\alpha\cos\beta$. Prove that

$$\tan\left(\frac{\theta+\alpha}{2}\right).\tan\left(\frac{\theta-\alpha}{2}\right) = \tan^2\left(\frac{\beta}{2}\right)$$

(iii) Eliminate θ from the equations

$$(a - b)\sin(\theta + \varphi) = (a + b)\sin(\theta - \varphi),$$

$$a\tan\left(\frac{\theta}{2}\right) - b\tan\left(\frac{\varphi}{2}\right) = c$$

(iv) Prove that if

$$\frac{\sin(\theta-\beta)\cos\alpha}{\sin(\varphi-\alpha)\cos\beta} + \frac{\cos(\alpha+\theta)\sin\beta}{\cos(\varphi-\beta)\sin\alpha} = 0 \text{ and}$$

$$\frac{\tan\theta\tan\alpha}{\tan\varphi\tan\beta} + \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = 0, \text{ then}$$

$$\tan\theta = \frac{1}{2}(\tan\beta + \cot\alpha), \tan\varphi = \frac{1}{2}(\tan\alpha - \cot\beta).$$

(v) Find the greatest and least values of the function $\tan(\cos x)$ on the interval $\left[\dfrac{\pi}{2}, \pi\right]$

9. (i) Solve the equation $\sin x + \cos x = 1 - \sin 2x$.

(ii) (a) Transform into a product the expression $\sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\cos\alpha\cos\beta\cos\beta\cos\gamma - 2$.

(b) Compute $\dfrac{1}{2\sin 10^\circ} - 2\sin 70^\circ$ without using tables.

(c) Prove that $\cos\left(\dfrac{\pi}{5}\right) - \cos\left(\dfrac{2\pi}{5}\right) = \dfrac{1}{2}$

(d) Compute

$$\sin^4\left(\frac{\pi}{16}\right) + \sin^4\left(\frac{3\pi}{16}\right) + \sin^4\left(\frac{5\pi}{16}\right) + \sin^4\left(\frac{7\pi}{16}\right)$$

without using tables.

(iii) If $a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x = 0$ for all $x \in R$, show that $a_0 = a_1 = a_2 = a_3 = 0$

(iv) Find $x, y, z \in R$ satisfying

$$\frac{4\sqrt{x^2+1}}{x} = \frac{5\sqrt{y^2+1}}{y} = \frac{6\sqrt{z^2+1}}{z} \text{ and}$$

$xyz = x + y + z$.

SOLUTIONS

1. (i) We have

$$2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)-\left(2\cos^2\left(\frac{\alpha+\beta}{2}\right)-1\right)=\frac{3}{2}$$

or $4\cos^2\left(\frac{\alpha+\beta}{2}\right)-4\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)+1=0$

Hence

$$\cos\left(\frac{\alpha+\beta}{2}\right)=\frac{4\cos\left(\frac{\alpha-\beta}{2}\right)\pm\sqrt{16\cos^2\left(\frac{\alpha-\beta}{2}\right)-16}}{8}$$

Since the radicand is equal to $-16\sin^2\left(\frac{\alpha-\beta}{2}\right)$ and $\cos\left(\frac{\alpha+\beta}{2}\right)$ is real, the expression $-16\sin^2\left(\frac{\alpha-\beta}{2}\right)$ must be greater than, or equal to zero. But this expression cannot exceed zero. Therefore, we have $\sin\left(\frac{\alpha-\beta}{2}\right)=0$. But since $0 < \alpha < \pi$ and $0 < \beta < \pi$, we have $\alpha = \beta$ and consequently, $\cos\alpha=\frac{1}{2}$ and $\alpha=\beta=\frac{\pi}{3}$

(ii) By hypothesis $2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)=a$,

$2\sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)=b$.

Hence $\tan\left(\frac{\theta+\varphi}{2}\right)=\frac{b}{a}$

But $\cos x=\dfrac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$, $\sin x=\dfrac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$

Therefore

$$\cos(\theta+\varphi)=\frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}}=\frac{a^2-b^2}{a^2+b^2},$$

$$\sin(\theta+\varphi)=\frac{2.\frac{b}{a}}{1+\frac{b^2}{a^2}}=\frac{2ab}{a^2+b^2}$$

(iii) We have

$1+\tan^2\theta=\frac{\cos\beta}{\cos\alpha}$, $1+\tan^2\varphi=\frac{\cos\beta}{\cos\gamma}$

Hence

$$\frac{\tan^2\theta}{\tan^2\varphi}=\frac{\cos\beta-\cos\alpha}{\cos\alpha}\cdot\frac{\cos\gamma}{\cos\beta-\cos\gamma}$$

On the other hand, it is given that

$$\frac{\tan^2\theta}{\tan^2\varphi}=\frac{\tan^2\alpha}{\tan^2\gamma}$$

Therefore, we have

$$\frac{\cos\beta-\cos\alpha}{\cos\beta-\cos\gamma}\cdot\frac{\cos\gamma}{\cos\alpha}=\frac{\tan^2\alpha}{\tan^2\gamma}$$

From this equality, we get

$$\cos\beta=\frac{\cos^2\alpha\sin^2\gamma-\cos^2\gamma\sin^2\alpha}{\cos\alpha\sin^2\gamma-\sin^2\alpha\cos\gamma}$$

$$=\frac{\sin^2\gamma-\sin^2\alpha}{\cos\alpha\sin^2\gamma-\sin^2\alpha\cos\gamma}$$

But, $\tan^2\frac{\beta}{2}=\frac{1-\cos\beta}{1+\cos\beta}$

$$=\frac{\cos\alpha\sin^2\gamma-\sin^2\alpha\cos\gamma-\sin^2\gamma+\sin^2\alpha}{\cos\alpha\sin^2\gamma-\sin^2\alpha\cos\gamma+\sin^2\gamma-\sin^2\alpha}$$

$$\frac{\sin^2\alpha(1-\cos\gamma)-\sin^2\gamma(1-\cos\alpha)}{\sin^2\gamma(1+\cos\alpha)-\sin^2\alpha(1+\cos\gamma)}$$

$$=\frac{8\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\sin^2\frac{\gamma}{2}-8\sin^2\frac{\gamma}{2}\cos^2\frac{\gamma}{2}\sin^2\frac{\alpha}{2}}{8\sin^2\frac{\gamma}{2}\cos^2\frac{\gamma}{2}\cos^2\frac{\alpha}{2}-8\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\cos^2\frac{\gamma}{2}}$$

$$=\frac{\sin^2\left(\frac{\alpha}{2}\right)\sin^2\left(\frac{\gamma}{2}\right)\left(\cos^2\left(\frac{\alpha}{2}\right)-\cos^2\left(\frac{\gamma}{2}\right)\right)}{\cos^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\gamma}{2}\right)\left(\sin^2\left(\frac{\gamma}{2}\right)-\sin^2\left(\frac{\alpha}{2}\right)\right)}$$

$$=\tan^2\left(\frac{\alpha}{2}\right).\tan^2\left(\frac{\gamma}{2}\right)$$

since $\cos^2\left(\frac{\alpha}{2}\right)-\cos^2\left(\frac{\gamma}{2}\right)=\sin^2\left(\frac{\gamma}{2}\right)-\sin^2\left(\frac{\alpha}{2}\right)$

(iv) Solving the given equation with respect to $\cos x$ we find

$\cos x(\sin^2\beta\cos\alpha - \sin^2\alpha\cos\beta)$

$= \cos^2\alpha\sin^2\beta - \sin^2\alpha\cos^2\beta = \cos^2\alpha - \cos^2\beta$

But,

$\sin^2\beta\cos\alpha - \sin^2\alpha\cos\beta$

$= \cos\alpha(1 - \cos^2\beta) - \cos\beta(1 - \cos^2\alpha)$

$= \cos\alpha - \cos\beta + \cos\alpha\cos\beta(\cos\alpha - \cos\beta)$

$= (\cos\alpha - \cos\beta)(1 + \cos\alpha\cos\beta)$

therefore $\cos x=\dfrac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}$

Further $\tan^2\left(\frac{x}{2}\right)=\frac{1-\cos x}{1+\cos x}$

$$=\frac{1+\cos\alpha\cos\beta-\cos\alpha-\cos\beta}{1+\cos\alpha\cos\beta+\cos\alpha+\cos\beta}$$

$$=\frac{(1-\cos\alpha)(1-\cos\beta)}{(1+\cos\alpha)(1+\cos\beta)}=\tan^2\left(\frac{\alpha}{2}\right)\tan^2\left(\frac{\beta}{2}\right)$$

and consequently

$$\tan\left(\frac{x}{2}\right)=\pm\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)$$

(v) Put $\tan\left(\frac{\theta}{2}\right)=x$, $\tan\left(\frac{\varphi}{2}\right)=y$

Then $\cos\theta = \frac{1-x^2}{1+x^2} = \cos\alpha\cos\beta$,

$\cos\varphi = \frac{1-y^2}{1+y^2} = \cos\alpha_1.\cos\beta$

Further, from (1) & (2), we get

$$x^2 = \frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta}, y^2 = \frac{1-\cos\alpha_1\cos\beta}{1+\cos\alpha_1\cos\beta},$$

Now

$$\tan^2\left(\frac{\beta}{2}\right) = x^2y^2 = \frac{(1-\cos\alpha\cos\beta)(1-\cos\alpha_1\cos\beta)}{(1+\cos\alpha\cos\beta)(1+\cos\alpha_1\cos\beta)}$$

Adding unity to both members of the equality, we get $\frac{2}{1+\cos\beta} = \frac{2(1+\cos\alpha\cos\alpha_1\cos^2\beta)}{(1+\cos\alpha\cos\beta)(1+\cos\alpha_1\cos\beta)}$

Assuming $\cos\beta \neq 0$, we obtain

$\cos\alpha + \cos\alpha_1 = 1 + \cos\alpha\cos\alpha_1\cos^2\beta$

$\cos\alpha + \cos\alpha_1 = 1 + \cos\alpha\cos\alpha_1(1 - \sin^2\beta)$,

$\cos\alpha\cos\alpha_1\sin^2\beta$

$= 1 + \cos\alpha\cos\alpha_1 - \cos\alpha - \cos\alpha_1$

$= (1 - \cos\alpha)(1 - \cos\alpha_1)$

and consequently, indeed

$$\sin^2\beta = \left(\frac{1}{\cos\alpha} - 1\right)\left(\frac{1}{\cos\alpha_1} - 1\right)$$

2. (i) Rewrite the given equalities in the following way

$\sin\theta(b\cos\alpha - a\cos\beta) = \cos\theta(b\sin\alpha - a\sin\beta)$

$\sin\theta(d\sin\alpha - c\sin\beta) = \cos\theta(c\cos\beta - d\cos\alpha)$

Eliminating θ, we find

$(b\cos\alpha - a\cos\beta)(c\cos\beta - d\cos\alpha)$

$= (b\sin\alpha - a\sin\beta)(d\sin\alpha - c\sin\beta)$

Hence,

$bc\cos\alpha\cos\beta - ac\cos^2\beta - bd\cos^2\alpha + ad\cos\alpha\cos\beta$

$= bd\sin^2\alpha - ad\sin\alpha\sin\beta - bc\sin\alpha\sin\beta + ac\sin^2\beta$

or $(bc + ad)\cos\alpha\cos\beta + (bc + ad)\sin\alpha\sin\beta$

$= bd + ac$

Finally, $\cos(\alpha-\beta) = \frac{bd+ac}{bc+ad}$

(ii) (a) we have

$$\frac{e^2-1}{1+2e\cos\alpha+e^2} = \frac{1+2e\cos\beta+e^2}{e^2-1}$$

$$= \frac{2e^2+2e\cos\beta}{2e^2+2e\cos\alpha} = \frac{e+\cos\beta}{e+\cos\alpha}$$

(By the property of proportions, from the equality ($\frac{a}{b}=\frac{c}{d}$ follows $\frac{a+c}{b+d}=\frac{a}{b}$)

Similarly, we have

$$\frac{e^2-1}{1+2e\cos\alpha+e^2} = \frac{1+2e\cos\beta+e^2}{e^2-1}$$

$$= \frac{-2-2e\cos\beta}{2+2e\cos\alpha} = -\frac{1+e\cos\beta}{1+e\cos\alpha}$$

Then

$$\left(\frac{e+\cos\beta}{e+\cos\alpha}\right)^2 = \frac{(1+e\cos\beta)^2}{(1+e\cos\alpha)^2}$$

$$\Rightarrow \left(\frac{e+\cos\beta}{e+\cos\alpha}\right)^2 = \frac{e^2+\cos^2\beta-1-e^2\cos^2\beta}{e^2+\cos^2\alpha-1-e^2\cos^2\alpha} = \frac{\sin^2\beta}{\sin^2\alpha}$$

Consequently,

$$\frac{e^2-1}{1+2e\cos\alpha+e^2} = -\frac{1+e\cos\beta}{1+e\cos\alpha} = \pm\frac{\sin\beta}{\sin\alpha}$$

(b) From given equality, it follows that

$$\frac{e+\cos\beta}{e+\cos\alpha} = -\frac{1+e\cos\beta}{1+e\cos\alpha}$$

Consequently,

$$\frac{e+\cos\beta-1-e\cos\beta}{e+\cos\beta+1+e\cos\beta} = \frac{e+\cos\beta+1+e\cos\beta}{e+\cos\beta-1-e\cos\beta}$$

(From the equality $\frac{a}{b}=\frac{c}{d}$ follows $\frac{a+c}{b+d}=\frac{a-c}{b-d}$)

Further

$$\frac{(1-e)(1-\cos\beta)}{(1+e)(1+\cos\alpha)} = \frac{(1+e)(1+\cos\beta)}{(1-e)(1-\cos\alpha)}$$

or

$$(1-\cos\beta)(1-\cos\alpha) = \frac{(1+e)^2}{(1-e)^2}(1+\cos\beta)(1+\cos\alpha)$$

Finally $\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = \pm\frac{1+e}{1-e}$

(iii) By hypothesis, we have

$a\cos\alpha + b\sin\alpha = c$, $a\cos\beta + b\sin\beta = c$.

Adding these equalities termwise we find,

$$2a\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2b\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = 2c$$

Hence

$$\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{c}{a\cos\left(\frac{\alpha+\beta}{2}\right) + b\sin\left(\frac{\alpha+\beta}{2}\right)}$$

Now subtracting the given equalities termwise, we obtain

$$-2a\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) + 2b\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) = 0$$

Since α and β are different solutions of the equation, then $\sin\left(\frac{\alpha-\beta}{2}\right) \neq 0$. Consequently, the last equality yields $\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{b}{a}$.

Let us return to compute $\cos^2\left(\frac{\alpha-\beta}{2}\right)$.

We have

$$\cos^2\left(\frac{\alpha-\beta}{2}\right) = \frac{c^2}{\cos^2\left(\frac{\alpha+\beta}{2}\right)\left(a+b\tan\left(\frac{\alpha+\beta}{2}\right)\right)^2}$$

$$= c^2\left(1+\frac{b^2}{a^2}\right)\frac{1}{\left(a+b\left(\frac{b}{a}\right)\right)^2} = \frac{c^2}{a^2+b^2}$$

(iv) We have

$$\frac{a+c}{b+d}=\frac{\cos x+\cos(x+2\theta)}{\cos(x+\theta)+\cos(x+3\theta)}$$

$$=\frac{\cos(x+\theta)\cos\theta}{\cos(x+2\theta)\cos\theta}=\frac{b}{c}$$

Hence $\frac{a+c}{b}=\frac{b+d}{c}$

(v) We have

$$\frac{\cos(\beta-\gamma)-\cos(\alpha-\beta)}{\cos(\alpha+\beta)-\cos(\beta+\gamma)}=\frac{\cos(\gamma-\alpha)-\cos(\beta-\gamma)}{\cos(\beta+\gamma)-\cos(\gamma+\alpha)}$$

$$=\frac{\cos(\alpha-\beta)-\cos(\gamma-\alpha)}{\cos(\gamma+\alpha)-\cos(\alpha+\beta)}=x$$

Hence

$$=\frac{\sin\left(\frac{\alpha+\gamma}{2}-\beta\right)}{\sin\left(\frac{\alpha+\gamma}{2}+\beta\right)}=\frac{\sin\left(\frac{\beta+\alpha}{2}-\gamma\right)}{\sin\left(\frac{\beta+\alpha}{2}+\gamma\right)}=\frac{\sin\left(\frac{\gamma+\beta}{2}-\alpha\right)}{\sin\left(\frac{\gamma+\beta}{2}+\alpha\right)}$$

$$=\frac{\tan\beta-\tan\left(\frac{\alpha+\gamma}{2}\right)}{\tan\beta+\tan\left(\frac{\alpha+\gamma}{2}\right)}=\frac{\tan\gamma-\tan\left(\frac{\beta+\alpha}{2}\right)}{\tan\gamma+\tan\left(\frac{\beta+\alpha}{2}\right)}$$

$$=\frac{\tan\alpha-\tan\left(\frac{\beta+\gamma}{2}\right)}{\tan\alpha-\tan\left(\frac{\beta+\gamma}{2}\right)}$$

But from the equalities

$\frac{a-b}{a+b}=\frac{a'-b'}{a'+b'}=\frac{a''-b''}{a''+b''}$ follows $\frac{a}{b}=\frac{a'}{b'}=\frac{a''}{b''}$

Therefore, we have

$$\frac{\tan\alpha}{\tan\frac{1}{2}(\beta+\gamma)}=\frac{\tan\beta}{\tan\frac{1}{2}(\alpha+\gamma)}=\frac{\tan\gamma}{\tan\frac{1}{2}(\alpha+\beta)}$$

3. **(i)** From the third equality we obtain

$\sin^2\theta\sin^2\varphi=(\cos\theta\cos\varphi-\sin\beta\sin\gamma)^2$

Using the first two equalities, we find

$$\left(1-\frac{\sin^2\beta}{\sin^2\alpha}\right)\left(1-\frac{\sin^2\gamma}{\sin^2\alpha}\right)=\left(\frac{\sin\beta\sin\gamma}{\sin^2\alpha}-\sin\beta\sin\gamma\right)^2$$

After some transformations this equality yields

$\tan^2\alpha=\tan^2\gamma+\tan^2\beta$

(ii) We have

$a\sin^2\theta+b\cos^2\theta=1,\ a\cos^2\varphi+b\sin^2\varphi=1$

Hence

$a\tan^2\theta+b=1+\tan^2\theta,\ b\tan^2\varphi+a=1+\tan^2\varphi$

Consequently

$(a-1)\tan^2\theta=1-b,\ (b-1)\tan^2\varphi=1-a$

$$\frac{\tan^2\theta}{\tan^2\varphi}=\left(\frac{1-b}{1-a}\right)^2$$

On the other hand, $\frac{\tan^2\theta}{\tan^2\varphi}=\frac{b^2}{a^2}$

From the last two equalities we get (assuming that a is not equal to b)

$a+b-2ab=0.$

(iii) Rewrite the first two equalities in the following way $\cos\theta\cos\alpha+\sin\theta\sin\alpha=a$ ...(1)

$\sin\theta\cos\beta-\cos\theta\sin\beta=b$ ...(2)

Multiplying (1) by $\sin\beta$ and (2) by $\cos\alpha$, and adding them, we find

$\sin\theta\cos(\alpha-\beta)=a\sin\beta+b\cos\alpha$ ...(3)

Again multiply (1) by $\cos\beta$ and (2) by $-\sin\alpha$ and then adding them, we find

$\cos\theta\cos(\alpha-\beta)=a\cos\beta-b\sin\alpha$ ...(4)

Squaring (3) & (4) and adding them, we get

$\cos^2(\alpha-\beta)=a^2-2ab\sin(\alpha-\beta)+b^2.$

(iv) From the first two equalities it is obvious that θ and φ are the roots of the equation

$x\cos\alpha+y\sin\alpha-2a=0$ (unknown α)

It is clear that θ and φ are also the roots of the equation $(2a-x\cos\alpha)^2=y^2\sin^2\alpha$

Transform the last equation in the following way

$x^2\cos^2\alpha-4ax\cos\alpha+4a^2=y^2(1-\cos^2\alpha),$

$(x^2+y^2)\cos^2\alpha-4ax\cos\alpha+4a^2-y^2=0$

Therefore the quantities $\cos\theta$ and $\cos\varphi$ are the roots of the following equation

$(x^2+y^2)z^2-4axz+4a^2-y^2=0$ and therefore

$$\cos\theta\cos\varphi=\frac{4a^2-y^2}{x^2+y^2},\ \cos\theta+\cos\varphi=\frac{4ax}{x^2+y^2}$$

We then have

$$4\sin^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\varphi}{2}\right)=4\left(\frac{1-\cos\theta}{2}\right)\cdot\left(\frac{1-\cos\varphi}{2}\right)=1$$

or $1-(\cos\theta+\cos\varphi)+\cos\theta\cos\varphi=1.$

Hence, $y^2=4a(a-x)$

(v) We have

$$\sin^2\alpha=4\sin^2\left(\frac{\varphi}{2}\right)\sin^2\left(\frac{\theta}{2}\right)=(1-\cos\varphi)(1-\cos\theta)$$

$$=\left(1-\frac{\cos\alpha}{\cos\beta}\right)\left(1-\frac{\cos\alpha}{\cos\gamma}\right)$$

Hence

$$1-\cos^2\alpha=1-\cos\alpha\left(\frac{\cos\beta+\cos\gamma}{\cos\beta\cos\gamma}\right)+\left(\frac{\cos^2\alpha}{\cos\beta\cos\gamma}\right)$$

i.e. $\cos^2\alpha\left(1+\frac{1}{\cos\beta\cos\gamma}\right)=\cos\alpha\left(\frac{\cos\beta+\cos\gamma}{\cos\beta\cos\gamma}\right)$

Assuming that $\cos\alpha$ is non-zero, we find

$$\cos\alpha=\frac{\cos\gamma+\cos\beta}{1+\cos\gamma\cos\beta}$$

Now it is easy to ckeck that

$$\tan^2\left(\frac{\alpha}{2}\right)=\tan^2\left(\frac{\beta}{2}\right)\tan^2\left(\frac{\gamma}{2}\right)$$

4. **(i)** Since $\cos 3x=\cos^3x-3\sin^2x\cos x,$

$\sin 3x=-\sin^3x+3\sin x\cos^2x$

Put $\sin 3x$ & $\cos 3x$ in given equation

$(\cos^3 x - 3\sin^2 x\cos x)\cos^3 x$
$+ (-\sin^3 x + 3\sin x\cos^2 x)\sin^3 x = 0$

$\cos^6 x - 3\cos^4 x\sin^2 x + 3\sin^4 x\cos^2 x - \sin^6 x = 0$

or $(\cos^2 x - \sin^2 x)^3 = 0$, $\cos 2x = 0$

(ii) Since $\sin 2x + 1 = (\sin x + \cos x)^2$, we have

$(\sin x + \cos x)^2 + (\sin x + \cos x) + \cos^2 x - \sin^2 x = 0$

Hence, $(\sin x + \cos x)(1 + 2\cos x) = 0$

or $\cos x(1 + \tan x)(1 + 2\cos x) = 0$

and so, $\tan x = -1$ and $\cos x = -\frac{1}{2}$

are the required solutions of our equation.

(iii) We have $\frac{\sin^2 x}{\cos^2 x} - \frac{1-\cos x}{1-\sin x} = 0$

Hence

$$\frac{(\cos^3 x - \sin^3 x) - (\cos^2 x - \sin^2 x)}{\cos^2 x(1-\sin x)} = 0$$

or $(1 - \tan x)(1 - \cos x) = 0$

Hence $\tan x = 1$ and $\cos x = 1$.

(iv) We have

$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

Therefore $\cos 6x = 4\cos^3 2x - 3\cos 2x$

On the other hand.

$$\cos^6 x = \left(\frac{1+\cos 2x}{2}\right)^3$$

The equation takes the following form

$4(1 + \cos 2x)^3 - (4\cos^3 2x - 3\cos 2x) = 1$

or $4\cos^2 2x + 5\cos 2x + 1 = 0$

Thus $\cos 2x = -1, \cos 2x = \frac{1}{4}$

(v) Since $\sin^2 x + \cos^2 x = 1$, we have

$\sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x = 1$ and

$\sin^4 x + \cos^4 x = 1 - \frac{1}{2}(\sin 2x)^2$

The equation takes the following form

$\sin^2 2x - 8\sin 2x + 4 = 0$

Hence

$\sin 2x = 4 \pm \sqrt{16-4}$, $\sin 2x = 4 \pm 2\sqrt{3}$

Rejecting one of the solutions, we get finally

$\sin 2x = 4 - 2\sqrt{3}$

5. (i) Applying the formula for the sum of sines, we get

$2\sin 5x\cos x = 0$ ....(1)

If x is a solution of (1), then at least one of the following equalities is true:

$\sin 5x = 0$ or $\cos x = 0$ ....(2)

Conversely, if x is a solution of one of equations (2), then, evidently, x is a solution of equation (1) as well. Thus, equation (1) is equivalent to the collection of equations (2). Equations (2) have the solutions

$x = \frac{\pi n}{5}$, $x = \frac{\pi}{2} + \pi n, n \in Z$

respectively.

All these values of x and only these values are the solutions of the original equation.

(ii) Let us transform the equation using the reduction formula and the formula for the difference of sines :

$\sin x - \cos 2x = 0$, $\sin x - \sin\left(\frac{\pi}{2} - 2x\right) = 0$,

$2\sin\frac{1}{2}\left(3x - \frac{\pi}{2}\right)\cos\frac{1}{2}\left(\frac{\pi}{2} - x\right) = 0$

The resulting equation is equivalent to the collection of two equations:

$\sin\frac{1}{2}\left(3x - \frac{\pi}{2}\right) = 0$ and $\cos\frac{1}{2}\left(\frac{\pi}{2} - x\right) = 0$

We solve the first equation :

$\frac{1}{2}\left(3x - \frac{\pi}{2}\right) = \pi k$, $x = \frac{\pi}{6} + \frac{2\pi k}{3}, k \in Z$

For the second equation, we have

$\frac{1}{2}\left(\frac{\pi}{2} - x\right) = \frac{\pi}{2} + \pi l$, $x = -\frac{\pi}{2} - 2\pi l, l \in Z$

It is easy to see that all solutions of the second equation are contained in the set of solutions of the first equation. Indeed, for $k = -1 - 3l$, $l \in Z$, we have $x = \frac{\pi}{6} + \frac{2\pi(-1-3l)}{3} = -\frac{\pi}{2} - 2\pi l$, that is, we obtain all solutions of the second equation.

The solution is $\left\{\left\{\frac{\pi}{6} + \frac{2\pi k}{3}\right\} | k \in Z\right\}$

(iii) Dividing both parts of the equation by $\sqrt{2}$, we obtain $\frac{1}{\sqrt{2}}\sin 2x + \frac{1}{\sqrt{2}}\cos 2x = -\frac{1}{\sqrt{2}}$

Taking account of the fact that

$\frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$, we write the equation in the form

$\sin 2x\cos\left(\frac{\pi}{4}\right) + \cos 2x\sin\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$

Using the formula for the sine of the sum of the arguments, we arrive at an equation

$\sin\left(2x + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

Hence we have $x = ((-1)^{n+1} - 1)\frac{\pi}{8} + \frac{n\pi}{2}, n \in Z$

These values of x constitute the set of all solutions of the original equation.

(iv) Dividing both sides of the equation by

$\sqrt{12^2 + 5^2} = 13$, we get

$\frac{12}{13}\cos x - \frac{5}{13}\sin x = -1$

One of the solutions of the system $\cos\varphi = \frac{12}{13}, \sin\varphi = \frac{5}{13}$ is $\varphi = \cos^{-1}\left(\frac{12}{13}\right)$. Taking this into account, we write the equation in the form $\cos x\cos\varphi - \sin x\sin\varphi = -1$ and, applying the formula for the cosine of the sum of the arguments, we get $\cos(x + \varphi) = -1$, whence we have $x + \varphi = \pi + 2\pi n$, $x = -\varphi + \pi(2n + 1)$, that is,

$$x = -\cos^{-1}\left(\frac{12}{13}\right) + \pi(2n+1), n \in Z$$

This formula yields all the solutions of the original equation.

(v) We make use of the formulas $\cos 2x = 1 - 2\sin^2 x$ and $\sin 3x = \sin x(3 - 4\sin^2 x)$.

It is easy to obtain the second formula by transforming the right-hand side of the equality $\sin 3x = \sin(x - 2x)$ by the formula for the sine of the sum and then by the formulas for a double argument.

After the substitution, the original equation assumes the form

$\sin x(3 - 4\sin^2 x) + 1 - 2\sin^2 x = 1$.

From this we get $4\sin^3 x + 2\sin^2 x - 3\sin x = 0$. Denoting $t = \sin x$, we obtain $4t^3 + 2t^2 - 3t = 0$. This equation has the roots $t_1 = 0$,

$t_2 = \frac{(\sqrt{13}-1)}{4}, t_3 = \frac{-(\sqrt{13}-1)}{4}$. This means that the original equation is equivalent to the collection of the equations

$$\sin x = 0,\ \sin x = \left(\frac{\sqrt{13}-1}{4}\right), \sin x = -\left(\frac{\sqrt{13}-1}{4}\right)$$

We consecutively find the solutions for the equations obtained :

$x = \pi n$, $n \in Z$; $x = (-1)^n$

are $\sin\left(\frac{\sqrt{13}-1}{4}\right) + \pi n, n \in Z;$

the third equation has no solutions since $-\frac{(\sqrt{13}+1)}{4} < -1$. The values of x we have obtained and only these values are solutions of the original equation.

6. (i) We have

$\sin 2x\cos x + \cos 2x\sin x + \sin 2x - m\sin x = 0$

Hence

$\sin x[2\cos^2 x + \cos 2x + 2\cos x - m] = 0$

$\sin x[4\cos^2 x + 2\cos x - (m + 1)] = 0$

And so, one solution is $\sin x = 0$.

The other is obtained by the formula

$$\cos x = \frac{-1 \pm \sqrt{4m+5}}{4}$$

Hence, first of all, it follows that there must be $4m + 5 \geq 0$

Further, for one of the roots to exist it is required that $|-1 + \sqrt{4m+5}| \leq 4$, *i.e.*

$-4 \leq -1 + \sqrt{4m+5} \leq +4$ or $-3 \leq \sqrt{4m+5} \leq 5$,

i.e., $m \leq 5$. For the other root to exist it is necessary that

$|-1 - \sqrt{4m+5}| \leq 4, -4 \leq -1 - \sqrt{4m+5} \leq 4,\ m \leq 1$

Thus if $m < -\frac{5}{4}$, then $\cos x$ has no real values; at $m = -\frac{5}{4}$ it has one real value $\left(\cos x = -\frac{1}{4}\right)$; for $-\frac{5}{4} < m \leq 1$ $\cos x$ has two real values $\left(\cos x = -\frac{+1 \pm \sqrt{4m+5}}{4}\right)$ and for $1 \leq m \leq 5$ $\cos x$ again has one real value $\left(\cos x = \frac{-1 + \sqrt{4m+5}}{4}\right)$ and at $m > 5$ it has no real values .

(ii) Rewrite the equation as

$$\frac{1}{\cos(x-\alpha)}\{(1+k)\cos x\cos(2x-\alpha)\} - (1 + k\cos 2x)\cos(x - \alpha)\} = 0$$

But

$$\cos x\cos(2x-\alpha) = \frac{1}{2}\cos(3x-\alpha) + \frac{1}{2}\cos(x-\alpha)$$

$$\cos 2x\cos(x-\alpha) = \frac{1}{2}\cos(3x-\alpha) + \frac{1}{2}\cos(x+\alpha)$$

Therefore

$$\frac{1}{\cos(x-\alpha)}\{(1+k)[\cos(3x-\alpha) + \cos(x-\alpha)]\}$$

$$-2\cos(x - \alpha) - k[\cos(3x - \alpha) + \cos(x + \alpha)] = 0$$

or $$\frac{1}{\cos(x-\alpha)}\{\cos(3x-\alpha) - \cos(x-\alpha)\} + k[\cos(x - \alpha) - \cos(x + \alpha)] = 0$$

$$\frac{2\sin x}{\cos(x-\alpha)}\{k\sin\alpha - \sin(2x-\alpha)\} = 0$$

Hence,

$\sin x = 0$ and $\sin(2x - \alpha) = k\sin\alpha$

(iii) Applying the formula $\cos^2\varphi = \frac{1+\cos 2\varphi}{2}$, write the equation in the form

or $\cos 2(\alpha + x) + \cos 2(\alpha - x) = 2a - 2$

whence $\cos 2\alpha\cos 2x = a - 1$,

$$\cos 2x = \frac{a-1}{\cos 2\alpha} \qquad .....(1)$$

On the other hand,

$$\cot x = \pm\sqrt{\frac{1+\cos 2x}{1-\cos 2x}},$$

and therefore from (1) we find

$$\cot x = \pm\sqrt{\frac{a-1+\cos 2\alpha}{1-a+\cos 2\alpha}}$$

Equation (1) holds if

$\cos 2\alpha \neq 0$ and $|\cos 2\alpha| \geq |a-1|$

(iv) Multiplying the right member of the equation by $\sin^2 x + \cos^2 x = 1$ we reduce it to the form

$(1-a)\sin^2 x - \sin x\cos x - (a+2)\cos^2 x = 0$ ...(1)

First let us assume that $a \neq 1$. Then from (1) it follows that $\cos x \neq 0$, since otherwise we have $\sin x = \cos x = 0$ which is impossible. Dividing both members of (1) by $\cos^2 x$ and putting $\tan x = t$ we get the equation

$(1-a)t^2 - t - (a+2) = 0$ ....(2)

Equation (1) is solvable if and only if the roots of equation (2) are real, *i.e.* if its discriminant is non-negative :

$D = -4a^2 - 4a + 9 \geq 0$ ....(3)

Solving inequality (3) we get

$$\frac{\sqrt{10}+1}{2} \leq a \leq \frac{\sqrt{10}-1}{2} \quad ....(4)$$

Let t_1 and t_2 be the roots of equation (2). Then the corresponding solutions of equation (1) have the form $x_1 = \tan^{-1}t_1 + k\pi$, $x_2 = \tan^{-1}t_2 + k\pi$

Now let us consider the case when $a = 1$.

In this case equation (1) is written in the form $\cos x(\sin x + 3\cos x) = 0$ and has the following solutions :

$$x_1 = \frac{\pi}{2} + k\pi,\ x_2 = -\tan^{-1}3 + k\pi$$

(v) Applying the formulas

$$\sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2,\ \cos^2 x = \frac{1+\cos 2x}{2}$$

and putting $\cos 2x = t$, we rewrite the given equation in the form

$t^2 - 6t + 4a^2 - 3 = 0$ ....(1)

The original equation has solutions for a given value of a if and only if, for this value of a, the roots t_1 and t_2 of the equation (1) are real. Solving equation (1), we get

$$t_1 = 3 - 2\sqrt{3-a^2},\ t_2 = 3 + 2\sqrt{3-a^2}$$

Hence, the roots of equation (1) are real if

$|a| \leq \sqrt{3}$ ....(2)

If condition (2) is fulfilled, then $t_2 > 1$ and, therefore, this root can be discarded. Thus, the problem is reduced to finding the values of a satisfying condition (2), for which $|t_1| \leq 1$, *i.e.*

$-1 \leq 3 - 2\sqrt{3-a^2} \leq 1$ ....(3)

From (3) we find

$-4 \leq 2\sqrt{3-a^2} \leq -2$, whence $2 \geq \sqrt{3-a^2} \geq 1$

Since the inequality $2 \geq \sqrt{3-a^2}$ is fulfilled for $|a| \leq \sqrt{3}$, the system of inequalities (4) is reduced to the inequality

$\sqrt{3-a^2} \geq 1$, whence we find $|a| \leq \sqrt{2}$

Thus, the original equation is solvable if $|a| \leq \sqrt{2}$, and its solutions are

$$x = \pm\frac{1}{2}\cos^{-1}(3 - 2\sqrt{3-a^2}) + k\pi$$

(vi) The given problem is equivalent to the following problem : what values can the function $\lambda = \sec x + \operatorname{cosec} x$ assume if the argument x varies within the range $0 < c < \frac{\pi}{2}$?

Consider the function

$$\lambda^2 = (\sec x + \operatorname{cosec} x)^2$$

$$= \frac{1}{\cos^2 x} + \frac{2}{\sin x\cos x} + \frac{1}{\sin^2 x}$$

$$= \frac{1}{\sin^2 x\cos^2 x} + \frac{2}{\sin x\cos x} = \frac{4}{\sin^2 2x} + \frac{4}{\sin 2x}$$

As x increases from zero to $\frac{\pi}{2}$, each summand on the right-hand side varies in the following way: it first decreases from $+\infty$ to $\left(\text{for } 0 < x \leq \frac{\pi}{4}\right)$, then increases from 4 to $+\infty\left(\text{for } \frac{\pi}{4} \leq x < \frac{\pi}{2}\right)$; for $x = \frac{\pi}{4}$ both summands simultaneously attain their least value as well, and $\lambda^2 = 8$. Therefore, if $0 < x < \frac{\pi}{2}$, then $\lambda^2 \geq 8$, and since $\sec x$ and $\operatorname{cosec} x$ are positive in the first quadrant, we have $\lambda \geq 2\sqrt{2}$.

Alternative Solution : Note that we must confine ourselves to considering only the positive values of λ because for $0 < x < \frac{\pi}{2}$ the function $\sec x$ and $\operatorname{cosec} x$ are positive. Transforming the equation to the form $\sin x + \cos x = \lambda\sin x\cos x$,

we then square both members and obtain

$1 + 2\sin x\cos x = \lambda^2\sin^2 x\cos^2 x$

Now putting $\sin 2x = z$ we can write

$\lambda^2 z^2 - 4z - 4 = 0$,

whence

$$z = \frac{2 \pm \sqrt{4+4\lambda^2}}{\lambda^2} \quad ....(1)$$

By the hypothesis, we have $0 < x < \frac{\pi}{2}$, and therefore $z = \sin 2x > 0$. Thus, inequality (1) we must take plus sign *i.e.*

$$z=\frac{2+\sqrt{4+4\lambda^2}}{\lambda^2}$$

If now we take the values of λ satisfying the inequality

$$\frac{2+\sqrt{4+4\lambda^2}}{\lambda^2}\le 1 \qquad \text{....(2)}$$

then the equation $\sin 2x=\frac{2+\sqrt{4+4\lambda^2}}{\lambda^2}$ will have a solution x such that $0<x<\frac{\pi}{2}$. Obviously, this solution will also satisfy the original equation. But if inequality (2) is not satisfied, the required solution does not exist. We see that the problem is reduced to solving inequality (2). Getting rid of the denominator, we readily find $\lambda \ge 2\sqrt{2}$.

7. (i) Let $\frac{\pi}{2}\le x_1<x_2\le\pi$, then $0\le \sin x_2<\sin x_1\le 1$, and the point $P_{\sin x_1}$, $P_{\sin x_2}$, $P_{\sin x_3}$ lie in the first quadrant since $1<\frac{\pi}{2}$. Since the function $\cos x$ decreases on the interval $\left[0,\frac{\pi}{2}\right]$, we have,

$0<\cos(\sin x_1)<\cos(\sin x_2)\le 1$

But the points $P_{\cos(\sin x_1)}$ and $P_{\cos(\sin x_2)}$ also lie in the first quadrant and the function $\sin x$ increases on the interval $\left[0,\frac{\pi}{2}\right]$, therefore

$0<\sin(\cos(\sin x_1))<\sin(\cos(\sin x_2))<1$

That is, the function $f(x)=\sin(\cos(\sin x))$ is increasing on the interval $\left[\frac{\pi}{2},\pi\right]$. Consequently, the minimal value of $f(x)$ on this interval is equal to $f\left(\frac{\pi}{2}\right)=\sin(\cos 1)$, while the maximal value of $f(\pi)=\sin(\cos 0)=\sin 1$

(ii) $(\sin\alpha+\sin\beta)-[\sin(\alpha+\beta+\gamma)-\sin\gamma]$

$$2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)-2\cos\left(\gamma+\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$$

$$=2\sin\left(\frac{\alpha+\beta}{2}\right)\left[\cos\left(\frac{\alpha-\beta}{2}\right)-\cos\left(\gamma+\frac{\alpha+\beta}{2}\right)\right]$$

$$=2\sin\left(\frac{\alpha+\beta}{2}\right)\left[2\sin\left(\frac{\alpha+\gamma}{2}\right)\sin\left(\frac{\beta+\gamma}{2}\right)\right]$$

$$=4\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha+\gamma}{2}\right)\sin\left(\frac{\beta+\gamma}{2}\right)$$

= R.H.S.

(iii) Let us write the given equation in the form

$\sin^2 4x-2\sin 4x\cos^4 x=-\cos^2 x$

Adding $\cos^8 x$ to both sides of the equation, we get

$\sin^2 4x-2\sin 4x\cos^4 x+\cos^8 x=\cos^8 x-\cos^2 x$

or $(\sin 4x-\cos^4 x)^2=-\cos^2 x(1-\cos^6 x)$

The left hand side of the equation is non-negative, while the right hand side is non-positive ($\cos^2 x\ge 0$, $(1-\cos^6 x)\ge 0$), consequently, the equality will be valid only when the following conditions are fulfilled simultaneously.

$$\begin{cases}-\cos^2 x(1-\cos^6 x)=0\\(\sin 4x-\cos^4 x)=0\end{cases}$$

The first equation decomposes into two parts :

1. $\cos^2 x=0$ or $\cos x=0$

   where $x=\frac{\pi}{2}+\pi n, n\in Z$. The obtained values also satisfy the second equation since

   $$\sin\left(4\left(\frac{\pi}{2}+\pi n\right)\right)=\sin(2\pi+4\pi n)=0$$

2. $1-\cos^6 x=0$ or $\cos x=\pm 1$, whence $x=\pi n$, $n\in Z$. Substituting these values of x into the second equation, we get $(\sin 4\pi n-\cos^4\pi n)^2=0$ or $(0-1)^2=0$ which is wrong.

   Thus, the solution of the original equation consists of the numbers $x=\frac{\pi}{2}+\pi n, n\in Z$

(iv)(a) We have

$\sin 2n\alpha+\sin 2n\beta+\sin 2n\gamma$

$=4\sin n(\alpha+\beta).\sin n(\beta+\gamma).\sin n(\gamma+\alpha)$ ....(1)

Furthermore, we have

$\sin n(\alpha+\beta)=\sin n(\pi-\gamma)=(-1)^{n+1}\sin n\gamma$ ...(2)

Similarly,

$\sin n(\beta+\gamma)=(-1)^{n+1}\sin n\alpha$ ...(3)

and $\sin n(\gamma+\alpha)=(-1)^{n+1}\sin n\beta$ ...(4)

Substituting (2), (3) and (4) in (1), we get

$\sin 2n\alpha+\sin 2n\beta+\sin 2n\gamma$

$=4(-1)^{n+1}\sin n\alpha\sin n\beta\sin n\gamma$

We get the required result.

(b) All values of α and β are permissible here except those for which $\cos(\alpha+\beta)=0$ and $\cos\beta=A$. Noting that $\sin\alpha=\sin(\alpha+\beta-\beta)$, let us rewrite the original equality in the form

$\sin(\alpha+\beta)\cos\beta-\cos(\alpha+\beta)\sin\beta=A\sin(\alpha+\beta)$ ....(1)

Dividing both members of (1) by $\cos(\alpha+\beta)\ne 0$, we obtain

$\tan(\alpha+\beta)\cos\beta-\sin\beta=A\tan(\alpha+\beta)$

$$\Rightarrow \tan(\alpha+\beta)=\frac{\sin\beta}{\cos\beta-A}$$

(c) It is readily seen that, by virtue of the conditions of the problem, we have $\sin\alpha\cos\alpha\cos\beta\ne 0$ because, if otherwise, we have $|m|\le|n|$. Therefore, the equality to be proved makes sense. We represent

this equality in the form

$$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{m+n}{m-n}\tan\alpha, \quad \text{....(1)}$$

whence

$$\tan(\alpha+\beta)=\frac{m+n}{m-n}\tan\alpha \quad \text{....(2)}$$

Replace in (2) the tangents of the angles α and $(\alpha+\beta)$ by the ratios of the corresponding sines and cosines, reduce the fractions to a common denominator and discard it.

We then obtain

$$m[\cos\alpha\sin(\alpha+\beta)-\sin\alpha\cos(\alpha+\beta)]$$
$$-n[\sin\alpha\cos(\alpha+\beta)+\cos\alpha\sin(\alpha+\beta)]=0 \quad \text{....(3)}$$

$$\Rightarrow \quad m\sin\beta-n\sin(2\alpha+\beta)=0 \quad \text{....(4)}$$

Thus, the proof is reduced to establishing relation (4). Since relation (4) is fulfilled by the hypothesis of the problem, we conclude that (3) holds true which implies the validity of (2).

But (2) implies (1), and (1), in its turn m implies the required relation

$$\frac{1+\frac{\tan\beta}{\tan\alpha}}{m+n}=\frac{1-\tan\alpha\tan\beta}{m-n}$$

(d) Consider the identity

$$\cos(x+y+z)=\cos(x+y)\cos z-\sin(x+y)\sin z$$
$$=\cos x\cos y\cos z-\cos z\sin x\sin y-\cos y\sin x\sin z-\cos x\sin y\sin z$$

By the hypothesis of the problem, we have $\cos x\cos y\cos z\neq 0$, and therefore this identity implies

$$\cos(x+y+z)=\cos x\cos y\cos z(1-\tan x\tan y-\tan y\tan z-\tan z\tan x)$$

(e) $$\cot^2 2x-\tan^2 2x=\frac{\cos^2 2x}{\sin^2 2x}-\frac{\sin^2 2x}{\cos^2 2x}$$
$$=\frac{\cos^4 2x-\sin^4 2x}{\sin^2 2x\cos^2 2x}$$
$$=\frac{\cos^2 2x-\sin^2 2x}{\left(\frac{1}{4}\right)\sin^2 4x}=\frac{4\cos 4x}{\sin^2 4x}$$

Hence,

$$S=\frac{4\cos 4x}{\sin^2 4x}(1-2\sin 4x\cos 4x)$$
$$=\frac{4\cos 4x}{\sin^2 4x}(1-\sin 8x)$$

Since $(1-\sin 8x)=2\sin^2\left(\frac{\pi}{4}-4x\right)$,

we finally obtain the expression

$$\frac{8\cos 4x\sin^2\left(\frac{\pi}{4}-4x\right)}{\sin^2 4x}$$

8. (i) Put $\tan\frac{\theta}{2}=\alpha, \tan\frac{\theta_1}{2}=\beta$. Then the first two equalities take the form

$$x\alpha^2-2y\alpha+2a-x=0,\ x\beta^2-2a-x=0$$

Consequently α and β are the roots of the quadratic equation

$$xz^2-2yz+2a-x=0$$

Therefore $\alpha+\beta=\frac{2y}{x}, \alpha\beta=\frac{2a-x}{x}$

Furthermore $\alpha-\beta=2l$.

Let us now eliminate α and β from the last three equalities. We have identically

$$(\alpha+\beta)^2=(\alpha-\beta)^2+4\alpha\beta$$

Consequently,

$$\left(\frac{2y}{x}\right)^2=4l^2+4\left(\frac{2a-x}{x}\right)$$

After simplification, we get

$$y^2=2ax-(1-l^2)x^2$$

(ii) We have

$$\tan\left(\frac{\theta+\alpha}{2}\right)\tan\left(\frac{\theta-\alpha}{2}\right)=\frac{\tan^2\left(\frac{\theta}{2}\right)-\tan^2\left(\frac{\alpha}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)\tan^2\left(\frac{\alpha}{2}\right)}$$

But

$$\tan^2\left(\frac{\theta}{2}\right)=\frac{1-\cos\theta}{1+\cos\theta}=\frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta},$$
$$\tan^2\left(\frac{\alpha}{2}\right)=\frac{1-\cos\alpha}{1+\cos\alpha}$$

Consequently

$$\tan\left(\frac{\theta+\alpha}{2}\right).\tan\left(\frac{\theta-\alpha}{2}\right)=\frac{\frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta}-\frac{1-\cos\alpha}{1+\cos\alpha}}{1-\frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta}.\frac{1-\cos\alpha}{1+\cos\alpha}}$$
$$=\frac{1-\cos\beta}{1+\cos\beta}=\tan^2\left(\frac{\beta}{2}\right)$$

(iii) The first equality can be written as

$$a[\sin(\theta+\varphi)-\sin(\theta-\varphi)]=b[\sin(\theta-\varphi)+\sin(\theta+\varphi)]$$
$$\Rightarrow \quad a[2\cos\theta\sin\varphi]=b[2\sin\theta\cos\varphi]$$
$$\Rightarrow \quad a\tan\varphi=b\tan\theta$$
$$\frac{a}{b}\tan\varphi=\frac{2\tan\left(\frac{\theta}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)}$$

But from the second equality we are given

that $$\tan\left(\frac{\theta}{2}\right)=\frac{b\tan\left(\frac{\varphi}{2}\right)+c}{a}$$

$$\Rightarrow \quad \frac{a}{b}\frac{2\tan\left(\frac{\varphi}{2}\right)}{1-\tan^2\left(\frac{\varphi}{2}\right)}=\frac{2\left(\frac{b\tan\left(\frac{\varphi}{2}\right)+c}{a}\right)}{\left[1-\frac{\left(b\tan\left(\frac{\varphi}{2}\right)+c\right)^2}{a^2}\right]} \quad \text{...(i)}$$

Putting $\tan\frac{\varphi}{2} = x$ in (i) and after simplification, we have

$bc(1 + x^2) = -(b^2 + c^2 - a^2)x$ ...(ii)

But $\frac{2x}{1+x^2} = \sin\varphi$

Substituting the value of $\sin\varphi$ in (ii), we have

$\sin\varphi = \frac{2bc}{a^2 - b^2 - c^2}$.

(iv) From the first equality we have

$$\frac{(\tan\theta\cos\beta - \sin\beta)\cos\alpha}{(\tan\varphi\cos\alpha - \sin\alpha)\cos\beta} + \frac{(\cos\alpha - \tan\theta\sin\alpha)\sin\beta}{(\cos\beta + \tan\varphi\sin\beta)\sin\alpha} = 0$$

Hence

$$\sin\alpha\cos\beta\cos(\alpha - \beta)\tan\theta + \sin\beta\cos\alpha\cos(\alpha + \beta)\tan\varphi = 2\sin\beta\cos\beta\sin\alpha\cos\alpha \quad ...(*)$$

From the second equality we get

$$\frac{\tan\theta}{\tan\varphi} = -\frac{\cos(\alpha-\beta)\tan\beta}{\cos(\alpha+\beta)\tan\alpha}$$

Therefore we may put

$\tan\theta = \lambda\cos(\alpha - \beta)\tan\beta$,

$\tan\varphi = -\lambda\cos(\alpha + \beta)\tan\alpha$

Substituting the expressions for $\tan\theta$ and $\tan\varphi$ into the equality (*), we find

$$\lambda = \frac{1}{2\sin\alpha\sin\beta}$$

Thus

$$\tan\theta = \frac{\cos(\alpha-\beta)}{2\sin\alpha\cos\beta} = \frac{1}{2}(\cot\alpha + \tan\beta)$$

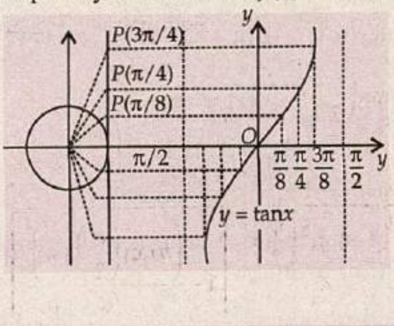
$$\tan\varphi = -\frac{\cos(\alpha+\beta)}{2\cos\alpha\sin\beta} = \frac{1}{2}(\tan\alpha - \cot\beta)$$

(v) Let $\frac{\pi}{2} \le x_1 < x_2 \le \pi$,, then

$-\frac{\pi}{2} < -1 \le \cos x_2 < \cos x_1 \le 0$, since the function $\cos x$ decreases from 0 to -1 on the interval $\left[\frac{\pi}{2}, \pi\right]$. Therefore the points $P_{\cos x_1}$ and $P_{\cos x_2}$ lie in the fourth quadrant, and

$-\tan 1 < \tan(\cos x_2) < \tan(\cos x_1) \le 0$,

Since $\tan x$ increases in the fourth quadrant. Consequently, the function $f(x) = \tan(\cos x)$



decreases on the interval $\left[\frac{\pi}{2}, \pi\right]$, its greatest value is $f\left(\frac{\pi}{2}\right) = \tan(0) = 0$, the least value being $f(\pi) = \tan(-1) = -\tan 1$.

9. (i) Putting $t = \sin x + \cos x$. Then we obtain

$t = 1 - (t^2 - 1)$

whence we get $t^2 + t - 2 = 0$, $t_1 = 1$, $t_2 = -2$.

We have $\sin x + \cos x = 1$ and $\sin x + \cos x = -2$. Let us solve each of the equation, say, by introducing an auxiliary angle. For the first equation we find $x = ((-1)^n - 1)\frac{\pi}{4} + \pi n, n \in Z$; the second equation has no solution since $|-2| > \sqrt{2}$

$\therefore$ The required solution is

$x = ((-1)^n - 1)\frac{\pi}{4} + \pi n, n \in Z$

(ii) (a) Denote the expression under consideration by S. Let us transform the first two summands according to formula and, finally substitute $1 - \cos^2\gamma$ for $\sin^2\gamma$.

We then obtain

$$S = -\frac{1}{2}(\cos 2\alpha + \cos 2\beta) - \cos^2\gamma + [\cos(\alpha+\beta) + \cos(\alpha - \beta)]\cos\gamma$$

Transforming the sum $\cos 2\alpha + \cos 2\beta$ into a product and opening the square brackets we get

$$S = -\cos(\alpha + \beta)\cos(\alpha - \beta) - \cos^2\gamma + \cos(\alpha + \beta)\cos\gamma + \cos(\alpha - \beta)\cos\gamma.$$

$S = -[\cos(\alpha - \beta) - \cos\gamma][\cos(\alpha + \beta) - \cos\gamma]$

Hence,

$$S = 4\sin\left(\frac{\alpha-\beta+\gamma}{2}\right)\sin\left(\frac{\gamma-\alpha+\beta}{2}\right)\sin\left(\frac{\alpha+\beta+\gamma}{2}\right)\sin\left(\frac{\alpha+\beta-\gamma}{2}\right)$$

(b) The expression in question can be transformed in the following way

$$\frac{1-4\sin 10°\sin 70°}{2\sin 10°} = \frac{1-2(\cos 60° - \cos 80°)}{2\sin 10°} = \frac{2\cos 80°}{2\cos 80°}$$

Thus, $\frac{1}{2\sin 10°} - 2\sin 70° = 1$

(c) $\cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{2\pi}{5}\right) = 2\sin\left(\frac{\pi}{10}\right)\sin\left(\frac{3\pi}{10}\right)$ ....(1)

Multiplying and dividing (1) by $2\cos\left(\frac{\pi}{10}\right)\cos\left(\frac{3\pi}{10}\right)$, and applying the formula for $\sin 2\alpha$, we obtain

$$2\sin\left(\frac{\pi}{10}\right)\sin\left(\frac{3\pi}{10}\right) = \frac{\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{3\pi}{5}\right)}{2\cos\left(\frac{\pi}{10}\right)\cos\left(\frac{3\pi}{10}\right)} \quad ...(2)$$

We know that

$$\cos\left(\frac{\pi}{10}\right)=\sin\left(\frac{\pi}{2}+\frac{\pi}{10}\right)=\sin\left(\frac{3\pi}{5}\right)$$

and $\cos\left(\frac{3\pi}{10}\right)=\sin\left(\frac{\pi}{2}-\frac{3\pi}{10}\right)=\sin\left(\frac{\pi}{5}\right)$

So, equation (2) becomes

$$2\sin\left(\frac{\pi}{10}\right)\sin\left(\frac{3\pi}{10}\right)=\frac{\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{3\pi}{5}\right)}{2\sin\left(\frac{3\pi}{5}\right)\sin\left(\frac{\pi}{5}\right)}=\frac{1}{2}$$

(d) We have

$$S=\frac{3}{2}-\frac{1}{2}\left(\cos\frac{\pi}{8}+\cos\frac{3\pi}{8}+\cos\frac{5\pi}{8}+\cos\frac{7\pi}{8}\right)$$
$$+\frac{1}{8}\left(\cos\frac{\pi}{4}+\cos\frac{3\pi}{4}+\cos\frac{5\pi}{4}+\cos\frac{7\pi}{4}\right)$$

The sums in the brackets are equal to zero because

$$\cos\frac{\pi}{8}=-\cos\frac{7\pi}{8},\ \cos\frac{3\pi}{8}=-\cos\frac{5\pi}{8}$$

and

$$\cos\frac{\pi}{4}=-\cos\frac{3\pi}{4},\ \cos\frac{5\pi}{4}=-\cos\frac{7\pi}{4}$$

Consequently, $S=\frac{3}{2}$.

(iii) Let $f(x) = a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x$

$f(0) = a_0 + a_1 + a_2 + a_3 = 0$ ...(1)

$f\left(\frac{\pi}{2}\right)=a_0-a_2=0 \Rightarrow a_0=a_2$ ....(2)

$f\left(\frac{\pi}{3}\right)=a_0+\frac{1}{2}a_1-\frac{1}{2}a_2-a_3=0$

$\Rightarrow \frac{1}{2}a_2+\frac{1}{2}a_1-a_3=0$

$\Rightarrow a_3=\frac{1}{2}(a_2+a_1)$ ....(3)

$f\left(\frac{\pi}{4}\right)=a_0+\frac{a_1}{\sqrt{2}}-\frac{a_3}{\sqrt{2}}=0$

$\Rightarrow a_2+\frac{(a_1-a_3)}{\sqrt{2}}=0$ or $a_2=\frac{(a_3-a_1)}{\sqrt{2}}$ ...(4)

Substituting in (1) the values obtained (2) and (3).

$2a_2+a_1+\frac{1}{2}(a_1+a_2)=0$

$\Rightarrow 5a_2 + 3a_1 = 0$ or $a_2=\frac{-3}{5}a_1$ ....(5)

From (4) and (5), we get

$\left(\frac{1}{\sqrt{2}}-\frac{3}{5}\right)a_1=\frac{1}{\sqrt{2}}a_3$ ....(6)

and from (3), (5) and (6), we get

$$\left(\frac{1}{\sqrt{2}}-\frac{3}{5}\right)a_1=\frac{1}{2\sqrt{2}}\left[\left(a_1-\frac{3}{5}a_1\right)\right]$$
$$=\frac{1}{2\sqrt{2}}\times\frac{2}{5}a_1=\frac{1}{\sqrt{2}.5}a_1$$

$\Rightarrow \left(\frac{1}{\sqrt{2}}-\frac{3}{5}-\frac{1}{\sqrt{2}.5}\right)a_1=0$

$\Rightarrow \frac{(5-3\sqrt{2}-1)}{5\sqrt{2}}a_1=0$

$\Rightarrow \frac{(4-3\sqrt{2})}{5\sqrt{2}}a_1=0$ but $\frac{4-3\sqrt{2}}{5\sqrt{2}}\neq 0$

$\therefore\ a_1 = 0$

$\therefore\ a_3=0 \quad \left[\text{as } a_3=\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{3}{5}\right)a_1\right]$

$\therefore\ a_2=\frac{-3}{5}a_1=0$

$a_0 = a_2 = 0$

Thus $a_0 = a_1 = a_2 = a_3 = 0$

(iv) Let $x = \tan\alpha$, $y=\tan\beta$, $z = \tan\gamma$, $\frac{-\pi}{2}<\alpha,\beta,\gamma<\frac{\pi}{2}$

$$\frac{4\sqrt{(\tan^2\alpha+1)}}{\tan\alpha}=\frac{5\sqrt{(\tan^2\beta+1)}}{\tan\beta}=\frac{6\sqrt{(\tan^2\gamma+1)}}{\tan\gamma}$$

$\Rightarrow \frac{4}{\sin\alpha}=\frac{5}{\sin\beta}=\frac{6}{\sin\gamma}$

Again, $\tan\alpha\tan\beta\tan\gamma = \tan\alpha + \tan\beta + \tan\gamma$

$\Rightarrow \tan\alpha(\tan\beta\tan\gamma - 1) = (\tan\beta + \tan\gamma)$

$\Rightarrow -\tan\alpha=\frac{(\tan\beta+\tan\gamma)}{1-\tan\beta\tan\gamma}=\tan(\beta+\gamma)$

$\Rightarrow \tan(k\pi - \alpha) = \tan(\beta + \gamma)$

$\Rightarrow \alpha + \beta + \gamma = k\pi$

Taking $k = 1$, we get $\alpha + \beta + \gamma = \pi$ which implies that there exists a Δ whose angles are α, β and γ and whose sides opposite to these angles are proportional to 4, 5 and 6 respectively.

Let the sides of such Δ be $4k$, $5k$ and $6k$.

s = semi-perimeter of the $\Delta=\frac{15k}{2}$

$$\tan\left(\frac{\alpha}{2}\right)=\sqrt{\frac{(s-5k)(s-6k)}{s(s-4k)}}=\sqrt{\frac{\frac{5k}{2}\times\frac{3k}{2}}{\frac{15}{2}k\times\frac{7}{2}k}}=\sqrt{\frac{1}{7}}$$

$$x=\tan\alpha=\frac{2t}{1-t^2}=\frac{2\sqrt{\frac{1}{7}}}{1-\frac{1}{7}}=\frac{\sqrt{7}}{3}$$

Similarly, $y=\tan\beta=\frac{5\sqrt{7}}{9}$ and $z=\tan\gamma=3\sqrt{7}$

$$\left[\tan\left(\frac{\beta}{2}\right)=\sqrt{\frac{(s-4k)(s-6k)}{s(s-5k)}}\right.$$
$$\left.\text{and }\tan\left(\frac{\gamma}{2}\right)=\sqrt{\frac{(s-4k)(s-5k)}{s(s-6k)}}\right]$$

where α, β, γ are measures of the angles A, B and C of ΔABC.

concept BOOSTERS

Class XII

Trigonometry - II

- MTG Editorial Board

This column is aimed at Class XII students so that they can prepare for competitive exams such as IIT, AIEEE, etc and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

PART-A

Trigonometric Equations

1. Solve the following equations :

(i) $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x\cos x$

(ii) $6\tan^2 x - 2\cos^2 x = \cos 2x.$

(iii) $5\sin 2x - 5\cos 2x = \tan x + 5.$

(iv) $\sin 2x + \cos 2x = -1.$

(v) $\sin^2 x - 3\sin x\cos x + 2\cos^2 x = 0.$

(vi) $\cos 7x - \sin 5x = \sqrt{3}(\cos 5x - \sin 7x)$

2. Solve the following equations :

(i) $\tan 2x + \frac{1}{\sin x} = \cot x + \frac{1}{\sin 5x}$

(ii) $\left(\cos\frac{x}{4} - 2\sin x\right)\sin x + \left(1 + \sin\frac{x}{4} - 2\cos x\right)\cos x = 0$

(iii) $\begin{cases} \sin x\cos y = \frac{1}{4}, \\ 3\tan x = \tan y. \end{cases}$

(iv) $8\sin^6 x + 3\cos 2x + 2\cos 4x + 1 = 0$

(v) $\sin\left(\frac{\pi}{10} + \frac{3x}{2}\right) = 2\sin\left(\frac{3\pi}{10} - \frac{x}{2}\right)$

(vi) $(\sin x + \sqrt{3}\cos x)\sin 4x = 2$

(vii) $(\cos 4x - \cos 2x)^2 = \sin 3x + 5$

Inverse Trigonometric functions

3.(i) Find the angle $\sin^{-1}(1/3) + \sin^{-1}(3/4)$.

(ii) Find the angle $2\tan^{-1}(-3)$.

(iii) Find the angle $\sin^{-1}(\sin 10)$.

(iv) Solve the equation $\sin^{-1} x = \pi$.

Applications to Geometry

4.(i) Prove that the the diagonals of a (convex) quadrilateral are perpendicular, if and only if, the sum of the squares of one pair of opposite sides equals that of the other.

(ii) ABC is a triangle. The bisectors of $\angle B$ and $\angle C$ meet AC and AB at D and E respectively and BD and CE intersect at O. If $OD = OE$, prove that either $\angle BAC = 60°$ or the triangle is isosceles.

(iii) The diagonals AC, BD of the quadrilateral $ABCD$ intersect at an interior point O. The areas of the triangles AOB and COD are s_1 and s_2 respectively and the area of the quadrilateral is s. Prove that $\sqrt{s_1} + \sqrt{s_2} \le \sqrt{s}$. When does this equality hold?

(iv) Given a circle of radius 1 unit and AB is a chord of the circle with length 1 unit. If C is any point on the major segment, show that $AC^2 + BC^2 \le 2(2+\sqrt{3})$.

5.(i) $ABCDE$ is a convex pentagon inscribed in a circle of radius 1 unit with AE as diameter. If $AB = a$, $BC = b$, $CD = c$ and $DE = d$, prove that $a^2 + b^2 + c^2 + d^2 + abc + bcd < 4$.

(ii) A rhombus has half the area of the square with the same side length. Find the ratio of the longer diagonal to that of the shorter one.

(iii) Given the base and vertical angle of a triangle, find the locus of its orthocentre and incentre.

Miscellaneous

6. Solve the following equations

(i) $5\cos 3x + 3\cos x = 3\sin 4x.$

(ii) $\cos^{-1}(x\sqrt{3}) + \cos^{-1} x = \frac{\pi}{2}$

(iii) $\sin^6 x + \cos^6 x = p$, where p is an arbitrary real number.

(iv) $\sin^{-1}\frac{3x}{5} + \sin^{-1}\frac{4x}{5} = \sin^{-1} x$

7.(i) Given $n^2\sin^2(\alpha + \beta)$

$$= \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\cos(\alpha - \beta)$$

Prove that $\tan\alpha = \frac{1\pm n}{1\mp n}\tan\beta$.

(ii) Eliminate θ from the equations
$\cos(\alpha-3\theta) = m\cos^3\theta$, $\sin(\alpha-3\theta) = m\sin^3\theta$.

(iii) Solve the equation $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$.

(iv) Solve the equation
$$\cot 2x + 3\tan 3x = 2\tan x + \frac{2}{\sin 4x}$$

(v) Compare the numbers $\sin(\cos 1)$ and $\cos(\sin 1)$.

8. Solve the following equations :

(i) $2\sin 17x + \sqrt{3}\cos 5x + \sin 5x = 0$

(ii) $$\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} - \frac{1}{\tan^2 x} - \frac{1}{\cot^2 x} - \frac{1}{\sec^2 x} - \frac{1}{\mathrm{cosec}^2 x} = -3$$

(iii) $\cot x - 2\sin 2x = 1$

(iv) $\sin^5 x - \cos^5 x = \frac{1}{\cos x} - \frac{1}{\sin x}$

(v) $\sin^8 x + \cos^8 x = \frac{17}{32}$

(vi) $\sin^3 x + \sin^3 2x + \sin^3 3x = (\sin x + \sin 2x + \sin 3x)^3$

PART - B

Straight objective Type

1. The area of a circle is A_1 and the area of a regular pentagon inscribed in the circle is A_2. Then $A_1 : A_2$ is

(a) $\frac{\pi}{5}\cos\left(\frac{\pi}{10}\right)$ (b) $\frac{2\pi}{5}\sec\left(\frac{2\pi}{5}\right)$

(c) $\frac{2\pi}{5}\mathrm{cosec}\left(\frac{2\pi}{5}\right)$ (d) $\frac{2\pi}{5}\sin\left(\frac{2\pi}{5}\right)$

2. If x_1, x_2, x_3, x_4 are roots of the equation
$x^4 - x^3\sin 2\beta + x^2\cos 2\beta - x\cos\beta - \sin\beta = 0$,
$\left(\sin\beta \neq \frac{1}{2}\right)$ then
$\tan^{-1}(x_1) + \tan^{-1}(x_2) + \tan^{-1}(x_3) + \tan^{-1}(x_4)$ can be equal to

(a) β (b) $\frac{\pi}{2} - \beta$

(c) $\pi - \beta$ (d) $-\beta$

3. If $[\sin^{-1}x] + [\cos^{-1}x] = 0$, where x is a non-negative real number and [.] denotes the greatest integer function, then the set of all values of x is

(a) $(\cos 1, 1)$ (b) $(-1, \cos 1)$

(c) $(\sin 1, 1)$ (d) $(\cos 1, \sin 1)$

4. Three equal circles, each of radius 'r' touch another externally. The radius of the circle touching all the three given circles internally is

(a) $(2+\sqrt{3})r$ (b) $\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)r$

(c) $\left(\frac{2-\sqrt{3}}{\sqrt{3}}\right)r$ (d) $(2-\sqrt{3})r$

5. In a ΔABC, $A = \frac{2\pi}{3}$, $(b-c) = 3\sqrt{3}$ cm and area of $\Delta ABC = \frac{9\sqrt{3}}{2}\text{cm}^2$, Then BC =

(a) $6\sqrt{3}$ cm (b) 9 cm

(c) 18 cm (d) 27 cm

6. In a ΔABC, if
$\cot A = \sqrt{ac}$, $\cot B = \sqrt{\frac{c}{a}}$, $\cot C = \sqrt{\frac{a^3}{c}}$, then which of the following can be true?

(a) $a + a^2 = 1 - c$ (b) $a + a^2 = 1 + c$

(c) $a + a^2 = 2 - c$ (d) $a + a^2 = 2 + c$

7. The equation $\sin^{-1}\left(\frac{3x}{5}\right) + \sin^{-1}\left(\frac{4x}{5}\right) = \sin^{-1}x$, has

(a) 3 roots whose sum is zero

(b) 2 roots

(c) 3 roots which are in G.P.

(d) 3 roots which are in H.P.

8. The value of $\sin^{-1}[\cos(\cos^{-1}(\cos x) + \sin^{-1}(\sin x))]$, where $x \in \left(\frac{\pi}{2}, \pi\right)$, is

(a) $\frac{\pi}{2}$ (b) π

(c) $-\frac{\pi}{2}$ (d) $-\frac{3\pi}{4}$

9. Points D, E are taken on the side BC of an acute angled ΔABC such that $BD = DE = EC$. If $\angle BAD = x$, $\angle DAE = y$, $\angle EAC = z$; then the value of $\frac{\sin(x+y)\sin(y+z)}{\sin x . \sin z}$ is

(a) 4 (b) 2

(c) 1 (d) 1/2

Assertion - Reason Type

10. **Statement - I** : If $\frac{1}{2} \le x \le 1$, then
$$\cos^{-1}x - \sin^{-1}\left[\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right] = \frac{\pi}{6}$$
Statement - II : $\sin^{-1}(2x\sqrt{1-x^2}) = 2\sin^{-1}x$, $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

11. **Statement - I** : In ΔABC, if
$2a^2 + 4b^2 + c^2 = 4ab + 2ac$, $\cos A = \frac{1}{4}$

Statement - II : In a ΔABC, if $\cos A = \frac{1}{4}$,

$$(a+b+c)(b+c+a)=\frac{3}{2}bc$$

12. **Statement - I :**

$$\operatorname{cosec}^{-1}\left(\frac{3}{2}\right)+\cos^{-1}\left(\frac{2}{3}\right)-2\cot^{-1}\left(\frac{1}{7}\right)-2\cot^{-1}(7)=-\frac{\pi}{2}$$

Statement - II : $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\ \forall\in[-1,1]$,

$\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}, \forall x\in R$

Linked Comprehension Type

Basic trigonometric functions are many to one functions. To define the inverse trigonometric functions, we restrict the domain of trigonometric functions in such a way that the functions are bijective. Now domain of different trigonometric functions are defined as follows :

Function	Domain	Range
$\sin x$	$\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$	$[-1, 1]$
$\cos x$	$[\pi, 2\pi]$	$[-1, 1]$
$\tan x$	$\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$	$[-\infty, \infty]$

Inverse of $\sin x$, $\cos x$, $\tan x$ are denoted by $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$ respectively.

13. $\sin^{-1}(x)+\sin^{-1}(-x)=$

(a) 0 (b) π

(c) $\frac{3\pi}{2}$ (d) 2π

14. $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(a) $\frac{5\pi}{12}$ (b) $\frac{13\pi}{12}$

(c) $\frac{31\pi}{12}$ (d) $\frac{29\pi}{12}$

15. Range of $\sin^{-1}x+\cos^{-1}x+\tan^{-1}x$ is

(a) $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$ (b) $(2\pi, 5\pi)$

(c) $\left[\frac{13\pi}{4},\frac{15\pi}{4}\right]$ (d) $(3\pi, 4\pi)$

SOLUTIONS

PART - A

1.(i) We transform the expression $\sin^4x+\cos^4x$ isolating a perfect square :

$\sin^4x+\cos^4x=\sin^4x+2\sin^2x\cos^2x+\cos^4x$
$-2\sin^2x\cos^2x=(\sin^2x+\cos^2x)^2-2\sin^2x\cos^2x$,

whence we get

$$\sin^4x+\cos^4x=1-\frac{1}{2}\sin^2 2x$$

Using the formula obtained, we write the equation in the form

$1-\frac{1}{2}\sin^2 2x=\frac{7}{4}\sin 2x$

Putting $\sin 2x=t$, we get

$2t^2+7t-4=0$, where $t_1=1/2$, $t_2=-4$.

The equation $\sin 2x=\frac{1}{2}$ has solutions

$x=(-1)^n\frac{\pi}{12}+\frac{\pi}{2}, n\in Z;$

The equation $\sin x=-4$ has no solutions.

(ii) The equation is $6\tan^2x-2\cos^2x=\cos 2x$ ....(*)

Let us transform the equation by the formulas

$2\cos^2x=1+\cos 2x,\ \tan^2x=\frac{1-\cos 2x}{1+\cos 2x}$

Note that both sides of the second formula are defined, although not for all, but atleast for the same values of x, namely, for all $x\neq\frac{\pi}{2}+\pi n, n\in Z$.

The left-hand side of the equation is defined for the same values of x. The substitution results in the equation

$6\left(\frac{1-\cos 2x}{1+\cos 2x}\right)-(1+\cos 2x)=\cos 2x$ .....(**)

Any solution of equation (*) is, evidently, a solution of equation (**) and, conversely, every solution of (**) is a solution of (*), that is, equations (*) and (**) are equivalent. Let us solve equation (**). Designating $\cos 2x$ by t and transforming (**), we get $(2t^2+9t-5)=0$, whence we have $t_1=0.5$, $t_2=-5$. Consequently, equation (*) is equivalent to the collection of the equations

$\cos 2x=0.5$, $\cos 2x=-5$.

The first of these equations has solutions

$x=n\pi\pm\frac{\pi}{6}, n\in Z,$

the second equation has no solutions.

(iii) The equation is $5\sin 2x-5\cos 2x=\tan x+5$ ...(*)

Let us express $\sin 2x$ and $\cos 2x$ in terms of $\tan x$ by formulas

$\sin 2x=\frac{2\tan x}{1+\tan^2x}, \cos 2x=\frac{1-\tan^2x}{1+\tan^2x}$

Substituting the right-hand side of these formulas for $\sin 2x$ and $\cos 2x$ in equation (*), we obtain

$5\left(\frac{2\tan x}{1+\tan^2x}\right)-5\left(\frac{1-\tan^2x}{1+\tan^2x}\right)=\tan x+5$ ....(**)

Let us now find whether equations (*) and (**) are equivalent. The functions $\sin 2x$ and $\cos 2x$ are defined for all x while the right-hand sides of formulas substituted for them are defined only for $x=\frac{\pi}{2}+\pi n, n\in Z$. Hence, as a result of the

substitution, the values $x=\frac{\pi}{2}+\pi n, n\in Z$ have got out of the consideration. But it is clear that neither of these values of x is a solution of the original equation (*) since its right-hand side is not defined for $x=\frac{\pi}{2}+\pi n, n\in Z$. Hence it follows that every solution of equation (*) is also a solution of equation (**). The converse is, evidently, also true. Thus we see that equations (*) and (**) are equivalent.

Let us solve equation (**). Putting $\tan x = t$, we get

$$\frac{10t}{1+t^2}-5\left(\frac{1-t^2}{1+t^2}\right)=t+5$$

Simple transformations lead us to the equation $t^3 - 9t + 10 = 0$ equivalent to the preceding. One of divisors of the constant term, namely, $t_1 = 2$, is a solution of this equation. Factoring now the left-hand side (say, by dividing the polynomial $t^3 - 9t + 10$ by the difference $t - 2$), we get $(t - 2)(t^2 + 2t - 5) = 0$. Solving the quadratic equation $t_2=\sqrt{6}-1, t_3=-\sqrt{6}-1$. Thus we see that the original equation (*) is equivalent to the collection of the equations

$\tan x = 2$, $\tan x=\sqrt{6}-1$, $\tan x=-\sqrt{6}-1$,

which have the following respective solutions:

$x = \tan^{-1}2 + \pi n, n \in Z,$

$x=\tan^{-1}(\sqrt{6}-1)+\pi n, n\in Z,$

$x=-\tan^{-1}(\sqrt{6}+1)+\pi n, n\in Z,$

These values of x constitute the set of all solutions of equation (*).

(iv) The equation is $\sin 2x + \cos 2x = -1$ ....(*)

Let us transform the equation with the aid of formulas as

$$\frac{2\tan x}{1+\tan^2 x}+\frac{1-\tan^2 x}{1+\tan^2 x}=-1 \qquad ....(**)$$

Equation (**) is not equivalent to the original equation (*). Indeed, the values $x=\frac{\pi}{2}+\pi k, k\in Z,$ are, evidently, no solutions of equation (**). At the same time, we can easily as certain by means of a substitution that all these values of x are solutions of equation (*).

Let us now consider the values $x\neq\frac{\pi}{2}+\pi k, k\in Z.$

It is clear that any solution of equation (*), satisfying the condition $x\neq\frac{\pi}{2}+\pi k,$ is a solution of equation (**) and, conversely, any solution of (**) is a solution of (*). Let us solve equation (**). We transform it and get

$\frac{2(\tan x+1)}{1+\tan^2 x}=0,$ whence we have $\tan x = -1,$

$x=-\frac{\pi}{4}+\pi k, k\in Z.$ Together with the values $x=-\frac{\pi}{4}+\pi k, k\in Z,$ these values of x constitute the set of all solutions of equation (*).

Answer : $\left\{\frac{\pi}{2}+\pi l; -\frac{\pi}{4}+\pi k \mid k\in Z\right\}$

(v) Let us consider x's such that $\cos x = 0$. It follows from the equation that in this case $\sin x = 0$, as well, and that is impossible. Consequently, there are no solutions among these values of x. Let us now take the values of x for which $\cos x \neq 0$. Dividing both sides of the given equation by $\cos x$, we obtain an equation $\tan^2 x-3\tan x+2=0$, which is equivalent to the original equation. Solving it as a quadratic equation with respect to $\tan x$, we find that $\tan x = 1$, $\tan x = 2$.

Hence we get the answer : $x=\frac{\pi}{4}+\pi n,$

$x = \tan^{-1}2 + n\pi, n \in Z.$

(vi) Rewrite the equation in the form

$$\frac{1}{2}\cos 7x+\frac{\sqrt{3}}{2}\sin 7x=\frac{\sqrt{3}}{2}\cos 5x+\frac{1}{2}\sin 5x$$

or, $\sin\left(\frac{\pi}{6}\right)\cos 7x+\cos\left(\frac{\pi}{6}\right)\sin 7x$

$$=\sin\left(\frac{\pi}{3}\right)\cos 5x+\cos\left(\frac{\pi}{3}\right)\sin 5x$$

i.e., $\sin\left(\frac{\pi}{6}+7x\right)=\sin\left(\frac{\pi}{3}+5x\right)$

But $\sin\alpha = \sin\beta$ if and only if either $\alpha - \beta = 2k\pi$ or $\alpha + \beta = (2m + 1)\pi$ $(k, m = 0, \pm1, \pm2, ...)$

Hence, $\frac{\pi}{6}+7x-\frac{\pi}{3}-5x=2k\pi$

or $\frac{\pi}{6}+7x+\frac{\pi}{3}+5x=(2m+1)\pi$

Thus, the roots of the equation are

$$\left.\begin{aligned} x&=\frac{\pi}{12}(12k+1),\\ x&=\frac{\pi}{24}(4m+1)\end{aligned}\right\}(k,m=0,\pm1,\pm2,\ ...)$$

2.(i) Transform the given equation :

$$\frac{\sin 2x}{\cos 2x}-\frac{\cos x}{\sin x}+\frac{1}{\sin x}-\frac{1}{\sin 5x}=0$$

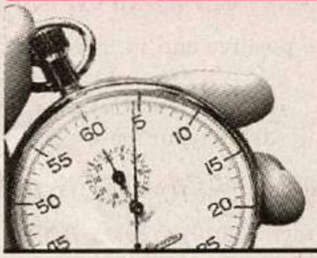
or $$\frac{\sin 2x\sin x-\cos 2x\cos x}{\sin x\cos 2x}+\frac{\sin 5x-\sin x}{\sin x\sin 5x}=0$$

Now we get

$$\frac{-\cos 3x}{\sin x\cos 2x}+\frac{2\cos 3x\sin 2x}{\sin x\sin 5x}=0,$$

CHALLENGING PROBLEMS

for various Engineering Exams



1. If $a\sin\theta - b\cos\theta = p$, $a\cos\theta + b\sin\theta = q$, then $\frac{p+a}{q+b} : \frac{q-b}{p-a} =$

(a) < 0 (b) > 0
(c) 1 (d) none of these

2. If $E = (1 - 3\sin x - 4\cos x)(1 + 3\sin x - 4\cos x)$, then

(a) minimum value of $E = \frac{216}{25}$
(b) minimum value of $E = -\frac{216}{25}$
(c) maximum value of $E = \frac{216}{25}$
(d) maximum value of $E = -\frac{216}{25}$

3. If $\lambda_n = \sin^n\theta + \cos^n\theta$ and $(\lambda_n - \lambda_{n-2}) : \lambda_{n-4} = f(\theta)$, then

(a) $f\left(\frac{\pi}{4}\right) = \frac{1}{4}$ (b) $f\left(\frac{\pi}{6}\right) = \frac{3}{16}$
(c) $f\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{6}\right)$ (d) none of these

4. In ΔABC, if $\cos A = \cos B \cos C$, then $\cot B \cot C =$

(a) 1/2 (b) 1
(c) 0 (d) none of these

5. If $k = 3\sin\left(x + \frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6} - x\right)$, then k

(a) $\le \sqrt{40 + 12\sqrt{3}}$ (b) $\ge \sqrt{40 + 12\sqrt{3}}$
(c) $\le \sqrt{10 + 3\sqrt{3}}$ (d) $\ge \sqrt{10 + 3\sqrt{3}}$

6. If $\cos(x - y) + 1 = 0$ and $\lambda = \cos x + \cos y$ & $\mu = \sin x + \sin y$, then

(a) $\mu > \lambda$ (b) $\mu < \lambda$
(c) $\mu = \lambda$ (d) none of these

7. If $2\tan\beta + \cot\beta = \tan\alpha$, then $\cot(\alpha - \beta)\cot\beta =$

(a) 1 (b) 1/2 (c) −1 (d) 2

8. If $\sin\theta = k\sin(\theta + \phi)$ and $\tan(\theta + \phi) = \frac{\sin\phi}{\cos\phi - \lambda}$,

(a) $\lambda = 1$ (b) $\lambda = 2k$
(c) $\lambda = \frac{k}{2}$ (d) $\lambda = k$

9. Number of solutions of $2\sin^{-1}x + \sin^{-1}(1 - x) = \frac{\pi}{2}$, is

(a) 1 (b) 2
(c) 0 (d) none of these

10. If $\cos^{-1}x < \sin^{-1}x$, then $x \in$

(a) $\left[0, \frac{1}{\sqrt{2}}\right)$ (b) $\left(\frac{1}{\sqrt{2}}, 1\right]$
(c) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (d) none of these

11. If $7\tan\alpha = 3\tan\beta = 1$, then $\cos 2\alpha =$

(a) $\sin 2\beta$ (b) $\sin 4\beta$
(c) $\cos 4\beta$ (d) none of these

12. If $8\cos\theta\cos 2\theta\cos 3\theta - 1 = \frac{\sin k\theta}{\sin\theta}$, then $k =$

(a) 3 (b) 5
(c) 7 (d) none of these

13. $\cos^{2010}x - \sin^{100}x = 1$ has general solution

(a) $n\pi$ (b) $(2n+1)\frac{\pi}{2}$
(c) $(4n+1)\frac{\pi}{2}$ (d) $2n\pi$

14. $\frac{1}{2}\tan^2\left(\frac{\pi}{8}\right) + \tan\left(\frac{\pi}{8}\right) =$

(a) $\tan^2\left(\frac{\pi}{3}\right)$ (b) $\tan^2\left(\frac{\pi}{4}\right)$
(c) $\tan\left(\frac{3\pi}{8}\right)$ (d) none of these

15. $\sin^4\theta - (k+2)\sin^2\theta - (k+3) = 0$ possesses a solution if $k \in$

(a) $(-\infty, \infty)$ (b) $[-3, -2]$
(c) $[-3, -2] \cup \{-1\}$ (d) none of these

16. If $\sin\theta = \frac{a-b}{a+b}$, then $\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = ?$

(a) $\sqrt{\frac{a}{b}}$ (b) $\sqrt{\frac{b}{a}}$

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(c) $\pm\sqrt{\frac{b}{a}}$ (d) none of these

17. If $A + B + C = \pi$ and $x = \Sigma \sin^2 \frac{A}{2}$, $y = \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, then

(a) $x = 1 + 2y$ (b) $2x = 1 - y$
(c) $x = 1 - 2y$ (d) none of these

18. If $2\cos 4x + 9\cos 2x - 7 = 0$, then $x =$

(a) $\frac{1}{2}\cos^{-1}\left(\frac{3}{4}\right)$ (b) $\cos^{-1}\left(\frac{3}{4}\right)$
(c) $\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)$ (d) none of these

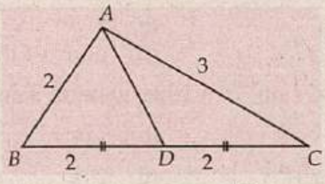
19. If $\tan^{-1} x + \sin^{-1} x = \frac{\pi}{2}$, then $x =$

(a) $\pm\sqrt{\frac{\sqrt{5}-1}{2}}$ (b) $\pm\sqrt{\frac{\sqrt{5}\pm 1}{2}}$
(c) $\pm\sqrt{\frac{\sqrt{5}+1}{2}}$ (d) none of these

20. If $\tan\theta + \frac{1}{2}\tan\theta + \frac{1}{2^2}\tan\theta + \ldots$ to ∞ $= 1 + \log_2\sqrt{6} + \log_2\left(\frac{3}{2}\right)^{-1/2}$ then values of θ are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

21. Find AD, in the figure given below



(a) $\frac{5}{2}$ (b) $\sqrt{5}$
(c) $\sqrt{2.5}$ (d) none of these

22. The number of solutions of $2(\sin^3 x + \cos^3 x) - 3(\sin x + \cos x) + 8 = 0$ is

(a) 0 (b) 1
(c) 3 (d) none of these

23. In ΔABC, if $2a(a - c) + 4b(b - a) + c^2 = 0$, then $B =$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{6}$ (d) none of these

24. If $\sin\theta + \sin\alpha = 2\sin(\theta + \alpha)$, then $\tan\left(\frac{\theta}{2}\right) : \cot\left(\frac{\alpha}{2}\right) =$

(a) 1 : 3 (b) 3 : 1
(c) 1 : 1 (d) none of these

25. If $\cos 7\frac{1}{2}^\circ + 4\cos 36^\circ = \sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d} + \sqrt{e} + \sqrt{f}$ so that a, b, c, d, e, f are positive and in increasing order then

(a) $a + b + c = d + e + f$ (b) $a + c + e = b + d + f$
(c) $a + f = b + e = c + d$ (d) none of these

26. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, $f(x + y) = f(x).f(y)$ $\forall x, y \in R$ and $f(1) = 1$, then $f(2010), f(2011), f(2012)$ are in

(a) A.P. (b) G.P.
(c) both A.P. & G.P. (d) H.P.

27. If $\sin A - \sqrt{6}\cos A = \sqrt{7}\cos A$, then $\sqrt{6}\sin A + \cos A =$

(a) $\sqrt{7}\cos A$ (b) $\cos A$
(c) $\sqrt{7}\sin A$ (d) $\sqrt{6}\cos A$

28. If α, β are solutions of $\cos^2 x + a\cos x + b = 0$ & $\sin^2 x + c\sin^2 x + d = 0$, then $\sin(\alpha + \beta) =$

(a) $\frac{2ac}{a^2 + c^2}$ (b) $\frac{2bd}{b^2 + d^2}$
(c) $\frac{a^2 + c^2}{b^2 + d^2}$ (d) none of these

29. In ΔABC, if $bc^2 = (a + b)(a - b)^2$, then

(a) $A = 2B$ (b) $A = 3B$
(c) $B = 3A$ (d) none of these

30. $[\tan^{-1}1 - \tan 1] = ?$ (where [.] denotes greatest integer function)

(a) -1 (b) 0
(c) 1 (d) none of these

SOLUTIONS

1. **(a)** : On squaring & adding, $a^2 + b^2 = p^2 + q^2$

$$\Rightarrow (a^2 - p^2) = q^2 - b^2$$

$$\Rightarrow \frac{a+p}{q+b} = \frac{q-b}{a-p} \quad \Rightarrow \frac{p+a}{q+b} : \frac{q-b}{p-a} = -1 < 0$$

2. **(b)** : $E = 1 - 8\cos x - 9\sin^2 x + 16\cos^2 x$ (on simplification)

$$= 1 - 8\cos x - 9(1 - \cos^2 x) + 16\cos^2 x$$

$$= 25\left(t^2 - \frac{8}{25}t\right) - 8 \quad [\text{where } t = \cos x]$$

$$= 25\left\{\left(t - \frac{4}{25}\right)^2 - \frac{16}{625}\right\} - 8$$

$$= 25\left(t - \frac{4}{25}\right)^2 - \frac{216}{25} \geq -\frac{216}{25}$$

$$\therefore \quad E_{\min} = -\frac{216}{25}.$$

3. **(c)** : $\lambda_n - \lambda_{n-2} = (\sin^n\theta + \cos^n\theta) - (\sin^{n-2}\theta + \cos^{n-2}\theta)$

$= \sin^{n-2}\theta(\sin^2\theta - 1) + \cos^{n-2}\theta(\cos^2\theta - 1)$

$= -\sin^2\theta\cos^2\theta(\sin^{n-4}\theta + \cos^{n-4}\theta)$

$= -\sin^2\theta\cos^2\theta.\lambda_{n-4}$

$\Rightarrow (\lambda_n - \lambda_{n-2}) : \lambda_{n-4} = -\sin^2\theta\cos^2\theta$

$\therefore f(\theta) = -\sin^2\theta\cos^2\theta$

$\therefore f\left(\frac{\pi}{4}\right) = -\frac{1}{4}, f\left(\frac{\pi}{6}\right) = -\frac{3}{16} = f\left(\frac{\pi}{3}\right)$

4. **(a)** : $\because \cos B\cos C = \cos A = \cos\{\pi - (B + C)\}$ $[\because A + B + C = \pi]$

$= -\cos(B + C) = -\cos B\cos C + \sin B\sin C$

$\Rightarrow 2\cos B\cos C = \sin B\sin C$

$\Rightarrow \cot B\cot C = \frac{1}{2}$

5. **(c)** : $k = 3\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right) + \left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right)$

$= \frac{1}{2}\{(3\sqrt{3}+1)\sin x + (3+\sqrt{3})\cos x\} \le \frac{1}{2}\sqrt{40+12\sqrt{3}}$

$\therefore k \le \sqrt{10+3\sqrt{3}}$

$[\because \sqrt{(3\sqrt{3}+1)^2 + (3+\sqrt{3})^2} = ... = \sqrt{40+12\sqrt{3}}$ and

$-\sqrt{a^2+b^2} \le a\sin x + b\cos x \le \sqrt{a^2+b^2}]$

6. **(c)** : $\cos(x - y) = -1 = \cos\pi \Rightarrow x = \pi + y$

$\therefore \lambda = \cos(\pi + y) + \cos y = -\cos y + \cos y = 0$

& $\mu = \sin(\pi + y) + \sin y = -\sin y + \sin y = 0$

7. **(d)** : $\cot\beta = 2(\tan\alpha - \tan\beta) - \tan\alpha$

$\Rightarrow \cot\beta + \tan\alpha = 2(\tan\alpha - \tan\beta)$

$\Rightarrow \cot\beta(1 + \tan\alpha\tan\beta) = 2(\tan\alpha - \tan\beta)$

$\Rightarrow \cot\beta = 2\tan(\alpha - \beta)$

$\therefore \cot(\alpha - \beta)\cot\beta = 2$

8. **(d)** : $\sin\{(\theta + \phi) - \phi\} = k\sin(\theta + \phi)$

$\Rightarrow \sin(\theta + \phi)\cos\phi - \cos(\theta + \phi)\sin\phi = k\sin(\theta + \phi)$

$\Rightarrow \sin(\theta + \phi)(\cos\phi - k) = \cos(\theta + \phi)\sin\phi$

$\Rightarrow \tan(\theta+\phi) = \frac{\sin\phi}{\cos\phi - k} \Rightarrow \lambda = k$

9. **(b)** : $2\sin^{-1}x + \sin^{-1}(1 - x) = \sin^{-1}x + \cos^{-1}x$

$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

$\Rightarrow \sin^{-1}x + \sin^{-1}(1 - x) = \cos^{-1}x$

$\Rightarrow \sin^{-1}\{x\sqrt{1-(1-x)^2} + (1-x)\sqrt{1-x^2}\} = \sin^{-1}\sqrt{1-x^2}$

$\Rightarrow x\sqrt{2x-x^2} + \sqrt{1-x^2} - x\sqrt{1-x^2} = \sqrt{1-x^2}$

$\Rightarrow x(\sqrt{2x-x^2} + \sqrt{1-x^2}) = 0$

$\Rightarrow x = 0$ or $2x - x^2 = 1 - x^2 \Rightarrow x = 0, \frac{1}{2}$

$\therefore$ Number of solutions = 2

10. **(d)** : Here, $-1 \le x \le 1$ ...(1)

[for $\cos^{-1}x$ & $\sin^{-1}x$ both to be defined]

$\because \cos^{-1}x < \sin^{-1}x \therefore \frac{\pi}{2} - \sin^{-1}x < \sin^{-1}x$

$\Rightarrow \sin^{-1}x > \frac{\pi}{4} \Rightarrow \sin(\sin^{-1}x) > \sin\frac{\pi}{4}$

$\Rightarrow x > \frac{1}{\sqrt{2}}$ ...(2)

From (1) & (2) $x \in \left(\frac{1}{\sqrt{2}}, 1\right]$

11. **(b)** : $\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{1-\frac{1}{49}}{1+\frac{1}{49}} = \frac{24}{25}$

$\sin 2\beta = \frac{2\tan\beta}{1+\tan^2\beta} = \frac{2\left(\frac{1}{3}\right)}{1+\frac{1}{9}} = \frac{3}{5}$

$\cos 2\beta = \frac{1-\tan^2\beta}{1+\tan^2\beta} = \frac{1-\frac{1}{9}}{1+\frac{1}{9}} = \frac{4}{5}$

$\therefore \sin 4\beta = 2\sin 2\beta\cos 2\beta = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25} = \cos 2\alpha$

12. **(c)** : We have

$\sin k\theta = 4.2\sin\theta\cos\theta\cos 2\theta\cos 3\theta - \sin\theta$

$= 4\sin 2\theta\cos 2\theta\cos 3\theta - \sin\theta$

$= 2\sin 4\theta\cos 3\theta - \sin\theta$

$= (\sin 7\theta + \sin\theta) - \sin\theta = \sin 7\theta$

$\Rightarrow k = 7$

13. **(a)** : $\cos^{2010}x = 1 + \sin^{100}x$

L.H.S. ≤ 1 and R.H.S ≥ 1

$\Rightarrow \sin x = 0 \Rightarrow x = n\pi$

14. **(d)** : $1 = \tan\left(\frac{\pi}{4}\right) = \frac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}$

$\therefore 2\tan\left(\frac{\pi}{8}\right) = 1 - \tan^2\left(\frac{\pi}{8}\right)$

$\Rightarrow \frac{1}{2}\tan^2\left(\frac{\pi}{8}\right) + \tan\left(\frac{\pi}{8}\right) = \frac{1}{2}$

15. **(b)** : $\sin^2\theta = \frac{k+2 \pm \sqrt{(k+2)^2 + 4(k+3)}}{2}$

$$= \frac{k+2\pm(k+4)}{2} = -1, k+3$$

$\because \quad \sin^2\theta \neq -1 \quad \therefore \quad \sin^2\theta = k+3$

$\Rightarrow \quad 0 \le k+3 \le 1 \Rightarrow \quad -3 \le k \le -2$

$\therefore \quad k \in [-3, -2]$

16. (c) : $\dfrac{a+b}{a-b} = \dfrac{1}{\sin\theta} = \dfrac{1}{\cos\left(\dfrac{\pi}{2}-\theta\right)}$

By componendo & dividendo

$$\frac{2b}{2a} = \frac{1-\cos\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}+\theta\right)} = \frac{2\sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}$$

$$\Rightarrow \tan^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \frac{b}{a} \quad \therefore \quad \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \pm\sqrt{\frac{b}{a}}$$

17. (c) : $x = \dfrac{1}{2}\left(2\sin^2\left(\dfrac{A}{2}\right) + 2\sin^2\left(\dfrac{B}{2}\right)\right) + \sin^2\left(\dfrac{C}{2}\right)$

$$= \frac{1}{2}(1-\cos A + 1 - \cos B) + \sin^2\left(\frac{C}{2}\right)$$

$$= 1 - \frac{1}{2}(\cos A + \cos B) + \sin^2\left(\frac{C}{2}\right)$$

$$= 1 - \frac{1}{2}\cdot 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \sin\left(\frac{C}{2}\right)\cdot\sin\left(\frac{C}{2}\right)$$

$$= 1 - \cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\left(\frac{A}{2}-\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right)\cdot\sin\left(\frac{\pi}{2}-\left(\frac{A}{2}+\frac{B}{2}\right)\right)$$

$[\because A+B+C=\pi]$

$$= 1 - \sin\left(\frac{C}{2}\right)\cdot\cos\left(\frac{A}{2}-\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right)\cdot\cos\left(\frac{A}{2}+\frac{B}{2}\right)$$

$$= 1 - \sin\left(\frac{C}{2}\right)\left\{\cos\left(\frac{A}{2}-\frac{B}{2}\right) - \cos\left(\frac{A}{2}+\frac{B}{2}\right)\right\}$$

$$= 1 - \sin\left(\frac{C}{2}\right)\cdot 2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right) = 1-2y$$

18. (a) : Put $\cos 2x = t$, then we have

$2(2t^2-1) + 9t - 7 = 0$

$\Rightarrow \quad 4t^2 + 9t - 9 = 0 \quad \Rightarrow \quad t = -3, \dfrac{3}{4}$

But, $\cos 2x \neq -3 \;\therefore\; \cos 2x = \dfrac{3}{4} \quad \therefore \quad x = \dfrac{1}{2}\cos^{-1}\left(\dfrac{3}{4}\right)$

19. (d) : $\tan^{-1}x = \dfrac{\pi}{2} - \sin^{-1}x = \cos^{-1}x$

$$\Rightarrow \sec^{-1}\sqrt{1+x^2} = \sec^{-1}\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad 1+x^2 = \frac{1}{x^2} \quad \Rightarrow \quad x^4 + x^2 - 1 = 0$$

$$\therefore \quad x^2 = \frac{-1\pm\sqrt{1-4\cdot1\cdot(-1)}}{2} = \frac{-1\pm\sqrt{5}}{2}$$

$$\because \quad x^2 \not< 0 \quad \therefore \quad x^2 = \frac{\sqrt{5}-1}{2}$$

$$\therefore \quad x = \pm\sqrt{\frac{\sqrt{5}-1}{2}}$$

But, if $x < 0$ then L.H.S. of given equation becomes –ve and given equation is not satisfied

$$\therefore \quad x = +\sqrt{\frac{\sqrt{5}-1}{2}}$$

20. (a) : $\tan\theta\left(1+\dfrac{1}{2}+\dfrac{1}{4}+\ldots\infty\right) = 1 + \log_2\sqrt{6} + \log_2\sqrt{\dfrac{2}{3}}$

$$\frac{1}{\left(1-\frac{1}{2}\right)}(\tan\theta) = 1 + \log_2\left(\sqrt{6}\times\sqrt{\frac{2}{3}}\right)$$

$\Rightarrow \quad 2\tan\theta = 1 + \log_2 2 = 2 \Rightarrow \quad \theta = n\pi + \dfrac{\pi}{4}, n \in I$

On putting n = ..., –2, –1, 0, 1, 2, ... etc., values of θ will be in A.P.

21. (c) : Using cosine Rule in ΔABD & ΔABC, we get

$$\cos B = \frac{4+4-AD^2}{2(2)(2)} = \frac{8-AD^2}{8} \;\&\; \cos B = \frac{4+16-9}{2\cdot(2)(4)} = \frac{11}{16}$$

on solving, $AD = \sqrt{2.5}$

22. (a) : Let $\sin x + \cos x = t$

$\therefore \quad 1 + 2\sin x\cos x = t^2 \qquad \ldots(1)$

From given equation, we get

$2(\sin x + \cos x)(\sin^2 x + \cos^2 x - \sin x\cos x) - 3t + 8 = 0$

$\Rightarrow \quad 2t(1-\sin x\cos x) - 3t + 8 = 0$

$\Rightarrow \quad 2t - t(t^2-1) - 3t + 8 = 0 \quad$ [using (1)]

$\Rightarrow \quad t^3 = 8 \quad \Rightarrow \quad t = 2$

$\Rightarrow \quad \sin x + \cos x = 2$

$\Rightarrow \quad 1 + \sin 2x = 4 \qquad$ [on squaring]

$\Rightarrow \quad \sin 2x = 3 \qquad$ (impossible)

$\Rightarrow \quad$ No solution

23. (d) : On simplifying $(a-c)^2 + (a-2b)^2 = 0$

$\Rightarrow \quad a - c = 0$ and $a - 2b = 0$

$\Rightarrow \quad a = 2b = c$

$$\therefore \quad \cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{c^2+c^2-\frac{c^2}{4}}{2c.c} = \frac{7}{8}$$

$$\therefore \quad B = \cos^{-1}\left(\frac{7}{8}\right)$$

24. (a) :

$$2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)=2.2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta+\alpha}{2}\right)$$

$$\Rightarrow \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\alpha}{2}\right)+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\alpha}{2}\right)$$

$$=2\left(\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\alpha}{2}\right)-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\alpha}{2}\right)\right)$$

$$\Rightarrow 3\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\alpha}{2}\right)=\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\alpha}{2}\right)$$

$$\Rightarrow \tan\left(\frac{\theta}{2}\right):\cot\left(\frac{\alpha}{2}\right)=1:3$$

25. (c) : $\because \quad 4\cos 36^\circ = 4\left(\frac{\sqrt{5}+1}{4}\right)=\sqrt{5}+1$

and $\cot 7\frac{1}{2}^\circ = \frac{2\cos^2 7\frac{1}{2}^\circ}{2\sin 7\frac{1}{2}^\circ \cos 7\frac{1}{2}^\circ}=\frac{1+\cos 15^\circ}{\sin 15^\circ}$

$$=\frac{1+\cos(60^\circ-45^\circ)}{\sin(60^\circ-45^\circ)}$$

$$=\frac{1+\frac{1+\sqrt{3}}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}=\frac{2\sqrt{2}+1+\sqrt{3}}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$=\frac{2\sqrt{6}+4+2\sqrt{2}+2\sqrt{3}}{2}=\sqrt{6}+\sqrt{3}+\sqrt{2}+\sqrt{4}$$

$$\therefore \quad \cot 7\frac{1}{2}^\circ + 4\cos 36^\circ = \sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}$$

(in increasing order)

$\therefore \quad a = 1, b = 2, c = 3, d = 4, e = 5, f = 6$

$\therefore \quad a + f = b + e = c + d = 7$

26. (c) : $\because -\frac{\pi}{2}\le \sin^{-1}x \le \frac{\pi}{2}$ etc.

$$\therefore \quad \sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$$

$$\Rightarrow \sin^{-1}x=\sin^{-1}y=\sin^{-1}z=\frac{\pi}{2}$$

$\therefore \quad x = y = z = 1$

Given, $f(1) = 1$ & $f(x + y) = f(x).f(y)$

$\therefore \quad f(2) = f(1 + 1) = f(1).f(1) = 1^2 = 1$

$f(3) = f(2 + 1) = f(2).f(1) = 1.1 = 1$

.... etc.

$\Rightarrow \quad f(2010) = 1 = f(2011) = f(2012)$

$\Rightarrow \quad f(2010), f(2011), f(2012)$ are in both A.P. & G.P.

27. (c) : $\sin A = (\sqrt{7}+\sqrt{6})\cos A = \frac{7-6}{\sqrt{7}-\sqrt{6}}\cos A$

$$\Rightarrow \sin A = \frac{\cos A}{\sqrt{7}-\sqrt{6}}$$

$$\Rightarrow \sqrt{7}\sin A - \sqrt{6}\sin A = \cos A$$

$$\therefore \quad \sqrt{6}\sin A + \cos A = \sqrt{7}\sin A$$

28. (a) : Put $\cos x = t$

$\Rightarrow \quad t^2 + at + b = 0$ has roots $\cos\alpha$ & $\cos\beta$

$\therefore \quad \cos\alpha + \cos\beta = -a$

$$\Rightarrow 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)=-a \quad ...(1)$$

Similarly, from 2nd equation,

$$2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)=-c \quad ...(2)$$

(2) ÷ (1) gives $\tan\left(\frac{\alpha+\beta}{2}\right)=\frac{c}{a}$

$$\therefore \quad \sin(\alpha+\beta)=\frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)}=\frac{2ac}{a^2+c^2}$$

29. (b) : $\frac{b}{a-b}=\frac{a^2-b^2}{c^2}=\frac{(\sin^2 A-\sin^2 B)}{\sin^2 C}$

$$=\frac{\sin(A+B)\sin(A-B)}{\sin^2(A+B)} \quad [\because A+B+C=\pi]$$

$$\Rightarrow \frac{b}{a-b}=\frac{\sin(A-B)}{\sin(A+B)}$$

$\Rightarrow \quad b\sin(A + B) = (a - b)\sin(A - B)$

$\Rightarrow \quad b\{\sin(A + B) + \sin(A - B)\} = a\sin(A - B)$

$\Rightarrow \quad 2b\sin A\cos B = a\sin(A - B)$

$\Rightarrow \quad 2R\sin B.2\sin A\cos B = 2R\sin A\sin(A - B)$

$\Rightarrow \quad \sin 2B = \sin(A - B)$

$\therefore \quad A = 3B$

30. (a) : $\because \tan^{-1}1=\frac{\pi}{4}=\frac{3.14}{4}=0.785$ (approx.)

and $\tan 1 = \tan 57^\circ$ (approx.)

$= \sqrt{3} = 1.732$ (approx.)

$\therefore \quad \tan^{-1}1 - \tan 1 = 0.785 - 1.732$

$= -0.94$ (app.)

$\therefore \quad [\tan^{-1}1 - \tan 1] = [-0.94] = -1$

MATHS MUSING

SOLUTION SET - 85

1. **(b)** : If $c = kb$, $k > 1$, then

$$b = \frac{4020kb}{2010+kb} \Rightarrow k(4020-b) = 2010$$

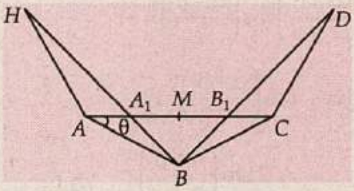
$\therefore$ k is divisor of $2010 = 2.3.5.67$ $\quad \therefore N = 2^4 - 1 = 15$

2. **(c)** : $z = re^{i\theta}$

$$\left|z+\frac{1}{z}\right|^2 = r^2 + \frac{1}{r^2} + 2\cos 2\theta \Rightarrow a = \frac{3}{2}, b = \frac{17}{4}$$

$$\therefore \quad a+b = \frac{23}{4}$$

3. **(b)** :



$$\theta = \frac{\pi}{8}$$

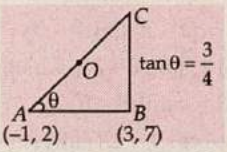
$$A_1B_1 = 2A_1M = 2\left(\cos\theta - \frac{1}{2\cos\theta}\right)AB$$

$$\frac{A_1B_1}{AB} = \frac{\cos 2\theta}{\cos\theta}, \left(\frac{A_1B_1}{AB}\right)^2 = \frac{1+\cos 4\theta}{1+\cos 2\theta} = 2-\sqrt{2}$$

4. **(d)** : The given determinant is the product

$$\begin{vmatrix} 0 & 1 & 1 \\ 0 & a+b & c+d \\ 0 & ab & cd \end{vmatrix}\begin{vmatrix} 0 & 0 & 0 \\ 1 & c+d & cd \\ 1 & a+b & ab \end{vmatrix} = 0$$

5. **(a, c)** :



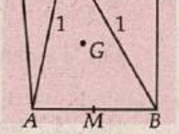
O is the centre. Using rotations, we have

$$z = -1+2i \pm (4+5i)\frac{5}{8}cis\theta$$

$$= -1+2i \pm \frac{1}{8}(4+5i)(4+3i) = -\frac{7}{8}+6i, \frac{23}{8}+3i$$

$$d = |z_1| = \sqrt{\left(\frac{7}{8}\right)^2 + 36} \quad \Rightarrow [d] = 6$$

$$d = |z_2| = \sqrt{\left(\frac{23}{8}\right)^2 + 9} \quad \Rightarrow [d] = 4$$



$$\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}, \overrightarrow{OD} = \vec{d}$$

$$AB = 2\sin\left(\frac{\theta}{2}\right), |\vec{a}-\vec{c}|^2 = 2AB^2 = 2 - 2\vec{a}\cdot\vec{c}$$

$$\therefore \quad \vec{a}\cdot\vec{c} = 1 - 4\sin^2\left(\frac{\theta}{2}\right) = 2\cos\theta - 1$$

$$(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} \cos\theta & 2\cos\theta - 1 \\ 1 & \cos\theta \end{vmatrix} = (1-\cos\theta)^2$$

6. **(c)** : $\sin^2\theta\cos\alpha = (1-\cos\theta)^2 \Rightarrow \cos\alpha = \tan^2\left(\frac{\theta}{2}\right)$

$$= (\sqrt{2}-1)^2 = 3 - 2\sqrt{2}, \text{ since } \theta = \frac{\pi}{4}$$

7. **(c)** : If G is the foot of perpendicular from O on the plane ABC, then $OG^2 = OM^2 - MG^2$

$$\therefore \quad OG = \sqrt{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)} = \sqrt{\cos\theta}$$

$$\cos\beta = \frac{GM}{OM} = \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = \tan\frac{\pi}{12} = 2-\sqrt{3}, \text{ since } \theta = \frac{\pi}{6}$$

8. **(b)** : Volume $= \frac{1}{3}AB^2\cdot OG = \frac{1}{3}\cdot 4\sin^2\left(\frac{\theta}{2}\right)\sqrt{\cos\theta}$

$$= \frac{2}{3}(1-\cos\theta)\sqrt{\cos\theta} = \frac{1}{3\sqrt{2}}, \text{ since } \theta = \frac{\pi}{3}$$

9. **(a) → (t); (b) → (r); (c) → (p); (d) → (q)**

(a) coeff. x^{10} in $(x + x^2 + ... + x^6)^4$

$$= \text{coeff. } x^6 \text{ in } (1-x)^6(1-x)^{-4} = \binom{9}{3} - 4 = 80$$

$$\therefore \quad \text{Probability} = \frac{80}{6^4} = \frac{5}{81}$$

(b) Consider the 3 rows : 1, 4, 7, 10
2, 5, 8
3, 6, 9

x and y are from any of the 3 rows or one from the first and the other from the second row

$$\therefore \quad \text{Probability} = \frac{\binom{4}{2} + 2\binom{3}{2} + \binom{4}{1}\binom{3}{1}}{\binom{10}{2}} = \frac{8}{15}$$

$$\text{(c)} \quad \therefore \quad \text{Probability} = \frac{\binom{10-3+1}{3}}{\binom{10}{3}} = \frac{\binom{8}{3}}{\binom{10}{3}} = \frac{7}{15}$$

(d) Number of triangles not having common side with the octagon is $\frac{8}{6}(8-4)(8-5) = 16$

10. $f(x) = \dfrac{6(ax^2+bx+1)}{(px^2+qx+1)}$

$f(2) = 3, f'(2) = 0, f(-2) = 4, f'(-2) = 0$

Determining, $a = \frac{1}{4}, b = -3, c = \frac{1}{4}, d = -5$

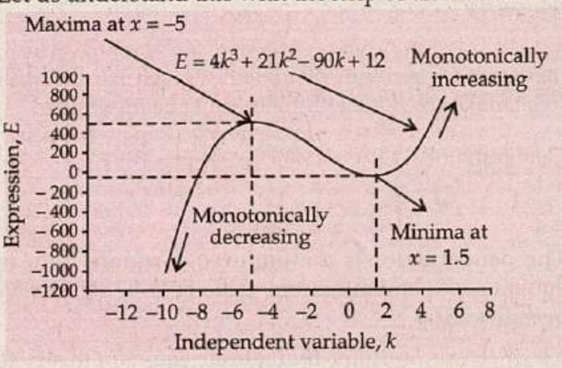
$$f(x) = \frac{6(x^2-12x+4)}{(x^2-20x+4)}, f(1) = \frac{14}{5}$$

$m = 14, n = 5, m - n = 9$

CONDITIONAL MAXIMA & MINIMA

Although calculus is used for getting maxima or minima of a given algebraic expression, but in competitive examinations, where multiple options are available, it is better to search for some efficient shortcut methods. This articles gives approaches to find maxima and minima using less time consuming methods. In the article, first letters of our names *i.e.* k, p, h and s are used as independent variables.

Finding maxima or minima of an algebraic expression is a subject matter of calculus, where first differentiation of a given expression with respect to independent variable is made zero, to get value of independent variable, which makes algebraic expression optimum. This is followed by taking a second derivative of the expression, for which the given value of independent variable may give a positive or negative value. If value of second derivative at the given value of independent variable is negative, maxima of expression is obtained. Let us understand this with the help of an illustration.



Find maximum value of expression $4k^3+21k^2-90k+12$. First derivative of this expression is $12k^2 + 42k - 90$. If this is equated to zero, k comes out be 3/2 and -5. Second derivative of this expression is $24k + 42$. For $k = 3/2$, second derivative is positive, so expression has a minimal at this value of k and the value is -62.25. Contrary to this, at $k = -5$, second derivative is negative, so expression takes a maximum value 487 at $k = -5$. Sometimes curve plotting is also attempted and figure indicates salient features of algebraic expression in x-y plane.

The problem of maxima and minima becomes more difficult, if it involves more than one independent variables and variables are related to each other by some relation or condition. Find minimum value of $k + p$, if $kp = 25$; k, p are positive real numbers. As per approach discussed above, expression is represented in one independent variable as $k + 25/k$. First derivative $(1 - 25/k^2)$ is equated to zero to give $k = 5$. Second derivative of given expression is $50/k^3$. It is positive, so the expression is a minimum. The approach can be shortened, if following rule is adopted.

Rule 1 : If all independent variables are symmetrical in given condition as well as optimization expression, equal value of each variable gives optimum results. A symmetrical expression means exchanging variables does not change expression.

Find maximum value kp, if $k + p = 12$. As expression is symmetrical for both the variables, optimum value will come for $k = p$ (= 6) as per rule 1. Maximum value of kp is 36. A further extension to above problem is having more than 2 variables. Find minimum value of $(1 + k)(1 + p)(1 + h)(1 + s)$, if k, p, h, s are positive real numbers and $kphs = 16$. Again, this is a symmetrical expression in 4 variables. Optimum value will occur, if $k = p = h = s$ (= 2). Minimum value of given expression is 81. However, above mentioned method fails for the problem where symmetry of variables is disturbed.

Find maximum value of $3k + 4p$, if $kp = 27$; k, p are positive real numbers. It is clear that expression is not symmetric, although condition is symmetric. So above rule of equal values of variables is not applicable here. For such expressions, another rule is applicable.

Rule 2 : Arithmetic mean is more than or equal to geometric mean. Equality holds if numbers are equal. Arithmetic mean of two numbers '$3k$' and '$4p$' is $\dfrac{(3k+4p)}{2}$ and geometric mean of $3k$ and $4p$ is

$\sqrt{(3k \cdot 4p)} = \sqrt{(12kp} = \sqrt{(12.27)} = 18$. Now as per rule 2,

$$\frac{3k+4p}{2} \geq 18, \text{ or } (3k+4p) \geq 36.$$

So, minimum value of $(3k + 4p)$ is 36. However, problems can be made more complex: Find maximum value of $k^3 p^2$, if $2k + 3p = 10$; k, p are positive real numbers. Here there is no symmetry in expression or in condi-

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tion. For such problems, a more exhaustive rule has to be applied.

Rule 3 : If $k^a p^b h^c s^d$ is given expression and $ek + fp + gh + js$ is given condition or its vice versa, optimum value of expression will occur if

$$\frac{ek}{a}=\frac{fp}{b}=\frac{gh}{c}=\frac{js}{d}.$$

As per rule 3, to get optimum value, $\frac{2k}{3}=\frac{3p}{2}$.

Using given condition, $2k+3p=10$, $k=3$, $p=\frac{4}{3}$.

So, maximum value of expression, k^3p^2 is 48. Following illustrations gives application of above mentioned rules along with some alternate methods for solution.

1. If k and p are real and $kp + 2(k + p) = 21$, what is minimum value of $(k + p)$?

(a) 6 (b) 19/3 (c) 7 (d) 17/6

(e) cannot be determined

Soln.(a) : Equation and requirements both are symmetrical, so as per rule 1, $k = p$ is a solution.

$$(k-p)=0$$
$$\Rightarrow (k+p)^2-4kp=0$$
$$\Rightarrow (k+p)^2+8(k+p)-84=0$$
$$\Rightarrow (k+p)(k+p+8)=84=14\times 6$$
$$\Rightarrow (k+p)=6.$$

Alternative Solution : An alternate approach can also be thought for this problem.

$$kp+2k+2p+4=25$$
$$\Rightarrow (k+2)(p+2)=25.$$

As per rule 2, $\frac{(k+2)+(p+2)}{2}>\sqrt{(k+2)(p+2)}$.

So, $(k+p)\geq 6$ and minimum value of $(k+p)=6$.

Note : This approach is specific to this problem because given expression can be written as product of two linear algebraic expressions.

2. If k and p are positive real numbers, such that $6kp + 10k + 15p = 39$. Find the minimum value of $2k + 3p$.

(a) 43/6 (b) 6 (c) 41/7 (d) 39/5

(e) 84/11

Soln. (b) : For solution, $h = 2k$, $s = 3p$, then problem reduces to finding minimum value of $(h + s)$ for given condition $hs + 5(h + s) = 39$. Obviously, this is similar to first question. All expressions are symmetrical in independent variables, and optimum value occurs for $h = s$. Using condition, $h = s\ (= 3)$ and $h + s = 6$.

Alternative Solution : $6kp+10k+15p+25=64$

$$\Rightarrow (2k+5)(3p+5)=64$$
$$\Rightarrow 2k+5=8=3p+5$$
$$\Rightarrow 2k=3, 3p=3.$$

3. If k and p are non-negative real numbers such that $k + p = h$, then minimum value of the expresion $(k + p + 2kp)/(1 + k + p + kp)$ is

(a) $\frac{2h}{(1+h^2)}$ (b) $\frac{2h}{(2+h)}$

(c) $\frac{2h^2}{(1+h^2)}$ (d) $\frac{2h}{(1+h)}$

(e) $\frac{2h}{(1+2h)}$

Soln. (e) : As given expression is symmetrical in both the independent variables, rule 1 is applicable.

So $k = p\ (= h/2)$ gives minimum value of function and is equal to $\frac{2h}{2+h}$.

4. If $3 < k < 8$, $6 < p < 12$, $7 < h < 14$, and k, p, h are integers, find maximum value of $\frac{h}{p+h-k}$.

(a) 7/5 (b) 7/6 (c) 14/13 (d) 14/11

(e) 14

Soln. (b) : Maximum value of given expression is same as finding minimum value of $1+\frac{(p-k)}{h}$. From nature of expression, p should acquire minimum value, while k and h should be given maximum value. So, $k = 8$, $p = 6$, $h = 14$. $\therefore$ The maximum value is 7/6.

5. What is maximum value of $\frac{4}{9k^6+8k^3+3}$?

(a) 36/13 (b) 36/7 (c) 4/3 (d) 9/5

(e) 36/11

Soln. (e) : For the given expression, if k^3 is replaced by p, denominator becomes $9p^2+8p+3$. If given expression is to be maximized, denominator must be minimized.

$$\text{Denominator } =(3p)^2+2\times 3p\times\frac{4}{3}+\left(\frac{4}{3}\right)^2-\left(\frac{4}{3}\right)^2+3$$
$$=\left(3p+\frac{4}{3}\right)^2+\frac{11}{9}$$

The denominator is a minimum, if square term is equal to zero and minimum value is 11/9. Expression becomes 36/11.

Alternative Solution : If quadratic equation theory is applied, an expression of type $ak^2 + bk + c$ is optimum, at $k=-\frac{b}{2a}$ and minimum value is $c-\left(\frac{b^2}{4a}\right)$. Using this knowledge, minimum of denominator = 11/9.

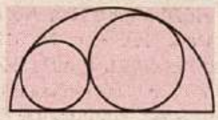
Although well set rules are there in calculus and algebra for finding conditional, maximal and minimal values of given expressions, competitive examinations demand a quicker, efficient, perfect and definite approach for finding correct solution. Definitely, the rules specified above and the type of problems discussed will help students to get correct option in multiple choice questions.

ISI 2010

OLYMPIAD CORNER

Challenging problems for Olympiads, IIT-JEE and other contests.

1. Two externally tangent circles of radii R_1 and R_2 are internally tangent to a semicircle of radius 1, as in the figure. Prove that $R_1 + R_2 \le 2(\sqrt{2} - 1)$



2. Show that for all points P of Γ, it is possible to construct a triangle of sides PA, PB, PC, with area $\frac{\sqrt{3}}{4}$.

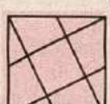
3. Let a, b, c, d, e, f be the lengths of edges of a given tetrahedron and S be its surface area. Prove that
$S \le \frac{\sqrt{3}}{6}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2)$.

4. Find all solutions of the equation
$\cos 12x = 5\sin 3x + 9\tan^2 x + \cot^2 x$

5. Show that if $x, y, z > 0$,
$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \ge \frac{9}{4}.$$

6. An acute triangle ABC is given. Points A_1 and A_2 are taken on the side BC (with A_2 between A_1 and C), B_1 and B_2 on the side AC (with B_2 between B_1 and A) and C_1 and C_2 on the side AB (with C_2 between C_1 and B) so that
$$\angle AA_1A_2 = \angle AA_2A_1 = \angle BB_1B_2 = \angle BB_2B_1 = \angle CC_1C_2 = CC_2C_1.$$
The lines AA_1, BB_1, and CC_1 bound a triangle, and the lines AA_2, BB_2 and CC_2 bound a second triangle. Prove that all six vertices of these two triangles lie on a single circle.

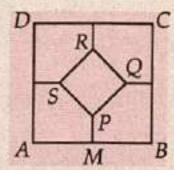
7. The large square has area 1. The inside lines join a vertex of the square to the mid-point of a side as shown. What is the area of the small central square?



8. Let r be the inradius of a tetrahedron $A_1A_2A_3A_4$, and let r_1, r_2, r_3, r_4 be the inradii of triangles $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$ respectively. Prove that $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} \le \frac{2}{r^2}$, with equality if the tetrahedron is regular.

9. A linoleum company currently produces a product in which the pattern is a repetition of the figure, below.

$ABCD$ and $PQRS$ are concentric squares. The diagonals of $PQRS$ are parallel to the sides of $ABCD$. If the length of AB is one unit and if the length of PQ is 1/2 unit, compute the length of PM, where M is the midpoint of AB.



10. Let Z denote the set of all integers. Consider a function $f : Z \to Z$ with the properties :
$f(92 + x) = f(92 - x)$
$f(19.92 + x) = f(19.92 - x)(19.92 = 1748)$
$f(1992 + x) = f(1992 - x)$
for all $x \in Z$. Is it possible that all positive divisors of 92 occur as values of f?

SOLUTIONS

1. Let O_1, O_2, and O denote the centres of the circles, and let A_1, A_2, B_1 and B_2 denote the points of tangency of these circles with the semicircle, as shown in the diagram. The $O_1O_2 = R_1 + R_2$, $O_1A_1 = R_1$, and $O_2A_2 = R_2$, so
$$A_1A_2 = \sqrt{(O_1O_2)^2 - (O_1A_1 - O_2A_2)^2}$$
$$= \sqrt{(R_1 + R_2)^2 - (R_1 - R_2)^2} = 2\sqrt{R_1R_2}.$$
Also $OB_1 = 1$, $OB_2 = 1$, $O_1B_1 = R_1$ and $O_2B_2 = R_2$, so that $OO_1 = 1 - R_1$ and $OO_2 = 1 - R_2$. Therefore
$A_1A_2 = OA_1 + OA_2$
$$= \sqrt{(OO_1)^2 - (O_1A_1)^2} + \sqrt{(OO_2)^2 - (O_2A_2)^2}$$

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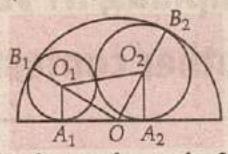
$$=\sqrt{(1-R_1)^2-R_1^2}+\sqrt{(1-R_2)^2-R_2^2}$$
$$=\sqrt{1-2R_1}+\sqrt{1-2R_2}.$$

Thus $\sqrt{1-2R_1}+\sqrt{1-2R_2}=2\sqrt{R_1R_2}$.



Squaring, then dividing each term by 2 and rearranging the terms, we get

$\sqrt{(1-2R_1)(1-2R_2)}=2R_1R_2+R_1+R_2-1$

Squaring both sides and simplify :

$$8R_1R_2=(2R_1R_2+R_1+R_2)^2 \qquad ...(1)$$

so $2\sqrt{2R_1R_2}=2R_1R_2+R_1+R_2$

Thus $R_1+R_2=2\sqrt{R_1R_2}(\sqrt{2}-\sqrt{R_1R_2})$ ...(2)

$$\le (R_1+R_2)(\sqrt{2}-\sqrt{R_1R_2}),$$

and therefore $\sqrt{R_1R_2}\le\sqrt{2}-1$

Now consider the function $f(x)=2x(\sqrt{2}-x)$.

$f(x)$ is increasing on the interval $\left(0,\frac{1}{\sqrt{2}}\right)$ since $f'(x)=2\sqrt{2}-4x>0$ for x in the interval.

Since $0<\sqrt{R_1R_2}\le\sqrt{2}-1<\frac{1}{\sqrt{2}}$ and $R_1+R_2=f(\sqrt{R_1R_2})$ from (2), R_1+R_2 attains its maximum when $\sqrt{R_1R_2}=\sqrt{2}-1$.

Hence $R_1+R_2\le 2(\sqrt{2}-1)[\sqrt{2}-(\sqrt{2}-1)]=2(\sqrt{2}-1)$.

Equality holds when $R_1=R_2$.

2. Now, set $x=AP=\frac{1}{\sqrt{3}}\sqrt{5-4s}$,

$y=BP=\frac{1}{\sqrt{3}}\sqrt{5+2s+2\sqrt{3}c}$, and

$z=CP=\frac{1}{\sqrt{3}}\sqrt{5+2s-2\sqrt{3}c}$.

By reflection and rotational geometry the distances AP, BP, CP will be a permutation of those obtained when $\pi/6\le\theta\le\pi/2$, so that $\frac{1}{\sqrt{3}}\le x\le 1$.

Also $y=\frac{1}{\sqrt{3}}\sqrt{5+4\left(\frac{1}{2}s+\frac{\sqrt{3}}{2}c\right)}=\frac{1}{\sqrt{3}}\sqrt{5+4\cos\left(\theta-\frac{\pi}{6}\right)}$

so that $\frac{\sqrt{7}}{\sqrt{3}}\le y\le\sqrt{3}$ and

$z=\frac{1}{\sqrt{3}}\sqrt{5+4\left(\frac{1}{2}s-\frac{\sqrt{3}}{2}c\right)}=\frac{1}{\sqrt{3}}\sqrt{5+4\sin\left(\theta-\frac{\pi}{3}\right)}$

so that $\frac{\sqrt{5}}{\sqrt{3}}\le z\le\frac{\sqrt{7}}{\sqrt{3}}$. Thus

$$(x+y)_{\min}=\frac{1+\sqrt{7}}{\sqrt{3}}>z_{\max}=\frac{\sqrt{7}}{\sqrt{3}}$$

and $(y+z)_{\min}=\frac{\sqrt{7}+\sqrt{5}}{\sqrt{3}}>x_{\max}=1$

and $(z+x)_{\min}=\frac{1+\sqrt{5}}{\sqrt{3}}>y_{\max}=\sqrt{3}$,

and x, y, z (*i.e.*, AP, BP, CP) can form the sides of a triangle.

From Heron's formula the area, F, of this triangle is given by

$$F^2=\frac{1}{2}(x+y+z)\frac{1}{2}(-x+y+z)\cdot\frac{1}{2}(x-y+z)\cdot\frac{1}{2}(x+y-z)$$
$$=\frac{1}{16}[(y+z)^2-x^2][x^2-(y-z)^2]$$
$$=\frac{1}{16}\left[\frac{1}{3}(5+2s+2\sqrt{3}c+5+2s-2\sqrt{3}c-5+4s)\right.$$
$$\left.+2\cdot\frac{1}{3}(5+2s+2\sqrt{3}c)^{1/2}(c+2s-2\sqrt{3}c)^{1/2}\right]$$
$$\cdot\left[\frac{1}{3}(5-4s-5-2s-2\sqrt{3}c-5-2s+2\sqrt{3}c)\right.$$
$$\left.+2\cdot\frac{1}{3}(5+2s+2\sqrt{3}c)^{1/2}(5+2s-2\sqrt{3}c)^{1/2}\right]$$
$$=\frac{1}{48}[(5+8s)+2\sqrt{(5+2s)^2-12c^2}]$$
$$[-(5+8s)+2\sqrt{(5+2s)^2-12c^2}]$$
$$=\frac{1}{48}[100+80s+16s^2-48c^2-25-80s-64s^2]$$
$$=\frac{1}{48}[75-48]=\frac{9}{16},\ \text{since } c^2+s^2=1.$$

Thus $F=\frac{3}{4}$.

3. In tetrahedron $ABCD$ we put $AB=a$, $AC=b$, $AD=c$, $BC=d$, $CD=e$ and $BD=f$, and we denote the areas of ΔABC, ΔACD, ΔABD, and ΔBCD by S_1, S_2, S_3 and S_4 respectively.

Then the surface area S of the tetrahedron is equal to the sum of S_1, S_2, S_3 and S_4, *i.e.*

$$S=S_1+S_2+S_3+S_4 \qquad ...(1)$$

Using well known geometric inequalities, we get

$$a^2+b^2+d^2\ge 4\sqrt{3}S_1 \qquad ...(2)$$
$$b^2+c^2+e^2\ge 4\sqrt{3}S_2 \qquad ...(3)$$
$$a^2+c^2+f^2\ge 4\sqrt{3}S_3 \qquad ...(4)$$
$$d^2+e^2+f^2\ge 4\sqrt{3}S_4 \qquad ...(5)$$

From (2) + (3) + (4) + (5), we get

$2(a^2+b^2+c^2+d^2+e^2+f^2)\ge 4\sqrt{3}S$, by (1)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Hence, we have

$$S \le \frac{\sqrt{3}}{6}(a^2+b^2+c^2+d^2+e^2+f^2) \text{ as required.}$$

4. We will show that the expression on the right has a minimum value of 1, which is obviously the maximum value of $\cos 12x$.

$9\tan^2 x + \cot^2 x \ge 2\sqrt{9} = 6$, with equality when

$$9\tan^2 x = \cot^2 x \qquad ...(1)$$

whence $3\tan x = \cot x$, since both have the same sign.

Thus $\tan x = \pm\frac{1}{\sqrt{3}}$ and $x = 30° + 180°n$,

or $x = 150° + 180°m$, where n, m are integers.

$$5\sin 3x \ge -5 \qquad ...(2)$$

with equality when $\sin 3x = -1$, and $x = 90° + 120°l$, where l is an integer.

Combining (1) and (2), we have

$$5\sin 3x + 9\tan^2 x + \cot^2 x \ge 1$$

with equality when $x = 210° + 360°n$

or $x = 330 + 360°n$.

Also note that $1 \ge \cos 12x$ with equality when $x = 30°n$. Since this is consistent with the values of x that minimize the right hand side of the expression we have for our solution that

$x = 210° + 360°n$

$x = 330° + 360°n$, for an integer n.

5. Without loss of generality suppose that $\min(x, y) \ge z > 0$.

Let $s = \frac{x+y}{2z}$ and $t = \frac{xy}{z^2}$

It is sufficient to show that

$$(2s+t)\left(\frac{1}{4s^2} + \frac{4s^2-2t+2+4s}{(1+2s+t)^2}\right) \ge \frac{9}{4} \qquad ...(1)$$

whenever $1 \le t \le s^2$.

$$2s+t = \frac{x+y}{z} + \frac{xy}{z^2} = \frac{xy+yz+zx}{z^2} \text{ and}$$

$$\frac{4s^2-2t+2+4s}{(1+2s+t)^2} = \frac{z^2[(x+y)^2-2xy+2z^2+2z(x+y)]}{(z^2+xy+yz+zx)^2}$$

$$= \frac{z^2[(y+z)^2+(z+x)^2]}{[(y+z)(z+x)]^2} = z^2\left(\frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right)$$

and thus (1) is just the original inequality; furthermore

$$(x-y)^2 \ge 0 \Rightarrow 4xy \le (x+y)^2 \Rightarrow t = \frac{xy}{z^2} \le \frac{(x+y)^2}{4z^2} = s^2,$$

and $\min(x, y, z) = z$ implies that $t \ge 1$

For fixed s and $1 \le t \le s^2$ let

$$f(t) = t^3 - (17s^2 - 6s - 2)t^2 + (16s^4 - 36s^3 + 2s^2 + 8s + 1)t + 32s^5 - 4s^4 - 12s^3 - s^2 + 2s$$

It is easy to check that (1) is equivalent to $f(t) \ge 0$,

$$(2s+t)[(1+2s+t)^2 + 4s^2(4s^2-2t+2+4s)] \ge 9s^2(1+2s+t)^2,$$

which simplifies to $f(t) \ge 0$.

Now $\frac{d^2f}{dt^2} = 6t - 2(17s^2-6s-2) \le 6s^2 - 2(17s^2-6s-2)$

$= -4(7s^2 - 3s - 1) < 0$

for $s \ge 1$ [and thus f is concave down for $s \ge 1$].

Also $f(1) = 32s^5 + 12s^4 - 48s^3 - 16s^2 + 16s + 4$

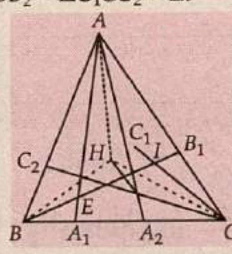
$= 4(s-1)(8s^4 + 11s^3 - s^2 - 5s - 1) \ge 0$

and $f(s^2) = 2s^5 - 4s^3 + 2s = 2s(s^2-1)^2 \ge 0$, since $s \ge 1$.

Hence for any $s \ge 1$, $f(t) \ge 0$ for all $1 \le t \le s^2$, and this completes the solution of the problem.

6. Let AA_1, BB_1 meet at the point E; AA_1, CC_2 meet at the point F; and BB_1, CC_1 meet at the point I. Also

$$\angle A_1AA_2 = \angle B_1BB_2 = \angle C_1CC_2 = 2x \qquad ...(1)$$



The bisectors of the angles at A_1, B_1 and C_1 in triangles ΔA_1AA_2, ΔB_1BB_2 and ΔC_1CC_2 respectively are perpendicular to their respective bases. Hence they are the altitudes of ΔABC. Let H be the orthocentre of ΔABC.

Since $\angle A_1AH = \angle B_1BH = x$ and $\angle A_1AH = \angle C_1CH = x$ each one of the quadrilaterals $AHEB$, $AHDC$ is inscribable in a circle.

These two circles have a common chord, the segment AH and since $\angle ABH = \angle ACH = 90° - \angle BAC$, then the circles have equal radii.

Thus, since the inscribed angles $\angle EAH$, $\angle DAH$ are equal, the corresponding chords HE and HD are equal.

Therefore $HE = HD$. Similarly, we prove that $HD = HI$, and so on for all six vertices of these two triangles of the problem.

Thus, all six vertices lie at the same distance from the point H, and the points are concyclic.

7. The easiest way to do this problem is to draw more lines to form a tilted "cross" : Now the eight little triangles in this picture are all the same size and shape, so they have the same area. Four of these triangles are inside the square and four are inside the cross, and this is the only difference between the square and the

cross. Therefore, the square and the cross have the same area, which is 1.

Since the small central square is obviously one-fifth of the cross, its area must be 1/5.

There are longer ways to do this problem using similar triangles.

8. Let S_1, S_2, S_3, S_4 be the areas of the faces $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$; let α, β, γ be the dihedral angles at the edges A_2A_3, A_2A_4, A_3A_4; and let h_1 be the altitude from the vertex A_1 of the tetrahedron and $h'_1 = A_1E$ be the altitude of the face $A_1A_2A_3$. Then

$$h_1 = h'_1 \sin\alpha = \frac{2S_4}{A_2A_3}\sqrt{(1+\cos\alpha)(1-\cos\alpha)}$$

$$= \frac{2}{A_2A_3}\sqrt{(S_4+S_4\cos\alpha)(S_4-S_4\cos\alpha)}, \quad ...(1)$$

and analogously

$$h_1 = \frac{2}{A_2A_4}\sqrt{(S_3+S_3\cos\beta)(S_3-S_3\cos\beta)}, \quad ...(2)$$

$$h_1 = \frac{2}{A_3A_4}\sqrt{(S_2+S_2\cos\gamma)(S_2-S_2\cos\gamma)}. \quad ...(3)$$

From (1), (2) and (3) we get

$$h_1 = \frac{2}{A_2A_3+A_2A_4+A_3A_4}\cdot Q \quad ...(4)$$

where

$$Q = \sqrt{(S_4+S_4\cos\alpha)(S_4-S_4\cos\alpha)} + \sqrt{(S_3+S_3\cos\beta)(S_3-S_3\cos\beta)} + \sqrt{(S_2+S_2\cos\gamma)(S_2-S_2\cos\gamma)}$$

According to Cauchy's inequality, we have

$$Q \le \{(S_4 + S_4\cos\alpha) + (S_3 + S_3\cos\beta) + (S_2 + S_2\cos\gamma)\}^{1/2} \cdot\{(S_4 - S_4\cos\alpha) + (S_3 - S_3\cos\beta) + (S_2 - S_2\cos\gamma)\}^{1/2}$$

$$= \{S_4 + S_3 + S_2 + (S_4\cos\alpha + S_3\cos\beta + S_2\cos\gamma)\}^{1/2} \cdot\{(S_4 - S_3 + S_2 - (S_4\cos\alpha + S_3\cos\beta + S_2\cos\gamma)\}^{1/2}$$

$$= (S_4 + S_3 + S_2 + S_1)^{1/2}(S_4 + S_3 + S_2 - S_1)^{1/2}$$

$$= S^{1/2}(S - 2S_1)^{1/2} \quad ...(5)$$

where $S = S_1 + S_2 + S_3 + S_4$. From (4) and (5), we get

$$h_1 \le \frac{2\sqrt{S(S-2S_1)}}{A_2A_3+A_2A_4+A_3A_4}.$$

Therefore [since $rS = h_1S_1$ = three times the volume of the tetrahedron] we obtain

$$r = \frac{h_1S_1}{S} \le \frac{2S_1}{A_2A_3+A_2A_4+A_3A_4}\cdot\frac{\sqrt{S(S-2S_1)}}{S}$$

$$= r_1\cdot\sqrt{\frac{S-2S_1}{S}}.$$

In the same manner, we have

$$r \le r_i\sqrt{\frac{S-2S_i}{S}},\ i = 1, 2, 3, 4,$$

hence $\displaystyle\sum_{i=1}^{4}\frac{1}{r_i^2} \le \frac{1}{r^2}\sum_{i=1}^{4}\frac{S-2S_i}{S} = \frac{2}{r^2}.$

9. Let O be the centre of the two squares. Then

$OM = \frac{1}{2}$ and $OP = \frac{1}{2}RP = \frac{1}{2}\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{4}.$

So $PM = OM - OP = \frac{1}{2} - \frac{\sqrt{2}}{4} = \frac{2-\sqrt{2}}{4}.$

10. The answer is No

Now $f: Z \to Z$ satisfies

(1) $f(92 + x) = f(92 - x)$

(2) $f(1748 + x) = f(1748 - x)$, and

(3) $f(1992 + x) = f(1992 - x)$.

Then, we have

$f(488 + x) = f(244 + 244 + x) = f(1992 - 1748 + 244 + x)$

$= f(1992 + 1748 - 244 - x)$, by (3)

$= f(1748 + 1992 - 244 - x)$

$= f(1748 - 1992 + 244 + x)$, by (2)

$= f(x)$ ...(4)

Then, we have

$f(40 + x) = f(1992 - 4.448 + x)$

$= f(1992 + x)$, by repeated application of (4),

$= f(1992 - x)$, by (2)

$= f(1992 - 4.488 - x)$, by repeated application of (4)

$= f(40 - x)$ ...(5)

So, $f(104 + x) = f(52 + 52 + x) = f(92 - 40 + 52 + x)$

$= f(92 + 40 - 52 - x)$, by (1)

$= f(40 + 92 - 52 - x)$

$= f(40 - 92 + 52 + x)$, by (5)

$= f(x)$ ...(6)

Now $8 = 3\cdot488 - 14\cdot104$. Therefore

$f(8 + x) = f(3.488 - 14.104 + x)$

$= f(-14.104 + x)$, by repeated application of (4)

$= f(x)$, by repeated application of (6).

This shows that f is periodic and all the possible values of f are in the list $f(0), f(1), f(2), ..., f(7)$. Finally

$f(4 + x) = f(92 - 8.11 + x) = f(92 + x)$, by periodicity

$= f(92 - x)$ by (1)

$= f(92 - 8\cdot11 + x) = f(4 - x)$.

In particular $f(7) = f(1)$, $f(6) = f(2)$, $f(5) = f(3)$. Hence all the possible values of f are $f(0)$, $f(1)$, $f(2)$, $f(3)$ and $f(4)$. In particular, f assumes no more than 5 function values. However, 92 has 6 positive divisors, namely 1, 7, 4, 23, 46 and 92.

Prized IIT seats lying vacant

In a revelation that may dishearten those who could not make it into the IITs in recent years, several high-in-demand seats at these premier institutes have been found lying vacant session after session.

In the Joint Entrance Exam of 2006 institutes and branches that were more in demand were artificially shown as filled though data later procured under the Right to Information Act showed that these seats were actually not occupied.

Again in JEE 2008, hundreds of seats, even in General Category, were vacant across IITs despite the fierce competition for entry. Reason -- IITs, until the last year, were conducting a single round of counselling even if it meant allowing prized seats in plum branches to go unclaimed despite demand.

Even today, five to 20 percent seats in General Category are estimated to be vacant in IITs, if the recent data supplied by the Ministry of Human Resource Development under the RTI Act is anything to go by. Data also hints at instances in the past, especially until 2005, where IIT seats at certain locations were being filled through faculty wards, who never qualified for JEE.

It was for the first time in 2009 that the issue of vacancy of seats in the much-desired IITs came to light when it was found that 50 to 100 vacancies were allowed to exist every year despite there being a mad rush for entry to these technical institutes. Documents with a newspaper show that in IIT Kanpur, one seat in computer engineering was vacant (top 60 rankers get this branch); textile engineering and biotechnology seats were vacant in IIT Delhi; IIT Kharagpur had the highest number of vacant seats in five-year MSc courses, including in preferred streams like physics and chemistry (these seats go to candidates ranked between 500 and 3,000). IIT Roorkee also have vacancies.

Five to 20 percent General Category seats are vacant in IITs, reveals an RTI query

It was only after the information on vacancies in coveted branches became public that the IITs, following Supreme Court's intervention last year, conducted a second round of counselling. Some seats continue to go waste in the absence of a fully online counselling and a wait-listing provision.

In normal course, a candidate's option should be recorded during online admission counselling and he should be instantly allotted a seat on the basis of his preference and the availability, if there is one. In case the candidate wishes to be on the upward sliding (in the event of vacancies in future), he should be allowed to avail the option of sliding his admission preferences, with a possible constraint that the later stages of sliding may be effective in the respective institute in which he has already started studying. Those who don't get a seat should be put on the waiting list, and multiple rounds of admission conducted for them.

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers comments and suggestions regarding the problems and solutions offered are always welcome.

1. The area bounded by the curve $f(x) = x + \sin x$ and its inverse function between the ordinates $x = 0$ to $x = 2\pi$ is
(a) 4π sq. units (b) 8π sq. units
(c) 4 sq. units (d) 8 sq. units

2. If the solution of $\dfrac{dy}{dx}=\dfrac{ax+3}{2y+f}$ represents a circle, then the value of 'a' is
(a) 2 (b) –2 (c) 3 (d) –4

3. Solution of $\left(\dfrac{x+y-1}{x+y-2}\right)\dfrac{dy}{dx}=\left(\dfrac{x+y+1}{x+y+2}\right)$, given that $y = 1$ when $x = 1$, is
(a) $\log\left|\dfrac{(x-y)^2-2}{2}\right|=2(x+y)$
(b) $\log\left|\dfrac{(x-y)^2+2}{2}\right|=2(x-y)$
(c) $\log\left|\dfrac{(x+y)^2+2}{2}\right|=2(x-y)$
(d) None of these

4. Area bounded by $|x-1| < 2$ and $x^2 - y^2 = 1$ is
(a) $6\sqrt{2}+\frac{1}{2}\log|3+2\sqrt{2}|$ (b) $6\sqrt{2}+\frac{1}{2}\log|3-2\sqrt{2}|$
(c) $6\sqrt{2}-\log|3+2\sqrt{2}|$ (d) None of these

5. Area bounded by $y=\dfrac{(x-1)(x+1)}{(x-2)}$, x-axis and ordinates $x = 0$ and $x = 3/2$ is
(a) 4/5 sq. units (b) 7/8 sq. units
(c) 1 sq. unit (d) none of these

6. Evaluate $\displaystyle\int e^x \frac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}\,dx$

7. Evaluate $\displaystyle\int \frac{(x\cos x+1)}{\sqrt{2x^3e^{\sin x}+x^2}}\,dx$

8. Evaluate $\displaystyle\int \frac{(2x-1)\,dx}{x^4-2x^3+x+1}$

9. Evaluate $\displaystyle\int_0^1 \frac{x^{\cos a}-1}{\ln x}dx$, '$a$' being a real number other than an odd multiple of π.

10. Evaluate $\displaystyle\int \frac{\cos 6x+\cos 9x}{1-2\cos 5x}dx$

SOLUTIONS

1. **(d)** : Required area = $4A$

$$A=\int_0^{\pi}(x+\sin x)dx-\int_0^{\pi}x\,dx$$
$$=\frac{\pi^2}{2}-\cos\pi+\cos 0-\frac{\pi^2}{2}$$
= 2 sq. units.

2. **(b)** : $\dfrac{dy}{dx}=\dfrac{ax+3}{2y+f} \Rightarrow (2y+f)dy=(ax+3)dx$

$$\frac{2y^2}{2}+fy=\frac{ax^2}{2}+3x+c$$

$\therefore$ for curve to be circle, $a = -2$ where $f^2 + 9 + 4c > 0$.

3. **(d)** : $\left(\dfrac{x+y-1}{x+y-2}\right)\dfrac{dy}{dx}=\left(\dfrac{x+y+1}{x+y+2}\right)$

Put $x + y = t$

$$1+\frac{dy}{dx}=\frac{dt}{dx}$$
$$\frac{dt}{dx}-1=\left(\frac{t+1}{t+2}\right)\left(\frac{t-2}{t-1}\right)$$
$$\frac{dt}{dx}=\left(\frac{t^2-t-2}{t^2+t-2}\right)+1$$

on solving we get

$$\Rightarrow\ 2(y-x)+\log\left(\frac{(x+y)^2-2}{2}\right)=0$$

4. **(c)** : Graph of required curve is $-1 \le x \le 3$ and $x^2 - y^2 = 1$

$\therefore$ Required area is shaded area

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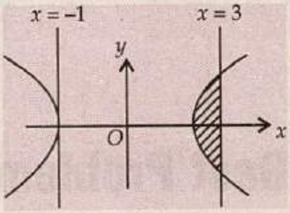
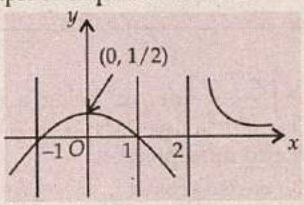


i.e., $2\int_1^3 \sqrt{x^2-1}\,dx$ on solving we will get,

Area $= 6\sqrt{2} - \log|3+2\sqrt{2}|$ sq. units

5. **(b)** : Graph of required curve is



$\therefore$ Required area is

$$\int_0^1 \frac{(x-1)(x+1)}{(x-2)}dx - \int_1^{3/2} \frac{(x-1)(x+1)}{(x-2)}dx$$

Hence area = 7/8 sq. units.

6. We know that $\int e^x(f(x)+f'(x))dx = e^x f(x)+c$

Then

$$\int e^x \frac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}dx$$

$$= \int e^x \left\{\frac{(1-x^{2n})}{(1-x^n)\sqrt{1-x^{2n}}} + \frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right\}dx$$

$$= \int e^x \left\{\frac{\sqrt{1-x^{2n}}}{1-x^n} + \frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right\}dx$$

$$= e^x . \frac{\sqrt{1-x^{2n}}}{1-x^n} + c$$

7. $I = \int \frac{(x\cos x+1)}{\sqrt{2x^3e^{\sin x}+x^2}}dx = \int \frac{(x\cos x+1)}{x\sqrt{2xe^{\sin x}+1}}dx$

Let $2xe^{\sin x} + 1 = t^2$

$\Rightarrow$ $(2xe^{\sin x}.\cos x + 2e^{\sin x})dx = 2tdt$

$\Rightarrow$ $e^{\sin x}(x\cos x + 1)dx = tdt$

$$\therefore \quad I = \int \frac{tdt}{xe^{\sin x}.t} = \int \frac{dt}{xe^{\sin x}} = \int \frac{2dt}{t^2-1}$$

$$= \int \frac{(1+t)+(1-t)}{(t^2-1)}dt = \int \frac{dt}{t-1} - \int \frac{dt}{t+1}$$

$$= \log(t-1) - \log(t+1) + c = \log\left(\frac{t-1}{t+1}\right) + c$$

where $t^2 = 2xe^{\sin x} + 1$

8. Since $\int (2x-1)dx = x^2 - x + c$

So, let $x^2 - x = t \Rightarrow (2x-1)dx = dt$

Now,

$$I = \int \frac{(2x-1)dx}{(x^4-2x^3+x+1)} = \int \frac{(2x-1)dx}{(x^2-x)^2-(x^2-x)+1}$$

$$= \int \frac{dt}{t^2-t+1} = \int \frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)+c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{2(x^2-x)-1}{\sqrt{3}}\right]+c$$

9. $\int_0^1 \frac{x^{\cos a}-1}{\ln x}dx$ is a function of 'a',

so let us call it $I(a)$.

Then $I(a) = \int_0^1 \frac{x^{\cos a}-1}{\ln x}dx$

Differentiating w.r. to 'a', we have

$$\therefore \quad I'(a) = \int_0^1 \frac{x^{\cos a}.\ln x\,(-\sin a)}{\ln x}dx \quad \left(\because \frac{d}{dx}(b^x) = b^x \ln b\right)$$

$$\therefore \quad I'(a) = -\int_0^1 x^{\cos a}.\sin a.dx = -\sin a\int_0^1 x^{\cos a}dx$$

$$= -\sin a.\left[\frac{x^{\cos a+1}}{\cos a+1}\right]_0^1 = -\sin a\left[\frac{1}{\cos a+1}\right]$$

$$I'(a) = \frac{-\sin a}{\cos a+1}$$

Integrating both sides w.r.t. 'a', we have

$$\therefore \quad I(a) = \int \frac{-\sin a.da}{\cos a+1} = \int \frac{d(\cos a+1)}{(\cos a+1)}$$

or $I(a) = \ln|1 + \cos a| + c$

as $a \to \frac{\pi}{2}$, $I(a) \to 0 \Rightarrow c = 0$

Thus $I(a) = \ln(1 + \cos a)$, where $a \neq$ odd multiple of π.

10. Let $I = \int \frac{\cos 6x + \cos 9x}{1-2\left(2\cos^2\frac{5x}{2}-1\right)}dx$

$$= \int \frac{2\cos\frac{15x}{2}.\cos\frac{3x}{2}}{3-4\cos^2\frac{5x}{2}}dx$$

$$= \int \frac{2\cos\left(\frac{15x}{2}\right).\cos\left(\frac{3x}{2}\right).\cos\left(\frac{5x}{2}\right)}{3\cos\left(\frac{5x}{2}\right)-4\cos^3\left(\frac{5x}{2}\right)}dx$$

$$= -\int(\cos 4x + \cos x)dx = -\frac{\sin 4x}{4} - \sin x + c.$$

(c) $\dfrac{1}{1+x}$ (d) none of these

31. This question contains Statement-1 and Statement-2 of the four choices given after the statements, choose the one that best describes the two statements.

Statement-1 : The equation $ax^2+bx+c=0$ cannot have rational roots if a, b, c are odd integers

Statement-2 : If an odd number does not leave remainder 1 when divided by 8, then it cannot be a perfect square.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

(c) Statement-1 is true, Statement-2 is false.

(d) Statement-1 is false, Statement-2 is true.

32. This question contains Statement-1 and Statement-2. Of the four choices given after the statements, choose the one the best describes the two statements.

Statement-1 : The equation of a line through the point (5, 7) and parallel to the line $y = \underset{t\to 0}{\text{Lt}}\dfrac{e^{xt}-1}{t}+7$ is $x-y+2=0$ by expansion or by L'Hospital rule.

Statement-2 : $\lim\limits_{t\to 0}\dfrac{e^{xt}-1}{t}=x,$ by expansion or L'Hospital's Rule.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true.

Directions : Questions No. 33 to 35 are based on the following paragraph

$P: y^2 = 8x,\ E: \dfrac{x^2}{4}+\dfrac{y^2}{15}=1$

33. Equation of a tangent common to both the parabola P and the ellipse E is

(a) $x-2y+8=0$ (b) $2x-y+8=0$

(c) $x+2y-8=0$ (d) $2x-y-8=0$

34. Equation of the normal at the point of contact of the common tangent, which makes an acute angle with the positive direction of x-axis, to the parabola P is

(a) $2x+y=24$ (b) $2x+y+24=0$

(c) $2x+y=48$ (d) $2x+y+48=0$

35. Point of contact of a common tangent to P and E on the ellipse is

(a) $\left(\dfrac{1}{2},\dfrac{15}{4}\right)$ (b) $\left(-\dfrac{1}{2},\dfrac{15}{4}\right)$

(c) $\left(\dfrac{1}{2},-\dfrac{15}{2}\right)$ (d) $\left(-\dfrac{1}{2},-\dfrac{15}{2}\right)$

ANSWER KEY

1. (a)	2. (a)	3. (d)	4. (c)	5. (a)	6. (d)	7. (b)
8. (c)	9. (c)	10. (a)	11. (a)	12. (b)	13. (a)	14. (d)
15. (b)	16. (b)	17. (a)	18. (c)	19. (a)	20. (c)	21. (d)
22. (c)	23. (b)	24. (a)	25. (c)	26. (c)	27. (c)	28. (a)
29. (c)	30. (b)	31. (a)	32. (a)	33. (a)	34. (a)	35. (b)

For detailed solution to sample paper visit www.vidyalankar.org

MOCK TEST PAPER

ISI 2010

Indian Statistical Institute

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PART - A

Multiple Choice Question type

1. Let $A(2, 2)$ and $B(7,7)$ be points in the plane. Define R as the region in the first quadrant consisting of those points C such that ΔABC is an acute triangle. What is the closest integer to the area of the region R?

(a) 25 (b) 39 (c) 51 (d) 60

2. Let $P(x) = (x - 1)(x - 2)(x - 3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P(Q(x)) = P(x).\ R(x)$?

(a) 19 (b) 22 (c) 24 (d) 27

3. For all positive integers n less than 2002, let

$$a_n = \begin{cases} 11, \text{ if } n \text{ is divisible by 13 and 14} \\ 13, \text{ if } n \text{ is divisible by 14 and 11} \\ 14, \text{ if } n \text{ is divisible by 11 and 13} \\ 0, \text{ otherwise} \end{cases}$$

The value of $\sum_{n=1}^{2001} a_n$ is

(a) 448 (b) 486 (c) 1560 (c) 2001

4. Positive integers a, b, and c are chosen so that $a < b < c$, and the system of equations $2x + y = 2003$ and $y = |x - a| + |x - b| + |x - c|$ has exactly one solution. What is the minimum value of c?

(a) 668 (b) 669 (c) 1002 (d) 2003

5. Mrs. Sharma gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Sharma noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last score Mrs. Sharma entered?

(a) 71 (b) 76 (c) 80 (d) 82

6. The roots x_1, x_2, x_3 of the equation $x^3 + ax + a = 0$, where 'a' is a non - zero real satisfy $\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \frac{x_3^2}{x_1} = -8$. The greatest root is

(a) $1+\sqrt{2}$ (b) $1+\sqrt{5}$

(c) $2\sqrt{5}-1$ (d) $2\sqrt{2}-1$

7. Given that x, y, z are positive reals satisfying $xyz = 32$, the minimum value of $x^2 + 4xy + 4y^2 + 2z^2$ is

(a) 92 (b) 96 (c) 48 (d) 44

8. The number of natural numbers 'n' such that $\frac{(n+1)^2}{n+7}$ is an integer is

(a) 4 (b) 5 (c) 3 (d) 6

9. In triangle AEC, $\angle ABC = 45°$. Point D is on BC so that $2BD = CD$ and $\angle DAB = 15°$. $\angle ACB$ is

(a) 54° (b) 60° (c) 72° (d) 75°

10. The number of positive integers 'n' such that $n + 9$, $16n + 9$, $27n + 9$ are all perfect squares is

(a) 1 (b) 2 (c) 3 (d) 4

11. The maximum number of right angles which can occur among the interior angles of a convex polygon is

(a) 1 (b) 2 (c) 3 (d) 4

12. The graphs of $y = -|x - a| + b$ and $y = |x-c|+d$ intersect at points (2, 5) and (8, 3). The value of $a+c$ is

(a) 7 (b) 8 (c) 10 (d) 13

13. The length of segment joining the midline of a trapezium equals 4 cm and the base angles are 40° and 50°. If the length of segment joining their mid-points equals 1cm, the length of base are

(a) 3, 5 (b) 2, 8 (c) 2, 6 (d) 4, 6

14. A convex quadrilateral $ABCD$ with area 2002 contains a point P in its interior such that $PA = 24$, $PB = 32$, $PC = 28$, and $PD = 45$. The perimeter of $ABCD$ is

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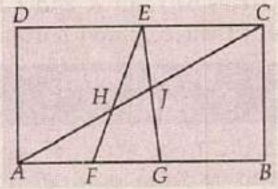
(a) $4\sqrt{2002}$ (b) $2(48+\sqrt{2002})$
(c) $2\sqrt{8633}$ (d) $4(36+\sqrt{113})$

15. How many maximum points of intersection can we get by arranging 8 straight lines and 4 circles in a plane ?
(a) 100 (b) 104 (c) 64 (d) 92

16. In rectangle $ABCD$, points F and G lie on AB so that $AF = FG = GB$ and E is the midpoint of DC. Also, AC intersects EF at H and EG at J. The area of rectangle $ABCD$ is 70. The area of the triangle EHJ, is



(a) $\frac{5}{2}$ (b) $\frac{35}{12}$ (c) 3 (d) $\frac{7}{2}$

17. The number of real ordered pairs (x, y) satisfying $x^2 + x = y^4 + y^3 + y^2 + y$ is
(a) 3 (b) 4 (c) 5 (d) 6

18. There are 500 stundets in a school. Two thirds of the students who do no wear spectacles do not bring lunch, three - quarters of the students who do not bring lunch do not wear spectacles. Altogether 60 spectacled students bring lunch too. The number of students who do not wear spectacles and do not bring lunch is
(a) 180 (b) 140 (c) 240 (d) 320

19. Given two sides a and b of a triangle, and it is known that the medians drawn to the given sides intersect at a right angle. What are the conditions for the triangle to exist?

(a) $\frac{1}{2}<\frac{a}{b}<2$ (b) $\frac{1}{4}<\frac{a}{b}<4$

(c) $\frac{1}{2\sqrt{2}}<\frac{a}{b}<2\sqrt{2}$ (d) $\frac{1}{4\sqrt{2}}<\frac{a}{b}<4\sqrt{2}$

20. A plane contains points A and B with $AB = 1$. Let S be the union of all disk of radius 1 in the plane that cover AB. What is the area of S?

(a) $2\pi+\sqrt{3}$ (b) $\frac{8\pi}{3}$

(c) $3\pi-\frac{\sqrt{3}}{2}$ (d) $\frac{10\pi}{3}-\sqrt{3}$

21. The number of real ordered triplets (x, y, z) satisfying the equations

$$\frac{4x^2}{1+4x^2}=y, \frac{4y^2}{1+4y^2}=z, \frac{4z^2}{1+4z^2}=x \text{ is}$$

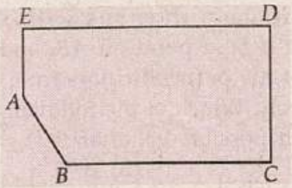
(a) 1 (b) 2 (c) 3 (d) 4

22. Given the sides b and c of a triangle. The third side is equal to the altitude drawn to it. Under what condition connecting b and c does the triangle exist?

(a) $\frac{2}{\sqrt{5}-1}\le\frac{b}{c}\le\frac{\sqrt{5}-1}{2}$ (b) $\frac{1}{2}\le\frac{b}{c}\le 2$

(c) $\frac{2}{1+\sqrt{5}}\le\frac{b}{c}\le\frac{1+\sqrt{5}}{2}$ (d) $\frac{1}{4}\le\frac{b}{c}\le 4$

23. A point P is selected a random from the interior of the pentagon with vertices $A = (0, 2)$, $B = (4, 0)$, $C = (2\pi + 1, 0)$, $D = (2\pi + 1, 0)$, and $E = (0, 4)$. What is the probability that $\angle APB$ is obtuse?



(a) $\frac{1}{5}$ (b) $\frac{1}{4}$ (c) $\frac{5}{16}$ (d) $\frac{3}{8}$

24. A square is constructed with two of its vertices on the bounding radii and two remaining vertices on the arc of a sector of a disc of radius 10 units, the sectorial angle bieng 60°. The area of the square is
(a) $100(2+\sqrt{3})$ (b) $100(2-\sqrt{3})$
(c) $50(3-\sqrt{2})$ (d) $50(3+\sqrt{2})$

25. Inside an equilateral triangle ABC an arbitrary point P is taken from which the perpendiculars PD, PE and PF are dropped onto BC, CA and AB respectively. The value of $\frac{PD+PE+PF}{BD+CE+AF}$ is

(a) $3\sqrt{3}$ (b) $\sqrt{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{3\sqrt{3}}$

26. Three equal circles are externally tangent to a circles of radius 1 and pairwise tangent to one another. The areas of the three curvilinear triangles formed by these circles is

(a) $12+7\sqrt{3}-\left(\frac{26}{3}+\sqrt{3}\right)\pi$

(b) $12+7\sqrt{3}-\left(\frac{23}{6}+2\sqrt{3}\right)\pi$

(c) $6+7\sqrt{3}-\left(\frac{11}{6}+\sqrt{3}\right)\pi$

(d) $6+7\sqrt{3}-\left(\frac{11}{6}+2\sqrt{3}\right)\pi$

27. A set S of points in the xy - plane is symmetric about the origin, both coordinate axes, and the line $y = x$. If (2, 3) is in S, what is the smallest number of points in S?

(a) 2 (b) 4 (c) 8 (d) 16

28. Let ΔXOY be a right - angle triangle with $\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, XY is

(a) 24 (b) 26 (c) 28 (d) 30

29. ABC is an isosceles triangle with $\angle B = \angle C = 78°$, D and E are points on AB, AC respectively, such that $\angle BCD = 24°$ and $\angle CBE = 51°$. $\angle BED$ is

(a) 48° (b) 36° (c) 12° (d) 40°

30. Let $ABCD$ be a rhombus with $AC = 16$ and $BD = 30$. Let N be a point on AB, and let P and Q be the feet of the perpendiculars from N to AC and BD respectively. Which of the following is closest to the minimum possible value of PQ?

(a) 6.5 (b) 6.75
(c) 7 (d) 7.25

PART - B

1. Find all values of a, b for which the system of equations $xyz + z = a$, $xyz^2 + z = b$, $x^2 + y^2 + z^2 = 4$ has only one real solution.

2. Find all pairs (x, y) of real numbers that satisfy the equation $\left(\sin^2 x+\frac{1}{\sin^2 x}\right)^2+\left(\cos^2 x+\frac{1}{\cos^2 x}\right)^2 =12+\frac{1}{2}\sin y$

3. Let the roots of the cubic equation $x^3 + ax^2 + bx + c = 0$ be real. Show that the difference between the greatest and the least of them is not less than $\sqrt{a^2-3b}$ nor greater than $2\sqrt{(a^2-3b)}$.

4. If in a triangle two bisectors are equal, prove that the triangle is isosceles.

5. In base R_1, two fractions F_1 and F_2 are expanded as 0.3737..... and 0.7373...... respectively; in another base R_2, they are 0.2525.......and 0.5252........ Find the sum of R_1 and R_2 written in base 10.

6. Find the fixed points of the function $f : R \to R$ which satisfies $f(f(x)) - f(x) = ax + b$ for all $x \in R$, where $a \neq 0$.

7. The real numbers a, b, c, x, y, z satisfy $a \geq b \geq c > 0$ and $x \geq y \geq z > 0$, prove that $\frac{a^2x^2}{(by+cz)(bz+cy)}+\frac{b^2y^2}{(cz+ax)(cx+az)}+\frac{c^2z^2}{(ax+by)(ay+bx)}$ is at least 3/4.

8. Find a relation among the sides of a triangle ABC if the median AM, the altitude BH and the angle bisector CD are concurrent at a point.

9. Find real values of x satisfying

$$\left[\frac{3x-1}{4}\right]+\left[\frac{3x+1}{4}\right]+\left[\frac{3x-1}{2}\right]=\frac{6x+3}{5}$$

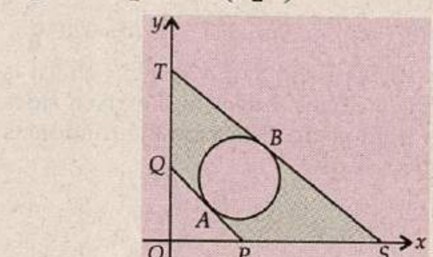
Where $[x]$ denotes the greatest integer not exceeding x.

10. The distance between the centres of two intersecting circles of radii R and r is equal to d. Find the area of their common portion.

SOLUTIONS
PART - A

1. **(c)** : For ΔABC to be acute, all angles must be acute. For $\angle A$ to be acute, point C must lies above the line passing through A and perpendicular to AB. The segment of that line in the first quadrant lies between $P(4, 0)$ and $Q(0, 4)$. For $\angle B$ to be acute, point C must lie below the line through B and perpendicular to AB. The segment of that line in the first quadrant lies between $S(14, 0)$ and $T(0, 14)$. For $\angle C$ to be acute, point C must lie outside the circle U that has AB as a diameter. Let O denote the origin. Region R, shaded below, has area equal to Area(ΔOST) – Area(ΔOPQ) – Area(Circle U)

$$=\frac{1}{2}.\ 14^2-\frac{1}{2}.\ 4^2-\pi\left(\frac{\sqrt{50}}{2}\right)^2$$



$$=90-\frac{25}{2}\pi=51$$

2. **(b)** : The polynomial $P(x)$ has degree 6, so $Q(x)$ must have degree 2. Therefore Q is uniquely determined by the ordered triplet $(Q(1), Q(2), Q(3))$. Then $x = 1, 2,$ or 3. We have $0 = P(x)$. $R(x) = P(Q(x))$. It follows that $(Q(1), Q(2), Q(3))$ is one of the 27 ordered triplet (i, j, k), where i, j, and k can be chosen from the set {1, 2, 3}. However, the choices (1,1, 1), (2, 2, 2), (3, 3, 3) and (1, 2, 3) lead to polynomials $Q(x)$ defined by $Q(x) = 1, 2, 3, x$ and $4 - x$, respectively, all of which have degree less than 2. The other 22 choices for $(Q(1), Q(2), Q(3))$ yield non-collinear points, so

in each case $Q(x)$ is a quadratic polynomial.

3. **(a)** : Since 2002 = 11, 13, 14, we have

$$a_n=\begin{cases}11, \text{if } n=13.14.i, \text{ where } i=1,2,\ldots\ldots,10\\ 13, \text{if } n=14.11.j, \text{ where } j=1,2\ldots\ldots,12\\ 14, \text{if } n=11.13.k, \text{where } k=1,2,\ldots\ldots,13\\ 0, \text{ otherwise}\end{cases}$$

Hence $\sum_{n=1}^{2001} a_n = 11.10+13.12+14.13=448$.

4. **(c)** : Since the system has exactly one solution, the graphs of the two equations must intersect at exactly one point. If $x < a$, the equation $y = |x-a| + |x-b| + |x-c|$ is equivalent to $y = -3x + (a + b + c)$. By similar calculations we obtain

$$y=\begin{cases}-3c+(a+b+c), \text{if } x<0\\ -x+(-a+b+c), \text{if } a\le x<b\\ x+(-a-b+c), \text{if } b\le x<c\\ 3x+(-a-b-c), \text{if } c\le x\end{cases}$$

Thus the graph consists of four lines with slopes $-3, -1, 1$ and 3, and it has corners at $(a, b + c - 2a)$, $(b, c - a)$, and $(c, 2c - a - b)$

On the other hand, the graph of $2x + y = 2003$ is a line whose slope is -2. If the graphs intersect at exactly one point, that point must be $(a, b + c - 2a)$. Therefore $2003 = 2a + (b + c - 2a) = b + c$

Since $b < c$, the minimum value of c is 1002.

5. **(c)** : Note that the integer average condition means that the sum of the score of the first n students is a multiple of n. The scores of the first two students must be both even or both odd and the sum of the scores of the first three students must be divisible by 3. The remainders when 71, 76, 80, 82, and 91 are divided by 3 are 2, 1, 2, and 1, respectively. Thus the only sum of the three scores divisible by 3 is $76 + 82 + 91 = 249$, so the first two scores entered are 76 and 82 (in some order), and the third scores is 91. Since 249 is 1 larger than a multiple of 4, the fourth score must be 3 larger than a multiple of 4, and the only possibility is 71, leaving 80 as the scores of the fifth student.

6. **(b)** : x_1, x_2, x_3 are the roots of

$x^3 + 0x^2 + ax + a = 0$

$\Rightarrow\ x_1 + x_2 + x_3 = 0$ ...(1)

$x_1x_2 + x_2x_3 + x_3x_1 = a$ ...(2)

$x_1x_2x_3 = -a$ ...(3)

Given : $\dfrac{x_1^2}{x_2}+\dfrac{x_2^2}{x_3}+\dfrac{x_3^2}{x_1}=-8$

$x_1^3x_3+x_2^3x_1+x_3^3x_2=-8x_1x_2x_3$

$\Rightarrow\ x_1^3x_3+x_2^3x_1+x_3^3x_2=8a$ ...(4)

Since x_1, x_2, x_3 are the roots of $x^3 + ax - a = 0$

$\Rightarrow\ x_1^3 + ax_1 + a = 0$ ...(5)

$\Rightarrow\ x_2^3 + ax_2 + a = 0$ ...(6)

$\Rightarrow\ x_3^3 + ax_3 + a = 0$ ...(7)

Multiply (5) by x_3, (6) by x_1 and (7) by x_2 and adding, we get

$$x_1^3x_3+x_2^3x_1+x_3^3x_2+a(x_1x_3+x_2x_1+x_2x_3)+a(x_1+x_2+x_3)=0$$

$\Rightarrow\ x_1^3x_3+x_2^3x_1+x_3^3x_2=-a(a)-a(0)$

$8a = -a^2$ [from (4)]

$\Rightarrow 8a + a^2 = 0$

$a(8 + a) = 0$

$a = 0$ or $a = -8$

It is given that a is a non zero real. So, $a = -8$

The equation is $x^3 - 8x - 8 = 0$ and one root is $x = -2$

The other roots are the roots of the equation

$x^2 - 2x - 4 = 0$.

$$x=\frac{2\pm\sqrt{4+16}}{2}=\frac{2\pm2\sqrt5}{2}=1\pm\sqrt5$$

The roots are $-2, 1+\sqrt5, 1-\sqrt5$.

7. **(b)** : Given x, y, z are positive reals and $xyz = 32$. We have to find the minimum value of $x^2 + 4xy + 4y^2 + 2z^2$.

It is a difficult problem in the sense that the A.M. – G.M. inequality be applied carefully. If we take all at a time and apply A.M. – G.M. inequality

We get $\dfrac{x^2+4xy+4y^2+2z^2}{4}\ge\sqrt[4]{32x^3y^3z^2}$.

But $x^3y^3z^2$ value cannot be found from $xyz = 32$

If we take terms in pairs $x^2 + 4y^2$ and $4xy + 2z^2$ and apply then also we cannot use $xyz = 32$.

First take $x^2 + 4y^2$ and apply A.M. – G.M. inequality

We have $x^2+4y^2\ge2\sqrt{x^2.4y^2}\Rightarrow x^2+4y^2\ge4xy$

Now

$$x^2+4y^2+4xy+2z^2\ge4xy+4xy+2z^2 \ge 3(4xy\times4xy\times2z^2)^{1/2}$$

$= 3\,(32x^2y^2z^2)^{1/3} = 3\,(32\times(32)^2)^{1/3} = 3\times32 = 96$

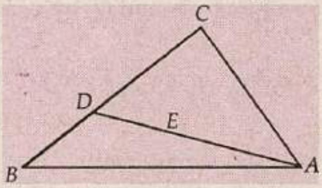
$\therefore$ The minimum value of the expression is 96.

8. **(a)** : $\dfrac{(n+1)^2}{n+7}$ is an integer this means $\dfrac{(n+7-6)^2}{n+7}$ must be an integer or $\dfrac{(n+7)^2-12(n+7)+36}{n+7}$ must be an integer or $n+7-12+\dfrac{36}{n+7}$ must be an integer.

or $n+5+\dfrac{36}{n+7}$ must be an integer.

$\therefore \quad n + 7 = 36, 18, 12, 9 \quad \Rightarrow n = 29, 11, 5, 2.$

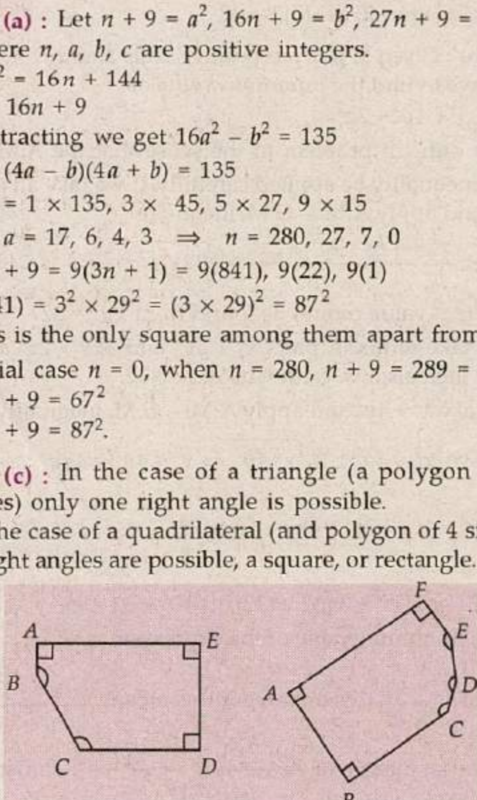
9. (d) : Let E be a point on AD such that CE is perpendicular to AD, and draw BE. Since $\angle ADC$ is an exterior angle of ΔADB, it follows that
$\angle ADC = \angle DAB + \angle ABD = 15° + 45° = 60°$
Thus, ΔCDE is a $30° - 60° - 90°$ triangle and



$DE = \frac{1}{2}CD = BD$. Hence ΔBDE is isosceles and $\angle EBD = \angle BED = 30°$. But $\angle EGB$ is also equal to $30°$ and therefore ΔBEG is isosceles with $BE = EG$. On the other hand $\angle ABE = \angle ABD - \angle EBD = 45° - 30° = 15° = \angle EAB$. Thus, ΔABE is isosceles with $AE = BE$.
Hence $AE = BE = EG$.
The right triangle AEC is also isosceles with $\angle EAG = \angle ECA = 45°$.
Hence, $\angle ACB = \angle ECA + \angle ECD = 45° + 30° = 75°$.

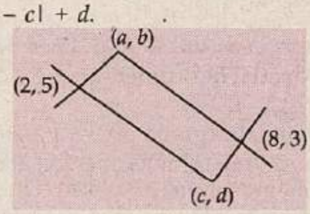
10. (a) : Let $n + 9 = a^2$, $16n + 9 = b^2$, $27n + 9 = c^2$
Where n, a, b, c are positive integers.
$16a^2 = 16n + 144$
$b^2 = 16n + 9$
Subtracting we get $16a^2 - b^2 = 135$
$\Rightarrow (4a - b)(4a + b) = 135$
$135 = 1 \times 135, 3 \times 45, 5 \times 27, 9 \times 15$
$\Rightarrow a = 17, 6, 4, 3 \Rightarrow n = 280, 27, 7, 0$
$27n + 9 = 9(3n + 1) = 9(841), 9(22), 9(1)$
$9(841) = 3^2 \times 29^2 = (3 \times 29)^2 = 87^2$
This is the only square among them apart from the trivial case $n = 0$, when $n = 280$, $n + 9 = 289 = 17^2$
$16n + 9 = 67^2$
$27n + 9 = 87^2$.

11. (c) : In the case of a triangle (a polygon of 3 sides) only one right angle is possible.
In the case of a quadrilateral (and polygon of 4 sides) 4 right angles are possible, a square, or rectangle.



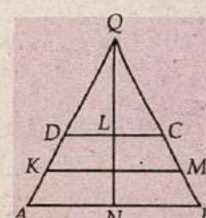
Assume a convex polygon of side $n > 4$. Let K interior angles be right angles. Each of the remaining interior angles must be less than two right angles.
Sum of the interior angles of a n sided polygon
$= (2n - 4)$ right angles.
$2n - 4 < k + 2(n - k)$
$\Rightarrow 2n - 4 < k + 2 - 2k$
$\Rightarrow k < 4$
Thus the maximum right angles will be only 3. A pentagon with 3 right angles has been shown in the figure.

12. (c) : 1st solution
The first graph is an inverted V- shaped" right angle with vertex at (a, b) and the second is a V-shaped right angle with vertex at (c, d). Thus (a, b), $(2, 5)$, (c, d), and $(8, 3)$ are consecutive vertices of a rectangle. The diagonals of this rectangle meet at their common midpoint, so the x-coordinate of this midpoint is $(2 + 8) /2 = (a + c)/2$. Thus $a + c = 10$.
2nd solution
Use the given information to obtain the equations $5 = -|2 - a| + b$, $5 = |2 - c| + d$, $3 = -|8 - a| + b$, and $3 = |8 - c| + d$.



Subtract the third from the first to eliminate b and subtract the fourth from the second to eliminate d. The two resulting equations $|8 - a| - 2|2 - a| = 2$ and $|2 - c| - |8 - c| = 2$ can be solved for a and c. To solve the former, first consider all $a \le 2$, for which the equation reduces to $8 - a - (2 - a) = 2$, which has no solution. Then consider all a in the interval $2 \le a \le 8$, for which the equation reduces to $8 - a - (a - 2) = 2$, which yields $a = 4$. Finally, consider all $a \ge 8$, for which the equation reduces to $8 - a(a - 2) = 2$, which has no solution. The other equation can be solved similarly to show that $c = 6$. Thus $a + c = 10$.

13. (a) : Extend AD, BC to meet at Q.
Since the base angles are 4° and 50°
$\angle AQB = 90°$
Clearly Δ's $ABQ \sim DCQ$



$$\frac{AB}{DC} = \frac{AQ}{DQ} = \frac{BQ}{CQ}$$

Let L, N be the mid points of DC, AB respectively
$\therefore \quad DC = 2DL$ and $AB = 2AN$

$$\therefore \quad \frac{AB}{DC} = \frac{AN}{DL} = \frac{AQ}{DQ} = \frac{BQ}{CQ}$$

$\Rightarrow$ ΔQDL and ΔQAN are similar.
$\Rightarrow$ Q, L, N are collinear.
$AN = BN = QN, DL = LC = QL.$
This is because AQB is a right angled triangle.

$$\frac{AB}{2}-\frac{CD}{2}=QN-QL=LN=1$$

$$\frac{AB+CD}{2}=4$$

$\Rightarrow$ $AB - CD = 2$ and $AB + CD = 8$
$\Rightarrow$ $AB = 5, CD = 3.$

14. (d) : We have $\text{Area}(ABCD) \le \frac{1}{2}AC.BD$ with equality if and only if $AC \perp BD$. Since

$$2002=\text{Area}(ABCD)\le\frac{1}{2}AC.BD$$

$$\le\frac{1}{2}(AP+PC).(BP+PD)=\frac{52\times77}{2}=2002$$

it follows that the diagonals AC and BD are perpendicular and intersect at P. Thus,

$$AB=\sqrt{24^2+32^2}=40, BC=\sqrt{28^2+32^2}=4\sqrt{113},$$

$$CD=\sqrt{28^2+45^2}=53 \text{ and } DA=\sqrt{45^2+24^2}=51.$$

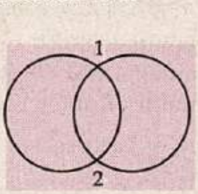
The perimeter of $ABCD$ is therefore

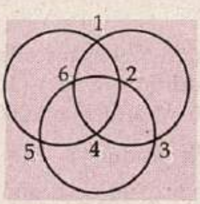
$$144+4\sqrt{113}=4(36+\sqrt{113}).$$

15. (b) : There are 8 straight lines and 4 circles. We have to find the maximum points of intersection. We go in stages

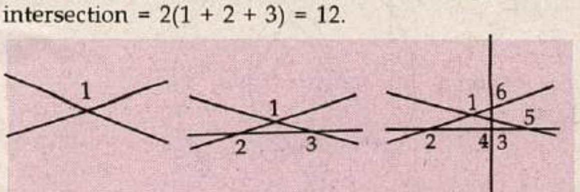
(i) To find the maximum points of intersection of 4 circles.
(ii) To find the maximum points of intersection of 8 straight lines.
(iii) To find the maximum points of intersection of 8 straight lines with the 4 circles.

When two circles intersects, the maximum points of intersection is 2.





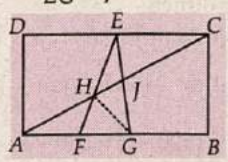
When three circles intersect the maximum points of intersection = 6 = 2 + 4 = 2(1 + 2).
When four circles intersect the maximum points of intersection = 2(1 + 2 + 3) = 12.



When two straight lines intersect, the maximum points of intersection = 1.
When three straight lines intersect, the maximum points of intersection = 1 + 2 = 3.
When four lines intersect, the maximum points of intersection = 1 + 2 + 3 = 6
When 8 lines intersect, the maximum points of intersection $= 1 + 2 + 3 + 4 + 5 + 6 + 7 = \frac{7\times8}{2} = 28$
A line intersects a circles in atmost 2 points, two circles in 2 + 2 = 4 points.
Three circles in 2 + 2 + 2 = 6 points. Four circles in 8 points. But there are 8 lines.
Total number of points $= 8 \times 8 = 64$.
When four circles and eight straight lines are placed then maximum points of intersection = 104.

16. (c) : The area of triangle EFG is $\left(\frac{1}{6}\right)(70)=\frac{35}{3}$.

Triangles AFH and CEH are similar, so $\frac{3}{2}=\frac{EC}{AF}=\frac{EH}{HF}$ and $\frac{EH}{EF}=\frac{3}{5}$. Triangles AGJ and CEJ are similar, so $\frac{3}{4}=\frac{EC}{AG}=\frac{EJ}{JG}$ and $\frac{EJ}{EG}=\frac{3}{7}$.



Since the areas of the triangles that have a common altitude are proportional to their bases, the ratio of the area of ΔEHJ to the area of ΔEHG is 3/7, and ratio of the area of ΔEHG to that of ΔEFG is 3/5. Therefore, the ratio of the area of ΔEHJ to the area of ΔEFG is $\left(\frac{3}{5}\right)\left(\frac{3}{7}\right)=\frac{9}{35}$.

Thus, the area of ΔEHJ is $\left(\frac{9}{35}\right)\left(\frac{35}{3}\right)=3$.

17. (d) : $x^2 + x = y^4 + y^3 + y^2 + y$
$4x^2 + 4x + 1 = 4y^4 + 4y^3 + 4y^2 + 4y + 1$
$(2x + 1)^2 = 4y^4 + 4y^3 + 4y^2 + 4y + 1$
Now

$$4y^4 + 4y^3 + 4y^2 + 4y + 1 = (2y^2 + y)^2 + (3y + 1)(y+1) \quad ...(1)$$

$$4y^4 + 4y^3 + 4y^2 + 4y + 1 = (2y^2 + y + 1)^2 - y(y - 2) \quad ...(2)$$

Consider $(3y + 1)(y + 1)$
This is positive when $y < -1$ or $y > 0$
$\therefore$ $4y^4 + 4y^3 + 4y^2 + 4y + 1 > (2y^2 + y)^2$
$y(y - 2) > 0$ for $y < 0, y < 2$
$4y^4 + 4y^3 + 4y^2 + 4y + 1 < (2y^2 + y + 1)^2$
$\therefore$ $(2y^2 + y)^2 < 4y^4 + 4y^3 + 4y^2 + 4y + 1 < (2y^2+y+1)^2$

This lies between two consecutive squares.

$\Rightarrow$ it cannot be a square.

But we must check for $y = -1, 0, 1, 2$.

When $y = -1$; $y^4 + y^3 + y^2 + y = 1 - 1 + 1 = 0$

$\therefore$ $x^2 + x = 0 \Rightarrow x(x + 1) = 0$, $x = 0$, $x = -1$

$\therefore$ $x = 0, y = -1$; $x = -1, y = -1$

When $y = 0$, $x^2 + x = 0$, $x = 0$, $x = -1$. We get $(0, 0)$, $(-1, 0)$.

When $y = 1$, $y^4 + y^3 + y^2 + y = 4$.

$x^2 + x = 4$, no integer solution.

When $y = 2$, $y^4 + y^3 + y^2 + y = 16 + 8 + 4 + 2 = 30$

$x^2 + x = 30$, $x(x + 1) = 30$, $x = 5, -6$

Solutions are $(0, -1)$, $(-1, -1)$, $(0, 0)$, $(-1, 0)$, $(5, 2)$, $(-6, 2)$.

18. (c) : Let the number of students wearing spectacles = a

Let the number of students bringing lunch = b

the number of students who either wear spectacles or bring lunch or both = $a + b - 60$.

The number of students who do not wear spectacles and do not bring lunch.

$= 500 - (a + b - 60) = 560 - a - b$

This is also equal to $\frac{2}{3}(500-a)$ or $\frac{3}{4}(500-b)$

$\Rightarrow \frac{2}{3}(500-a) = \frac{3}{4}(500-b) = 560-a-b$

There are two unknowns a, b to be evaluated.

$\frac{2}{3}(500-a) = \frac{3}{4}(500-b)$

$8(500 - a) = 9(500 - b)$

$8a - 9b + 500 = 0$ ....(1)

Also $3(500 - b) = 4(560 - a - b)$

$1500 - 3b = 2240 - 4a - 4b$

$4a + b = 740$ ...(2)

From (1) and (2) gives

$8a + 2b = 1480$ and $\dfrac{8a - 9b = -500}{11b = 1980}$

$b = 180$, $a = 140$.

19. (a) : In the triangle ABC the medians AD and BE intersect at a point O, $AC = b$ and $BC = a$. Let us find $AB = c$.

Let $OD = x$ and $OE = y$. Taking advantage of the property of medians we find from the triangles AOB, BOD and AOE that

$$4x^2 + y^2 = \frac{b^2}{4}, 4x^2 + 4y^2 = c^2, 4x^2 + 16y^2 = a^2$$

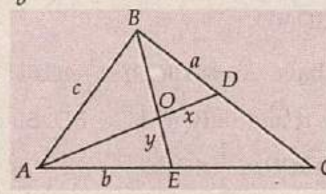
Eliminating x and y we obtain $c^2 = \dfrac{a^2 - b^2}{5}$

The conditions for existence of a triangle with sides a, b and c take the form

$5(a + b)^2 > a^2 + b^2$, $5(a - b)^2 < a^2 + b^2$.

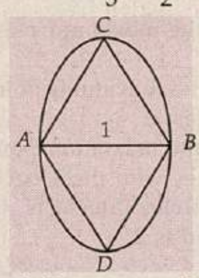
The first inequality is obviously fulfilled for any a and b, and the second one is transformed into the following relation : $a^2 - \frac{5}{2}ab + b^2 < 0$

Solving this inequality with respect to $\frac{a}{b}$ we finally obtain $\frac{1}{2} < \frac{a}{b} < 2$



20. (c) : The centre of the disc lies in a region R, consisting of all points within 1 unit of both A and B. Let C and D be the points of intersection of the circles of radius 1 centred at A and B. Because ΔABC and ΔABD are equilateral, arcs CAD and CBD are each 120°. Thus the sector bounded by BC, BD, and arc CAD has area $\frac{\pi}{3}$, as does the sector bounded by AC, AD and arc CBD. The intersection of the two sectors, which is the union of the two triangles, has area $\frac{\sqrt{3}}{2}$, so the area of R is $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

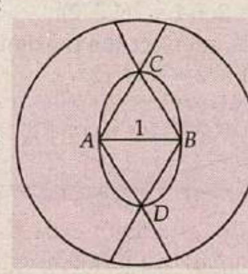
Region R



The region S consists of all points within 1 units of R. In addition to R itself, contains two 60° sectors of radius 1 and two 120° annuli of outer radius 2 and inner radius 1. The area of each sector is $\frac{\pi}{6}$, and the area of each annulus is

$\frac{\pi}{3}(2^2 - 1^2) = \pi$

Region S



Therefore, the area of S is

$$\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)+\left(\frac{\pi}{6}+\pi\right)=3\pi-\frac{\sqrt{3}}{2}.$$

21. (a) : The values of x, y, z are real.
Clearly $x = 0$, $y = 0$, $z = 0$ do not satisfy the equation
$\Rightarrow\ x \neq 0, y \neq 0, z \neq 0$
Multiplying all the equations, we have

$$\frac{64x^2y^2z^2}{(1+4x^2)(1+4y^2)(1+4z^2)}=xyz$$

$\Rightarrow\ 64xyz = (1 + 4x^2)(1 + 4y^2)(1 + 4z^2)$ ...(1)
Clearly $(1 - 2x)^2 \geq 0 \Rightarrow 1 + 4x^2 \geq 2$
$1 + 4y^2 \geq 4y \Rightarrow 1 + 4z^2 \geq 4z$
$\Rightarrow\ (1 + 4x^2)(1 + 4y^2)(1 + 4z^2) \geq 64xyz$
When $x = y = z$, the equality holds.
$\Rightarrow\ x = y = z$

Now $\dfrac{4x^2}{1+4x^2}=y$

$\Rightarrow\ 4x^2 = y(1 + 4x^2) \Rightarrow 4x^2 = x(1 + 4x^2)$
Since, $x \neq 0 \Rightarrow 4x = 1 + 4x^2$

or $(1-2x)^2=0$ or $x=\dfrac{1}{2}, y=\dfrac{1}{2}, z=\dfrac{1}{2}$

22. (c) : Denote by x the third side of the triangle which is equal to the altitude drawn to it. Using two expressions for the area of the given triangle, we get the equation

$$\frac{1}{2}x^2=\sqrt{\frac{b+c+x}{2}\cdot\frac{c+x-b}{2}\cdot\frac{x+b-c}{2}\cdot\frac{b+c-x}{2}} \quad ...(i)$$

Solving it, we get

$$x^2=\frac{1}{5}(b^2+c^2\pm 2\sqrt{3b^2c^2-b^4-c^4}) \quad ...(ii)$$

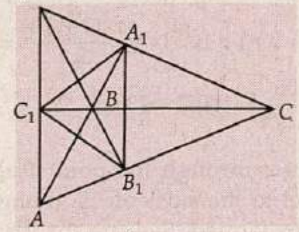
The necessary condition for solvability of the problem is $3b^2c^2 \geq b^4 + c^4$ . ...(iii)
It it is fulfilled, then both values of x^2 in (i) are positive. It can easily be verified that if (ii) is fulfilled, the inequalities $b + c > x \geq |b - c|$ are also fulfilled, the sign of equality appearing only in the case when $x = 0$. The latter takes place if in (i) we take a minus in front of the radical for $b = c$. Hence if $b = c$, the problem has a unique solution, namely

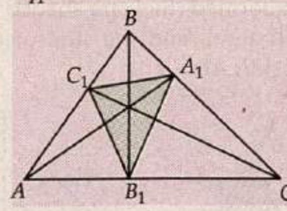
$$x=\frac{2}{\sqrt{5}}b$$

For $b \neq c$ the triangle exists only if inequality (ii) is fulfilled. Solving it with respect to b/c, we find that it is equivalent to the two inequalities

$$\frac{2}{1+\sqrt{5}}\leq\frac{b}{c}\leq\frac{1+\sqrt{5}}{2}$$

Consequently, for $b \neq c$ there exist two triangles if both inequalities (iii) are fulfilled with sign <, and only one triangle if at least one of the relation (iii) turns in an equality.





23. (c) : Since $\angle APB = 90°$ if and only if P lies on the semi - circle with centre (2, 1) and radius $\sqrt{5}$, the angle is obtuse if and only if the point P lies inside this semicircle. The semicircle lies entirely inside the pentagon, since the distance, 3, from (2, 1) to DE is greater than the radius OB of the circle. Thus, the probability that the angles is obtuse is the ratio of the area of the semicircle to the area of the pentagon.
Let $O = (0, 0)$, $A = (0, 2)$, $B = (4, 0)$, $C = (2\pi + 1, 0)$, $D = (2\pi + 1, 4)$ and $E = (0, 4)$. Then the area of the pentagon is

$$[ABCDE]=[OCDE]-[OAB]=4.(2\pi+1)-\frac{1}{2}(2.4)=8\pi$$

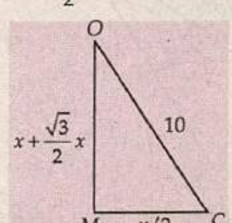
and the area of the semicircle is $\dfrac{1}{2}\pi(\sqrt{5})^2=\dfrac{5}{2}\pi$.

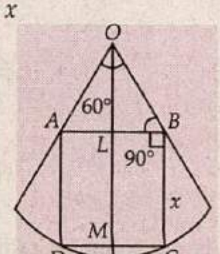
The probability is $\dfrac{\frac{5\pi}{8}}{8\pi}=\dfrac{5}{16}$.

24. (b) : From the centre O drop a perpendicular to AB and CD to cut them at L and M respectively.

$$\angle LOB=30°,\ \tan 30°=\frac{LB}{OL}$$

$\Rightarrow\ OL=\dfrac{\sqrt{3}}{2}x$ where $BC=x$





Using pythagoras theorem in ΔOMC, we get

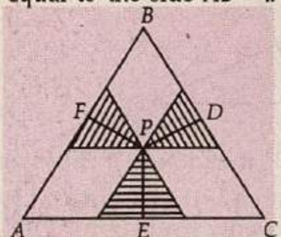
$$100=\left(x+\frac{\sqrt{3}}{2}x\right)^2+\frac{x^2}{4}$$

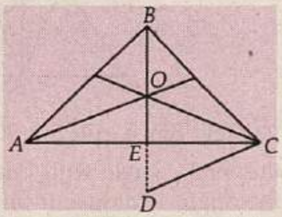
$$= \frac{x^2(2+\sqrt{3})^2}{4} + \frac{x^2}{4} = \frac{x^2}{4}((2+\sqrt{3})^2+1)$$

$$= \frac{x^2}{4}[4+3+1+4\sqrt{3}] = \frac{4x^2(2+\sqrt{3})}{4} = x^2(2+\sqrt{3})$$

$$\Rightarrow \quad x^2 = \frac{100}{2+\sqrt{3}} = 100(2-\sqrt{3}).$$

25. (c) : Draw through the point P three straight lines parallel to the sides of the triangle. The three triangles thus formed (they are shaded in the figure) are also equilateral, and the sum of their sides is equal to the side $AB = a$





of the triangle ABC. Consequently, the sum of their altitudes is equal to the altitude of ΔABC and hence

$$PD + PE + PF = \frac{a\sqrt{3}}{2}$$

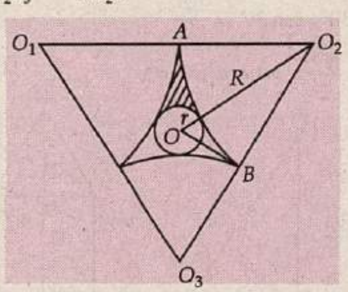
The sum $BD + CE + AF$ is equal to the sum of the sides of the shaded triangle added to the sum of the halves of these sides and thus,

$$BD + CE + AF = \frac{3}{2}a$$

Consequently, $\dfrac{PD+PE+PF}{BD+CE+AF} = \dfrac{1}{\sqrt{3}}$

26. (b) : Let O_1, O_2 and O_3 be centres of the three equal circles and O be the centre of the circle of radius r. Let us denote by $S_{O_1O_2O_3}$ the area of $\Delta O_1O_2O_3$ and by S_{AO_2B} the area of the sector AO_2B. Then the required area is equal to

$$S = \frac{1}{3}(S_{O_1O_2O_3} - 3S_{AO_2B} - \pi r^2)$$



If R is the common radius of the three circles, then $R = \dfrac{\sqrt{3}}{2}(R+r)$, whence we obtain

$$R = \frac{\sqrt{3}}{2-\sqrt{3}}r = (3+2\sqrt{3})r$$

Then we find

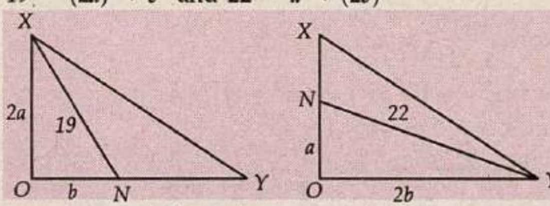
$$S_{O_1O_2O_3} = \frac{1}{2}2RR\sqrt{3} = \sqrt{3}R^2 = 3(12+7\sqrt{3})r^2$$

and $S_{AO_2B} = \dfrac{1}{6}\pi R^2 = \dfrac{\pi}{2}(7+4\sqrt{3})r^2$.

Finally we obtain $S = \left[12 + 7\sqrt{3} - \left(\dfrac{23}{6} + 2\sqrt{3}\right)\pi\right]r^2$

27. (d) : The set S is symmetric about the line $y = x$ and contains (2, 3), so it must also contain (3, 2). Also S is symmetric about the x-axis, so it must contain (2, −3) and (3, −2). Finally, since S is symmetric about the y-axis, it must contain (−2, 3), (−3, 2), (−2, −3), and (−3, −2). Since the resulting set of 8 points is symmetric about both coordinate axes, it is also symmetric about the origin.

28. (b) : Let $OM = a$ and $ON = b$, Then $19^2 = (2a)^2 + b^2$ and $22^2 = a^2 + (2b)^2$

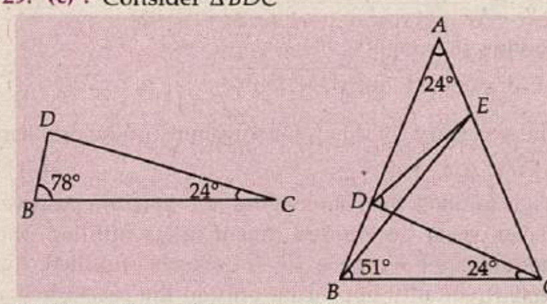


Hence $5(a^2 + b^2) = 19^2 + 22^2 = 845$

It follows that $MN = \sqrt{a^2+b^2} = \sqrt{169} = 13$

Since ΔXOY is similar to ΔMON and $XO = 2.\ MO$, we have $XY = 2.\ MN = 26$.

29. (c) : Consider ΔBDC



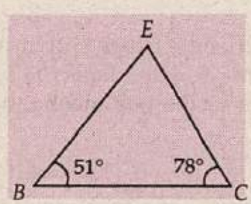
$\angle BDC = 180° - (78 + 24) = 78°$

$\Rightarrow \quad BC = CD$ ...(1)

Consider triangle BEC

$\angle BEC = 180° - (51° + 78°) = 51°$

$\Rightarrow \quad BC = EC$ ...(2)



From (1) and (2) we get $CD = EC$

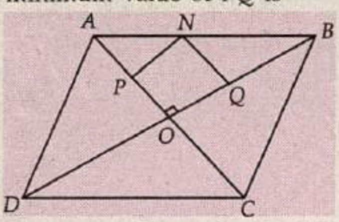
$\angle DCE = 78° - 24° = 54°$

$\therefore \quad \angle DEC = \dfrac{180° - 54°}{2} = 63°$

$\therefore \quad \angle DEB = \angle DEC - \angle BEC = 63° - 51° = 12°.$

30. (c) : Let O be the point of intersection of AC and ED. Then AOB is a right triangle with legs $OA = 8$ and $OB = 15$. Quadrilateral $OPNQ$ is a rectangle because it has right angles at O, P and Q. Thus $PQ = ON$, because the diagonals of a rectangle are of equal length. The minimum value of PQ is the minimum value of ON. This is achieved if and only if N is the foot of the altitude from O in triangle AOB. Writing the area of ΔAOB in two different ways yields $\frac{1}{2}AB.ON = \frac{1}{2}OA.OB.$

Hence the minimum value of PQ is



$$ON = \frac{OA.OB}{AB} = \frac{OA.OB}{\sqrt{OA^2 + OB^2}} = \frac{8 \times 15}{17} = \frac{120}{17} = 7.06$$

PART - B

1. If (x, y, z) satisfies the given system of equations, then so does $(-x, -y, z)$; since the system is to have only one solution, we must have $x = y = 0$. Thus $a = b = z$ and $z = \pm 2$. If $a = b = 2$, then

$xyz + z = 2$, $xyz^2 + z = 2$ and $x^2 + y^2 + z^2 = 4$

$\therefore \; xyz(1 - z) = 0$. Taking $z = 1$, then have $xy = 1$ and $x^2 + y^2 = 3$. This has four real solutions. Hence $a = b = 2$ is not a suitable choice.

So let us consider the case that $a = b = -2$. Then the system becomes

$xyz + z = -2$, $xyz^2 + z = -2$ and $x^2 + y^2 + z^2 = 4$

Here again we have $xyz(1 - z) = 0$.

Obviously $z \neq 0$; also $z \neq 1$ for otherwise we would have $xy = -3$ and $x^2 + y^2 = 3$ implying $(x + y)^2 = -3$ whence both x and y could not be real. Therefore either x or y is zero. Now it follows that $z = -2$

$\Rightarrow \; x = y = 0$

Thus the given system has a unique solution if and only if $a = b = -2$.

2. Let $a = \sin^2 x$ and $b = \cos^2 x$

Since $\dfrac{1}{2}\left(\dfrac{1}{a} + \dfrac{1}{b}\right) \geq \left(\dfrac{1}{a+b}\right)$ and $a + b = 1$ we have

$\left(\dfrac{1}{a} + \dfrac{1}{b}\right) \geq 4$

where equality holds only when $a = b = \dfrac{1}{2}$.

Therefore $12 + \dfrac{1}{2}\sin y = \left(a + \dfrac{1}{a}\right)^2 + \left(b + \dfrac{1}{b}\right)^2$

$$\geq \frac{1}{2}\left\{\left(a + \frac{1}{a}\right) + \left(b + \frac{1}{b}\right)\right\}^2$$

$$= \frac{1}{2}\left(a + b + \frac{1}{a} + \frac{1}{b}\right)^2 \geq \frac{1}{2}(1+4)^2 = \frac{25}{2}$$

with equality only when $a = b = \dfrac{1}{2}$.

So, we have $\sin y \geq 1$;

Hence $\sin y = 1$; this also implies $a = b = \dfrac{1}{2}$;

therefore $\sin^2 x = \dfrac{1}{2}$; *i.e.*, $\sin x = \pm \dfrac{1}{\sqrt{2}}$.

Thus the solutions are

$$x = \frac{\pi}{4} + \frac{k\pi}{2}, y = \frac{\pi}{2} + 2l\pi, \text{ where } k, l \in Z$$

3. The cubic equation is $t^3 + at^2 + bt + c = 0$

Let α, β, γ be the real roots. Let $\alpha \geq \beta \geq \gamma$

We have

$\alpha + \beta + \gamma = -a \quad$ ...(1)

$\alpha\beta + \beta\gamma + \gamma\alpha = b \quad$ ...(2)

$\alpha\beta\gamma = -c \quad$ ...(3)

$2(a^2 - 3b) = 2[(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)]$

$= 2\alpha^2 + 2\beta^2 + 2\gamma^2 - 2\alpha\beta - 2\beta\gamma - 2\gamma\alpha$

$= (\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2$

$= m^2 + l^2 + (l - m)^2 \geq 0$

$2(a^2 - 3b) = l^2 + m^2 + l^2 + m^2 - 2lm$

$= 2(l^2 + m^2) - 2lm = 2\left(m - \dfrac{l}{2}\right)^2 + \dfrac{3l^2}{2} \geq 0$

$= m^2 + l^2 + (l - m)^2 \geq 0$

Minimum of the expression occurs when $m = \dfrac{l}{2}$ maximum will occur at $m = 0$ or $m = l$.

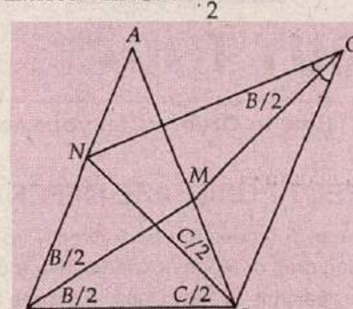
Thus $\dfrac{3}{2}l^2 \leq 2(a^2 - 3b) \leq 2l^2$

$\Rightarrow \; (a^2 - 3b)^{1/2} \leq l \leq \dfrac{2}{\sqrt{3}}(a^2 - 3b)^{1/2} < 2(a^2 - 3b)^{1/2}$

$\Rightarrow \; (a^2 - 3b)^{1/2} \leq \alpha - \gamma < 2(a^2 - 3b)^{1/2}.$

4. ABC be a triangle with equal bisectors BM and CN. Complete the parallelogram $BMGN$.

$\angle MBC = \angle MBA = \angle NGM = \dfrac{B}{2}$



To prove $B = C$; (*i.e.,*) $\frac{B}{2}=\frac{C}{2}$. If $\frac{B}{2}\neq\frac{C}{2}$ let $\frac{B}{2}>\frac{C}{2}$

Consider the triangle MBC and NBC

$\frac{B}{2}>\frac{C}{2}\Rightarrow CM>BN$ ...(1)

In ΔCGN we have $CN = NG$.

$\Rightarrow \angle NCG = \angle NGC$

$\Rightarrow \frac{C}{2}+\angle ACG=\frac{B}{2}+\angle MGC$

Since $\frac{B}{2}>\frac{C}{2}\Rightarrow \angle MGC<\angle AGC$

$\Rightarrow CM < MG \Rightarrow CM < BN$ ...(2)

from (1) and (2) contradict each other $\therefore \frac{C}{2}\not<\frac{B}{2}$

Similarly $\frac{B}{2}\not<\frac{C}{2} \Rightarrow \frac{B}{2}=\frac{C}{2}\Rightarrow B=C$

Thus the triangle is isosceles.

5. $F_1=\frac{3}{R_1}+\frac{7}{R_1^2}+\frac{3}{R_1^3}+\frac{7}{R_1^4}+......$

$=\frac{3}{R_1}\left(1+\frac{1}{R_1^2}+......\right)+\frac{7}{R_1^2}\left(1+\frac{1}{R_1^2}+......\right)$

$=\frac{3/R_1}{1-1/R_1^2}+\frac{7/R_1^2}{1-1/R_1^2}=\frac{3R_1+7}{R_1^2-1}$

Similarly $F_2=\frac{7R_1+3}{R_1^2-1}$. Again, in base R_2, we have

$F_1=\frac{2R_2+5}{R_2^2-1}$ and $F_2=\frac{5R_2+2}{R_2^2-1}$

Thus we have $\frac{3R_1+7}{R_1^2-1}=\frac{2R_2+5}{R_2^2-1}$ ....(1)

and $\frac{7R_1+3}{R_1^2-1}=\frac{5R_2+2}{R_2^2-1}$ ....(2)

Now (1) – (2) and (1) + (2) yield after simplification

$\frac{R_1+1}{R_2+1}=\frac{4}{3}$ and $\frac{R_1-1}{R_2+1}=\frac{10}{7}$

Thus by componendo and dividendo rule gives

$\frac{R_1+R_2+2}{R_2-R_1}=\frac{7}{1}$ and $\frac{R_1+R_2-2}{R_2-R_1}=\frac{17}{3}$

Therefore $\frac{R_1+R_2+2}{R_2-R_1-2}=\frac{21}{17}$. This by simplification yields

$R_1 + R_2 = 19$.

6. Let $y=f\left(-\frac{b}{a}\right)$; then

$f(y)-y=a\left(-\frac{b}{a}\right)+b=0\Rightarrow f(y)=y$

i.e., y is a fixed point of f. Now let z be any fixed point of f : then

$az + b = f(f(z)) - f(z) = z - z = 0$

Therefore $z=-\frac{b}{a}$. Thus it follows that f has a unique fixed point, viz., $-\frac{b}{a}$.

7. $a \ge b \ge c > 0$ and $x \ge y \ge z > 0$

The first term is $\frac{a^2x^2}{(by+cz)(bz+cy)}$

The denominator is $(by + cz)(bz + cy)$

$by + cz - (bz + cy) = b(y - z) - c(y - z)$

$y \ge z$ and $b \ge c$

This quantity is positive.

$\therefore\ by + cz \ge bz + cy$

or $(by + cz)(bz + cy) \le (by + cz)^2 \le 2[(by)^2 + (cz)^2]$

$\therefore \frac{a^2x^2}{(by+cz)(bz+cy)}\ge\frac{a^2x^2}{2\{(by)^2+(cz)^2\}}$

Let $(ax)^2 = A$, $(by)^2 = B$, $(cz)^2 = C$

$\therefore \frac{a^2x^2}{(by+cz)(bz+cy)}\ge\frac{A}{2(B+C)}$

Similarly $\frac{b^2y^2}{(by+ax)(cx+az)}\ge\frac{B}{2(C+A)}$

$\frac{c^2z^2}{(ax+by)(ay+bx)}\ge\frac{C}{2(A+B)}$

Adding all we get

$$\frac{a^2x^2}{(by+cz)(bz+cy)}+\frac{b^2y^2}{(cz+ax)(cx+az)}+\frac{c^2z^2}{(ax+by)(ay+bx)}$$

$$\ge\frac{1}{2}\left[\frac{A}{B+C}+\frac{B}{C+A}+\frac{C}{A+B}\right]$$

We have to prove $\frac{A}{B+C}+\frac{B}{C+A}+\frac{C}{A+B}\ge\frac{3}{2}$

We use a result in inequality called power mean theorem. If $m > 0$ or $m > 1$

$\frac{a^m+b^m+c^m}{3}\ge\left(\frac{a+b+c}{3}\right)^m$

Where a, b, c are distinct positive reals.

Take $B + C$, $C + A$, $A + B$. They are positive reals.

Take $m = -1$

$$\frac{(B+C)^{-1}+(C+A)^{-1}+(A+B)^{-1}}{3}\ge\left(\frac{B+C+C+A+A+B}{3}\right)^{-1}$$

or $\frac{\frac{1}{B+C}+\frac{1}{C+A}+\frac{1}{A+B}}{3}\ge\left(\frac{2(A+B+C)}{3}\right)^{-1}$

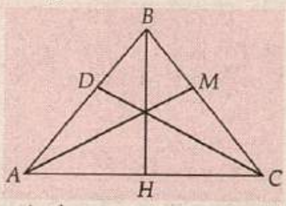
or $(A+B+C)\left[\frac{1}{B+C}+\frac{1}{C+A}+\frac{1}{A+B}\right]\ge\frac{9}{2}$

or $\frac{A+B+C}{B+C}+\frac{A+B+C}{C+A}+\frac{A+B+C}{A+B}\ge\frac{9}{2}$

or $\frac{A+B+C}{B+C}-1+\frac{A+B+C}{C+A}-1+\frac{A+B+C}{A+B}\ge\frac{9}{2}-3$

or $\frac{A}{B+C}+\frac{B}{C+A}+\frac{C}{A+B}\geq\frac{3}{2}$

8. The solution is based on Ceva's Theorem.
If three concurrent lines AP, BQ, CR are drawn from the vertices A, B, C of a triangle as shown in the figure then $\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=1$



Applying Ceva's theorem to our problem we get

$$\frac{AH}{HC}\cdot\frac{CM}{MB}\cdot\frac{BD}{DA}=1 \quad .....(1)$$

Since AM is the median, M is the midpoint of BC so that $CM = MB$, now (1) gives

$$\Rightarrow \frac{AH}{HC}\cdot\frac{BD}{DA}=1 \quad ....(2)$$

Using the internal bisector theorem.

Now (2) gives $\frac{AH}{HC}\cdot\frac{a}{b}=1 \Rightarrow \frac{AH}{HC}=\frac{b}{a}$

Let $AH = x$, $\Rightarrow CH = b - x$

Applying pythagoras theorem in triangles BAH, BCH and eliminating BH^2, we get

$c^2 - x^2 = a^2 - (b-x)^2$

$$\Rightarrow x=\frac{b^2+c^2-a^2}{2b},\ b-x=\frac{b^2+a^2-c^2}{2b}$$

so, we get $\frac{b^2+c^2-a^2}{b^2+a^2-c^2}=\frac{b}{a}$.

9. Given $\left[\frac{3x-1}{4}\right]+\left[\frac{3x+1}{4}\right]+\left[\frac{3x-1}{2}\right]=\frac{6x+3}{5}$

Let x be any real number

Let $[x]$ denote the integral part and (x) denote the fractional part. Now $0 \leq (x) < 1$

$\therefore\ x = [x] + (x) \Rightarrow [x] = x - (x)$

The given equaiton can be written as

$$\frac{3x-1}{4}-\left[\frac{3x-1}{4}\right]+\frac{3x+1}{4}-\left[\frac{3x+1}{4}\right]+\frac{3x-1}{4}-\left[\frac{3x-1}{2}\right]=\frac{6x+3}{5}$$

$$\Rightarrow \frac{3x-1}{4}+\frac{3x+1}{4}+\frac{3x-1}{2}-\frac{6x+3}{5}=\left[\frac{3x-1}{4}\right]+\left[\frac{3x+1}{4}\right]+\left[\frac{3x-1}{2}\right]$$

$$\frac{5(3x-1)+5(3x+1)+10(3x-1)-4(6x+3)}{20}=\left(\frac{3x-1}{4}\right)+\left(\frac{3x+1}{4}\right)+\left(\frac{3x-1}{2}\right)$$

$$\frac{36x-22}{20}=\left(\frac{3x-1}{4}\right)+\left(\frac{3x+1}{4}\right)+\left(\frac{3x-1}{2}\right)$$

$$\frac{18x-11}{10}=\left(\frac{3x-1}{4}\right)+\left(\frac{3x+1}{4}\right)+\left(\frac{3x-1}{2}\right)$$

Now $0\leq\left[\frac{3x-1}{4}\right]<1, 0\leq\left[\frac{3x+1}{4}\right]<1, 0\leq\left[\frac{3x-1}{2}\right]<1$

$$\Rightarrow 0\leq\left[\frac{3x-1}{4}\right]+\left[\frac{3x+1}{4}\right]+\left[\frac{3x-1}{2}\right]<3$$

$$\Rightarrow 0\leq\frac{18x-11}{10}<3$$

$0 \leq 18x - 11 < 30 \Rightarrow 11 \leq 18x < 41$

$$\Rightarrow \frac{11}{18}\leq x<\frac{41}{18}$$

Now $\frac{6x+3}{5}$, the RHS, is an integer.

Let $\frac{6x+3}{5}=n(\in z^+) \Rightarrow x=\frac{5n-3}{6}$

$n=1\Rightarrow x=\frac{1}{3}\notin\left[\frac{11}{18},\frac{41}{18}\right]$, $n=2\Rightarrow x=\frac{7}{6}\in\left[\frac{11}{18},\frac{41}{18}\right]$

$n=3\Rightarrow x=2\in\left[\frac{11}{18},\frac{41}{18}\right]$

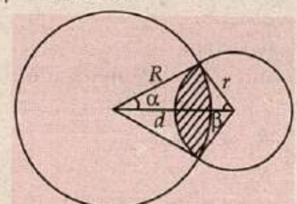
but it does not satisfy the equation

$n=4\Rightarrow x=\frac{17}{6}\notin\left[\frac{11}{18},\frac{41}{18}\right]$

$\therefore\ x=\frac{7}{6}$ is the only solution.

10. The required area is equal to the sum of the areas of two sectors with central angles 2α and 2β minus twice the area of the triangle with sides R, r, d :

$S = R^2\alpha + r^2\beta - Rd\sin\alpha$



For determining the angles α and β we have two equations $R\sin\alpha = r\sin\beta$ ...(i)

$R\cos\alpha + r\cos\beta = d$ ...(ii)

Solving (i) and (ii) we get : $\cos\alpha=\frac{d^2+R^2-r^2}{2Rd}$, $\cos\beta=\frac{d^2+r^2-R^2}{2rd}$.

Hence,

$$S=R^2 \text{ arc ccos}\frac{d^2+R^2-r^2}{2Rd}+r^2\text{arc cos}\frac{d^2+r^2-R^2}{2rd}-Rd\sqrt{1-\left(\frac{d^2+R^2-r^2}{2Rd}\right)^2}.$$

$= \text{Coeff. of } x^{-7} \text{ in } \left(1-\frac{1}{x}\right)\left(1-\frac{2}{x^2}\right)....\left(1-\frac{20}{x^{20}}\right)$

$= \text{Coeff. of } t^7 \text{ in } (1-t)(1-2t^2)\ ....\ (1-20t^{20}) \left[t=\frac{1}{x}\right]$

$= -7 + (1\cdot 6 + 2\cdot 5 + 3\cdot 4) - (1\cdot 2\cdot 4)$
$= -7 + 28 - 8 = 13.$

5. (d): Note that ${}^{48}C_1 + {}^{48}C_3 + {}^{48}C_5 + ... + {}^{48}C_{47} = 2^{47}$

Any common divisior must be of the form 2^k, $0 \le k \le 47$

As $24 = 2^3 \cdot 3$

${}^{48}C_1 = 48 = 2^4 \cdot 3$

The common divisior cannot be larger than 2^4.

But ${}^{48}C_p = \frac{48}{p}\,{}^{47}C_{p-1} = \frac{2^4\cdot 3}{p}\,{}^{47}C_{p-1}$

As this shows that ${}^{48}C_p$ is divisible by 16 where $p = 1, 3, 5, ..., 47$.

Hence, greater common divisior is 16.

6. (b): We have $a_n = 3^{n-2}\cdot {}^nC_2$

$$\sum\frac{3^n}{a_n} = \sum\frac{3^n}{3^{n-2}\cdot{}^nC_2} = \sum\frac{9}{n\frac{(n-1)}{2}} = 18\sum_{n=2}^{\infty}\frac{1}{n(n-1)}$$

$= 18 \times 1 = 18.$

7. (a, c): $$\sum_{p=1}^{16}\sum_{m=p}^{16}\binom{16}{m}\binom{m}{p} = \sum_{p=1}^{16}\sum_{m=p}^{16}\frac{\lfloor 16}{\lfloor 16-m\,\lfloor m-p\,\lfloor p}$$

$$= \sum_{p=1}^{16}\sum_{m=p}^{16}\binom{16}{p}\binom{16-p}{m-p} = \sum_{p=1}^{16}\binom{16}{p}\sum_{m=p}^{16}\binom{16-p}{m-p}$$

$$= \sum_{p=1}^{16}\binom{16}{p}\sum_{s=0}^{16-p}\binom{16-p}{s} = \sum_{p=1}^{16}\binom{16}{p}2^{16-p}$$

$$= 2^{16}\sum_{p=1}^{16}\binom{16}{p}\frac{1}{2^p} = 2^{16}\left\{\left(1+\frac{1}{2}\right)^{16}-1\right\} = 3^{16}-2^{16}$$

Now $3^{16} - 2^{16} = (3+2)(3^2+2^2)(3^4+2^4)(3^8+2^8)$.

8. (a, c) : The series can be identified with the fibonacci number by working at first two terms.

9. (a, d) : $$S_n = \sum_{k=0}^{n}\frac{1}{\lfloor n-k\,\lfloor n+k} = \frac{1}{\lfloor 2n}\sum_{k=0}^{n}\frac{\lfloor 2n}{\lfloor n-k\,\lfloor n+k}$$

$$\frac{1}{\lfloor 2k}\sum_{k=0}^{n}{}^{2n}C_k = \frac{1}{\lfloor 2k}\{{}^{2n}C_0 + {}^{2n}C_1 + ... + {}^{2n}C_n\}$$

which can be readily calculated.

10. (a, c)

11. (d): The highest exponent of 2 in $\lfloor 100$ is

$$\left(\frac{100}{2}\right)+\left(\frac{100}{4}\right)+\left(\frac{100}{8}\right)+\left(\frac{100}{16}\right)+\left(\frac{100}{32}\right)+\left(\frac{100}{64}\right)$$

$= 50 + 25 + 12 + 6 + 3 + 1 = 97$

The highest exponent of 3 in $\lfloor 100$ is

$$\left(\frac{100}{3}\right)+\left(\frac{100}{9}\right)+\left(\frac{100}{27}\right)+\left(\frac{100}{81}\right)$$

$= 33 + 11 + 3 + 1 = 48.$

So $\lfloor 100 = 2^{97}\cdot 3^{48}\cdot p^{\alpha}\cdot q^{\beta}\cdot r^{\gamma}......$

where p, q, r are others primes occuring in the prime factorization of $\lfloor 100$

Now $\lfloor 100 = (2^2\cdot 3)^{48}\cdot 2\cdot p^{\alpha}q^{\beta}r^{\gamma}$

giving that the highest value of k such that 12^k divides $\lfloor 100$ is 48.

Thus statement-1 is false, statement-2 is correct.

12. (c)

13. (a)

14. (c)

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15. (a) : $$\sum\sum_{0\le i<j\le 50} a_ia_j = \frac{1}{2}\left[\left(\sum_{i=1}^{50}a_i\right)^2 - \sum_{i=1}^{50}a_i^2\right]$$

And a_i's can be identified with binomial coefficients.

16. (d): Let $S = \sum\sum_{0\le i<j\le 50}(i+j)a_ia_j$

Writing S in the reverse order, we have

$$S = \sum\sum_{0\le i<j\le 50}(n-i+n-j)a_{n-i}a_{n-j}$$

(As a_i's are binomial coefficients we have ${}^nC_r = {}^nC_{n-r}$)

$$= \sum\sum_{0\le i<j\le 50}\{2n-(i+j)\}a_ia_j$$

$$= 2n\sum\sum_{0\le i<j\le 50}a_ia_j - \sum\sum_{0\le i<j\le 50}(i+j)a_ia_j = 2n\sum\sum_{0\le i<j\le 50}a_ia_j - S$$

$$\Rightarrow\ 2S = 2n\sum\sum_{0\le i<j\le 50}a_ia_j$$

$$\therefore\ S = n\sum\sum_{0\le i<j\le 50}a_ia_j$$

17. (d): Similar techniques as in previous two problems can be used to establish the result.

18. (b) **19. (b)** **20. (b)**

Set $r = 1, -1, i, -i$ and then form suitable combination to set the value of the expressions.

21. (c) **22. (b)** **23. (a)**

Change x to $-x$ and x to $1/x$ to produce two more identities and then it's simple matter to get the desired result.

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers comments and suggestions regarding the problems and solutions offered are always welcome.

1. If [.] stands for the greatest integer function, $\int_1^2 [3x]dx$ is equal to

(a) 3 (b) 4 (c) 5 (d) 6

2. If $\int_{\ln 2}^{x} \frac{du}{\sqrt{e^u - 1}} = \frac{\pi}{6}$, then the value of x is

(a) 4 (b) ln 8
(c) ln 4 (d) None of these

3. The equation of a curve is $y = f(x)$. The tangents at $(1, f(1))$, $(2, f(2))$ and $(3, f(3))$ make angles $\frac{\pi}{6}, \frac{\pi}{3}$ and $\frac{\pi}{4}$ respectively with positive direction of x-axis, then the value of $\int_2^3 f'(x)f''(x)dx + \int_1^3 f''(x)dx$ is equal to

(a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$
(c) 0 (d) None of these

4. Let $f(x)$ be a function defined by $f(x) = \int_1^x t(t^2 - 3t + 2)dt$, $1 \le x \le 3$. Then the range of $f(x)$ is

(a) [0, 2] (b) $\left[-\frac{1}{4}, 4\right]$
(c) $\left[-\frac{1}{4}, 2\right]$ (d) None of these

5. All the values of 'a' for which $\int_1^2 (a^2 + (4-4a)x + 4x^3)dx \le 12$ are given by

(a) $a = 3$ (b) $a \le 4$
(c) $0 \le a \le 3$ (d) None of these

6. Evaluate $\int \frac{(1+x)\sin x}{(x^2+2x)\cos^2 x - (1+x)\sin 2x}dx$

7. Show that the value of $\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}dx$, $n \in N$, is independent of n. Find the value of the integral.

8. Find the orthogonal trajectories of the family of circles having their centres on the y-axis and touching the x-axis.

9. Find the area bounded by $x = \cos^{-1}y$, x-axis and the lines $|x| = 1$.

10. Show that the solution of the differential equation $yp^2 - p = (p+2)(py-1)$, $p \equiv dy/dx$, represents two families, one of which is the family of curves having subnormal unity and the other is the family of parallel lines.

SOLUTIONS

1. **(b)** : $\int_1^2 [3x]dx = \int_1^{4/3} [3x]dx + \int_{4/3}^{5/3} [3x]dx + \int_{5/3}^{2} [3x]dx$

$= \int_1^{4/3} 3dx + \int_{4/3}^{5/3} 4dx + \int_{5/3}^{2} 5dx = 4$

2. **(c)** : Put $e^u - 1 = t^2$. Then $e^u du = 2tdt$, so that

$\frac{\pi}{6} = \int_1^{\sqrt{e^x-1}} \frac{2tdt}{t(t^2+1)} = 2\left[\tan^{-1}t\right]_1^{\sqrt{e^x-1}}$

$= 2\tan^{-1}\sqrt{e^x-1} - \frac{\pi}{2}$

$\Rightarrow \tan^{-1}\sqrt{e^x-1} = \frac{\pi}{3} \Rightarrow \sqrt{e^x-1} = \tan\frac{\pi}{3} = \sqrt{3}$

$\Rightarrow e^x = 4 \therefore x = \ln 4.$

3. **(a)** : Here $f'(1) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}, f'(2) = \tan\frac{\pi}{3} = \sqrt{3},$

$f'(3) = \tan\frac{\pi}{4} = 1$

By : Prof. Shyam Bhushan, Director, Narayana Institute, Jamshedpur.

Now, $\int_2^3 f'(x).f''(x)dx = \left[\frac{1}{2}\{f'(x)\}^2\right]_2^3$
$= \frac{1}{2}[\{f'(3)\}^2 - \{f'(2)\}^2]$

and $\int_1^3 f''(x)dx = [f'(x)]_1^3 = f'(3) - f'(1)$

$\therefore$ Value$= \frac{1}{2}[1^2 - (\sqrt{3})^2] + 1 - \frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

4. (c) : $f'(x) = x(x^2 - 3x + 2) = x(x-1)(x-2)$.

The sign scheme for $f'(x)$ is as below.

$-\quad +\quad -\quad +$
$0\quad 0\quad 2$

$\therefore$ $f'(x) \le 0$ in $1 \le x < 2$ and $f'(x) \ge 0$ in $2 \le x \le 3$.
$\therefore$ $f(x)$ is decreasing in [1, 2] and increasing in [2, 3]

$\therefore$ $\min f(x) = f(2) = \int_1^2 x(x^2 - 3x + 2)dx$
$= \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2 = -\frac{1}{4}$

max. $f(x)$ = the greatest among $\{f(1), f(3)\}$.

$f(1) = \int_1^1 x(x^2 - 3x + 2)dx = 0$, $f(3) = \int_1^3 x(x^2 - 3x + 2)dx = 2$

$\therefore$ $\max f(x) = 2$, so the range $= \left[-\frac{1}{4}, 2\right]$

5. (a) : $\int_1^2 (a^2 + (4 - 4a)x + 4x^3)dx$
$= a^2[x]_1^2 + (2 - 2a)[x^2]_1^2 + [x^4]_1^2 = a^2 + (2 - 2a)(3) + 15$
Given $a^2 - 6a + 21 \le 12 = a^2 - 6a + 9 \le 0 \Rightarrow (a-3)^2 \le 0$
$(a-3)^2 = 0 \Rightarrow a = 3$

6. Now $\int (1+x)\sin x\,dx = \sin x - (1+x)\cos x + C$

$\therefore$ $I = \int \frac{(1+x)\sin x}{[\sin x - (1+x)\cos x]^2 - 1}dx$
$= \frac{1}{2}\ln\left|\frac{\sin x - (1+x)\cos x - 1}{\sin x - (1+x)\cos x + 1}\right| + C$

7. Let $I_n = \int_0^\pi \frac{\sin\left(n + \frac{1}{2}\right)x\,dx}{\sin\frac{x}{2}}$

Then $I_{n+1} - I_n = \int_0^\pi \frac{\sin\left(n + \frac{1}{2}\right)x - \sin\left(n + \frac{3}{2}\right)x}{\sin\frac{x}{2}}dx$

$= -2\int_0^\pi \cos(n+1)x.dx = -\frac{2}{n+1}.[\sin(n+1)x]_0^\pi = 0$

Hence $I_0 = I_1 = I_2 = I_3 = \ldots\ldots$ Thus $I_n = I_0 = \int_0^\pi dx = \pi$.

8. The family of circles is given by $x^2 + y^2 - ay = 0$, $a \in R$ ......(1)

Differentiating w.r.t. x, we have
$$2x + 2y\frac{dy}{dx} - a\frac{dy}{dx} = 0 \Rightarrow a = 2\left(x\frac{dx}{dy} + y\right)$$

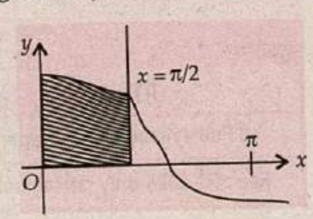
Putting this value in (1), we have
$$x^2 + y^2 - 2\left(x\frac{dx}{dy} + y\right)y = 0 \Rightarrow x^2 - y^2 - 2xy\frac{dx}{dy} = 0$$

For the orthogonal trajectories we replace

$\left(\frac{dy}{dx}\right)$ by $\left(-\frac{dx}{dy}\right)$, to get
$$x^2 - y^2 + 2xy\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

This is a homogeneous differential equation in x and y.
Putting $y = Vx$, we have
$$V + x\frac{dV}{dx} = \frac{V^2x^2 - x^2}{2x.Vx} = \frac{V^2 - 1}{2V}$$
$$\Rightarrow \quad x\frac{dV}{dx} = \frac{V^2 - 1 - 2V^2}{2V} = -\frac{1 + V^2}{2V}$$
$$\Rightarrow \quad \frac{dx}{x} = -\frac{2V}{1 + V^2}dV$$
$\Rightarrow$ $\log|x| + \log(1 + V^2) =$ constant
$\Rightarrow$ $|x|(1 + y^2/x^2) =$ constant $= b$ (say)
Which is required orthogonal trajectories

9. $x = \cos^{-1}y$
$\Rightarrow y = \cos x$, $x \in [0, \pi]$.
The required area (shaded portion) is shown in the adjacent figure.



Required area $= \int_0^1 \cos x\,dx = \sin x|_0^1 = \sin 1$ sq.units.

10. The given differential equation can be re-written as
$$\frac{dy}{dx}\left(y\frac{dy}{dx} - 1\right) = \left(\frac{dy}{dx} + 2\right)\left(y\frac{dy}{dx} - 1\right)$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dx} + 2 \text{ or } y\frac{dy}{dx} - 1 = 0 \Rightarrow \frac{dx}{dy} = 0 \text{ or } y\frac{dy}{dx} = 1$$
Now $\frac{dx}{dy} = 0 \Rightarrow x = c$, a family of parallel lines (lines parallel to y-axis)

$y\frac{dy}{dx} = 1 \Rightarrow$ the solution curve has subnormal unity.

The curves are $y^2 = 2x + k$, k is a constant.

$$=\frac{1}{2}\left|\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}\right| = |a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)|$$

The intersection of the tangents, at these points, are the points

$\{at_1t_2, a(t_1+t_2)\}$, $\{at_2t_3, a(t_2+t_3)\}$, $\{at_3t_1, a(t_3+t_1)\}$

The area of the Δ formed by these points

$$=\frac{1}{2}|a^2(t_1-t_2)(t_2-t_3)(t_3-t_1)|.$$

9. Find the locus of the point of intersection of two normals to a parabola which are at right angles to one another.

Soln.: The equation of the normal to the parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$

It passes through the point (h, k) if

$k = mh - 2am - am^3 \Rightarrow am^3 + m(2a-h) + k = 0$ ... (1)

Let the roots of the above equation be m_1, m_2 and m_3.

Let the perpendicular normals correspond to the values of m_1 and m_2 so that $m_1 m_2 = -1$.

From the equation (1), $m_1m_2m_3 = -\frac{k}{a}$

Since $m_1 m_2 = -1, m_3 = \frac{k}{a}$

Since m_3 is a root of (1), we have

$$a\left(\frac{k}{a}\right)^3 + \frac{k}{a}(2a-h) + k = 0$$

$\Rightarrow k^2 + a(2a-h) + a^2 = 0 \Rightarrow k^2 = a(h-3a)$

Hence the locus of (h, k) is $y^2 = a(x-3a)$.

10. Three normals from a point to the parabola $y^2 = 4ax$ meet the axis of the parabola in points whose abscissa are in A.P. Find the locus of the point.

Soln.: The equation of any normal to the parabola is

$y = mx - 2am - am^3$

It passes through the point (h, k) if

$am^3 + m(2a-h) + k = 0$ ... (1)

The normal cuts the axis of the parabola viz., $y = 0$ at point where $x = 2a + am^2$

Hence the abscissa of the points in which the normals through (h, k) meet the axis of the parabola are $x_1 = 2a + am_1^2$, $x_2 = 2a + am_2^2$, $x_3 = 2a + am_3^2$

Since x_1, x_2, x_3 are in A.P.

$(2a + am_1^2) + (2a + am_3^2) = 2(2a + am_2^2)$

$\Rightarrow m_1^2 + m_3^2 = 2m_2^2$ ... (2)

Also, from (1), $m_1 + m_2 + m_3 = 0$, ... (3)

$m_2m_3 + m_3m_1 + m_1m_2 = \frac{2a-h}{a}$ ... (4)

and $m_1m_2m_3 = -\frac{k}{a}$ ... (5)

From (3),

$(m_1+m_3)^2 = m_2^2 \Rightarrow m_1^2 + m_3^2 + 2m_1m_3 = m_2^2$

$\Rightarrow 2m_2^2 - 2.\frac{k}{am_2} = m_2^2 \Rightarrow m_2^2 = \frac{2k}{am_2}$

$\Rightarrow am_2^3 = 2k$

Since m_2 is a root of (1), $am_2^3 + m_2(2a-h) + k = 0$

$\Rightarrow 2k + m_2(2a-h) + k = 0$

$\Rightarrow \{m_2(h-2a)\}^3 = 27k^3 \Rightarrow \frac{2k}{a}(h-2a)^3 = 27k^3$

$\Rightarrow 27ak^2 = 2(h-2a)^3$.

Hence the locus of (h, k) is $27ay^2 = 2(x-2a)^3$.

MOCK TEST PAPER

ISI 2010

Indian Statistical Institute

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PART - A

Multiple Choice Question type

1. On a standard die one of the dots is removed at random with each dot equally likely to be chosen. The die is then rolled. What is the probability that the top face has an odd number of dots?

(a) $\frac{5}{11}$ (b) $\frac{10}{21}$ (c) $\frac{11}{21}$ (d) $\frac{6}{11}$

2. Three circles of radius s are drawn in the first quadrant of the xy - plane. The first circle is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis. A circle of radius $r > s$ is tangent to both axes and to the second and third circles. What is r/s?

(a) 5 (b) 6 (c) 8 (d) 9

3. The equiangular convex hexagon $ABCDEF$ has $AB = 1$, $BC = 4$, $CD = 2$, and $DE = 4$. The area of the hexagon is

(a) $\frac{15\sqrt{3}}{2}$ (b) 16 (c) $\frac{39\sqrt{3}}{4}$ (d) $\frac{43\sqrt{3}}{4}$

4. A point P is chosen at random in the interior of an equilateral triangle ABC. What is the probability that ΔABP has a greater area than each of ΔACP and ΔBCP?

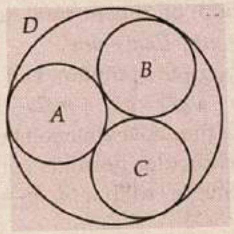
(a) $\frac{1}{6}$ (b) $\frac{1}{4}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

5. How many 15-letter arrangements of 5 A's, 5 E's, and 5 C's have no A's in the first 5 letters, no B's in the next 5 letters, and no C's in the last 5 letters?

(a) $\sum_{k=10}^{5}\binom{5}{k}^3$ (b) $3^5.2^5$

(c) 2^{15} (d) $\frac{15!}{(5!)^3}$

6. Circles A, B and C are externally tangent to each other and internally tangent to circles D, circles B and C are congruent. Circle A has radius 1 and passes, through the centre of D. What is the radius of circle B?



(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{7}{8}$

(c) $\frac{8}{9}$ (d) $\frac{1+\sqrt{3}}{3}$

7. Three points are chosen randomly and independently on a circle. What is the probability that all three pairwise distances between the points are less than the radius of the circle?

(a) $\frac{1}{36}$ (b) $\frac{1}{18}$ (c) $\frac{1}{12}$ (d) $\frac{1}{9}$

8. Call a number "prime - looking" if it is composite but not divisible by 2, 3 or 5. The three smallest prime - looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime - looking numbers are there less than 1000?

(a) 100 (b) 102 (c) 106 (d) 108

9. In triangle ABC, side AC and the perpendicular bisector of BC meet in point D, and BD bisects $\angle ABC$. If $AD = 9$ and $DC = 7$, what is the area of triangle ABD?

(a) 14 (b) 28 (c) $14\sqrt{5}$ (d) $28\sqrt{5}$

10. A circle centered at A with a radius of 1 and a circle centered at B with a radius of 4 are externally tangent. A third circle is tangent to the first two and to one of their common external tangents as shown. The radius of the third circle is

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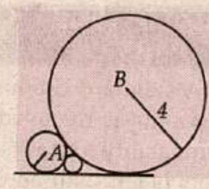


(a) $\frac{1}{3}$ (b) $\frac{2}{5}$ (c) $\frac{5}{12}$ (d) $\frac{4}{9}$

11. Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?

(a) $3+\frac{\sqrt{30}}{2}$ (b) $3+\frac{\sqrt{69}}{3}$

(c) $\frac{3+\sqrt{123}}{4}$ (d) $\frac{52}{9}$

12. All three vertices of an equilateral triangle are on the parabola $y = x^2$, and one of its sides has a slope of 2. The x-coordinates of the three vertices have a sum of m/n where m and n are relatively prime positive integers. What is the value of $m + n$?

(a) 14 (b) 15 (c) 16 (d) 17

13. The number of pairs of positive integers (m, n) with $m < n$ such that the difference between the mth and nth triangle numbers is equal to 2006, is

(a) 1 (b) 2 (c) 3 (d) 4

14. The 2007th term of the Arithmetical progression whose first three terms are $x, |x + 1|$ and $|x - 1|$ can be

(a) $\frac{4013}{2}$ (b) $\frac{8021}{2}$ (c) $\frac{8023}{2}$ (d) $\frac{9011}{2}$

15. The number of integers (m, n) such that $m^2 + (m + 1)^2 = n^4 + (n + 1)^4$ is

(a) exactly one (b) exactly two

(c) infinitely many (d) none of these

16. The number of integers 'n' such that $\frac{n^3-1}{5}$ is a prime number is

(a) exactly one (b) exactly three

(c) exactly five (d) none of these

17. An ordered triplet (a, b, c) where a, b, c are non zero reals is said to be good if each of a, b, c is the product of the other two. The number of such good triplets is

(a) 1 (b) 2 (c) 3 (d) 4

18. The number of values of x for which the numbers $\frac{14x+5}{9}$ and $\frac{17x-5}{12}$ both be integers is

(a) exactly one (b) exactly three

(c) infinitely (d) none of these

19. The number of ordered pair solutions (x, y) in integers of the equation $x^2 + 2y^2 = x^2y^2 - 2000$, is

(a) 4 (b) 6 (c) 8 (d) 10

20. Let x_1, x_2 be the roots of the equation $x^2+px-\frac{1}{2p^2}=0$ where x is the unknown and p is a real parameter. Then the least value of $x_1^4 + x_2^4$ is

(a) $2+\sqrt{2}$ (b) $2+2\sqrt{2}$

(c) $4+\sqrt{2}$ (d) $4+2\sqrt{2}$

21. The number of positive integral values of 'n' for which $(n^3 - 8n^2 + 20n - 13)$ is a prime number, is

(a) exactly one (b) exactly three

(c) exactly five (d) none of these

22. Given a system of equations $a + b = 2m^2$, $b + c = 6m$, $a + c = 2$ where 'm' is a real number. The range of values of m such that $a \le b \le c$, is given by

(a) $\frac{2}{3}\le m\le 2$ (b) $\frac{1}{3}\le m\le 1$

(c) $\frac{1}{3}\le m\le 2$ (d) $\frac{2}{3}\le m\le 1$

23. Let 'n' be an odd integer. Then the number $(n^3 + 3n^2 - n - 3)$ is

(a) always divisible by 48 for all n

(b) not divisible by 24 for some n

(c) always divisible by 96 for all n

(d) not divisible by 96 for any n

24. The number of pairs (x, y) of positive integers such that $x^3 - y^3 = xy + 61$,, is

(a) exactly one (b) exactly three

(c) infinitely many (d) none of these

25. If $ax^2 = by^2 = cz^2$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$.

Then $\sqrt[3]{ax^2+by^2+cz^2}$ equals

(a) $\sqrt{a}+\sqrt{b}+\sqrt{c}$ (b) $\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}$

(c) $\sqrt[3]{a^2}+\sqrt[3]{b^2}+\sqrt[3]{c^2}$ (d) $\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}$

26. Let $a, b > 0$ satisfy $4\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)-20\left(\frac{a}{b}+\frac{b}{a}\right)+33\le 0$. Then $\frac{\max\{a,b\}}{\min\{a,b\}}$ equals

(a) 1 (b) 2 (c) 3 (d) 4

27. A circle of raduis 5 cm passes through two adjacent vertices of a square. A tangent to a circle drawn from a third vertex of the square is twice

the length of the side of the square. The side of the square is

(a) $\sqrt{10}$ (b) $\sqrt{5}$ (c) $2\sqrt{10}$ (d) $2\sqrt{5}$

28. The number of negative roots of the equation $x^4 - x^3 - 6x^2 - 2x + 9 = 0$ is

(a) 0 (b) 1 (c) 2 (d) 3

29. The number of pairs (x, y) where x, y are integers such that $x^3 + 11^3 = y^3$ is

(a) 2 (b) 4 (c) 6 (d) 8

30. The number of solutions of $|[x] - 2x| = 4$, where $[x]$ is the greatest integer function, is

(a) 1 (b) 2 (c) 3 (d) 4

PART - B

1. Determine all triplets (x, y, z) of positive integers which are solutions of $2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4 = 576$

2. Find a point P inside a triangle ABC such that the product of its distances from the sides is a maximum.

3. a, b, c, d are real numbers such that $a^2 + b^2 + (a - b)^2 = c^2 + d^2 + (c - d)^2$, show that $a^4 + b^4 + (a - b)^4 = c^4 + d^4 + (c - d)^4$,

4. Find all triplets (x, y, z) of positive integers satisfying $2^x + 2^y + 2^z = 2336$.

5. Inside a unit square, all isosceles triangles whose base is a side of the square, and whose vertex is the midpoint of the opposite side are drawn. Find the area of the octagon determined by the intersection of these four triangles.

6. How to inscribe a square in a given triangle so that one side may lie along a side of the triangle.

7. (i) Determine the set of positive integers n for which 3^{n+1} divides $2^{3^n} + 1$.

(ii) Prove that 3^{n+2} does not divide $2^{3^n} + 1$ for any positive integer n.

8. Let 'a' be a real number such that $a^5 - a^3 + a = 2$. Prove that $3 < a^6 < 4$.

9. Suppose a and b are two positive real numbers such that the roots of the cubic equation $x^3 - ax + b = 0$ are all real. If α is a root of this cubic with minimal absolute value prove that $\frac{b}{a} < \alpha \le \frac{3b}{2a}$.

10. Let ABC be a triangle and h_a be the altitude through A. Prove that $(b + c)^2 \ge a^2 + 4h_a^2$.

(As usual a, b, c denote the sides BC, CA, AB respectively.)

SOLUTIONS
PART - A

1. **1st Solution :** A standard die has a total of 21 dots. For $1 \le n \le 6$, a dot is removed from the face with n dots with probability $n/21$. Thus the face that originally has n dots is left with an odd number of dots with probability $n/21$ if n is even and $1 - n/21$ if n is odd. Each face is the top face with probability 1/6. Therefore the top face has an odd number of dots with probability

$$\frac{1}{6}\left(\left(1-\frac{1}{21}\right)+\frac{2}{21}+\left(1-\frac{2}{21}\right)+\frac{4}{21}+\left(1-\frac{5}{12}\right)\right)$$

$$=\frac{1}{6}\left(3+\frac{3}{21}\right)=\frac{1}{6}\cdot\frac{66}{21}=\frac{11}{21}$$

2nd Solution :

The probability that the top face is odd is 1/3 if a dot is removed from an odd face, and the probability that the top face is odd is 2/3 if a dot is removed from an even face. Because each dot has the probability 1/21 of being removed, the top face is odd with probability

$$\left(\frac{1}{3}\right)\left(\frac{1+3+5}{21}\right)+\left(\frac{2}{3}\right)\left(\frac{2+4+6}{21}\right)=\frac{33}{63}=\frac{11}{21}$$

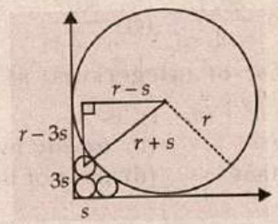
2. Consider a right triangle as shown. By the Pythagorean Theorem,

$(r + s)^2 = (r - 3s)^2 + (r - s)^2$

so $r^2 + 2rs + s^2 = r^2 - 6rs + 9s^2 + r^2 - 2rs + s^2$

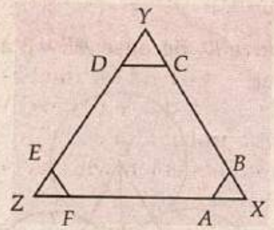
and $0 = r^2 - 10rs + 9s^2 = (r - 9s)(r - s)$

But $r \ne s$, so $r = 9s$ and $r/s = 9$



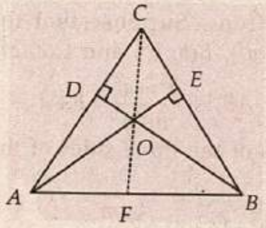
Alternatively, Because the ratio r/s is independent of the value of s, assume that s = 1 and proceed as in the previous solution.

3. **(c) :** Extend FA and CB to meet at X, BC and ED to meet at Y, and DE and AF to meet at Z. The interior angles of the hexagon are 120°. Thus the triangles XYZ, ABX, CDY and EFZ are equilateral. Since $AB = 1$, $BX = 1$, Since $CD = 2$, $CY = 2$. Thus $XY = 7$ and $YZ = 7$. Since $YD = 2$ and $DE = 4$, $EZ = 1$. The area of the hexagon can be found by subtracting the areas of the three small triangles from the area of the large triangle.



$$7^2\left(\frac{\sqrt{3}}{4}\right)-1^2\left(\frac{\sqrt{3}}{4}\right)-2^2\left(\frac{\sqrt{3}}{4}\right)-1^2\left(\frac{\sqrt{3}}{4}\right)=\frac{43\sqrt{3}}{4}$$

4. **1st Solution** : Since the three triangles ABP, ACP, and BCP have equal bases, their areas are proportional to the length of their altitudes.
Let O be the centroid of ΔABC, and draw medians AOE and BOD. Any point above BOD will be farther from AB than from BC, and any point above AOE will be farther from AB than from AC. Therefore the condition of the problem is met if and only if P is inside the quadrilateral $CDOE$.



If CO is extended to F on AB, then ΔABC is divided into six congruent triangles, of which two comprise quadrilateral $CDOE$. Thus $CDOE$ has one - third the area of ΔABC, so the required probability is 1/3.

2nd Solution:

By symmetry, each of ΔABP, ΔACP, and ΔBCP is largest with the same probability, so the probability must be 1/3 for each.

5. **(a)** : Since the first group of five letters contains no A's it must contain k B's and $(5-k)$C's for some integer k with $0 \le k \le 5$. Since the third group of five letters contains no C's, the remaining k C's must be in the second group, along with $(5-k)$ A's. Similarly, the third group of five letters must contain kA's and $(5-k)$B's. Thus each arrangement that satisfies the conditions is determined uniquely by the location of the kB's in the first group, the kC's in the second group, and the kA's in the third group.

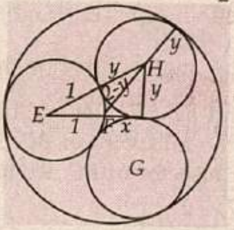
For each k, the letters can be arranged in $\binom{5}{k}^3$ ways,

so the total number of arrangements is $\sum_{k=0}^{5}\binom{5}{k}^3$.

6. Let E, H and F be the centres of circles A, B and D, respectively, and let G be the point of tangency of circle B and C. Let $x = FG$ and $y = GH$. Since the centre of circle D lies on circle A and the circles have a common point of tangency, the radius of circle D is 2 which is the diameter of circle A. Applying the Pythagorean theorem to right triangles EGH and FGH gives

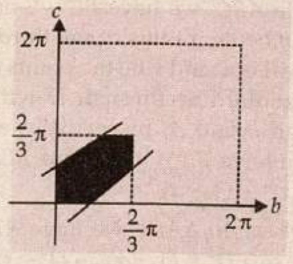
$(1+y)^2 = (1+x)^2 + y^2$ and $(2-y)^2 = x^2 + y^2$

From which it follows that $y = x + \frac{x^2}{2}$ and $y = 1 - \frac{x^2}{4}$



The solutions of this system are $(x, y) = (2/3, 8/9)$ and $(x, y) = (-2, 0)$. The radius of circle B is the positive solution for y which is 8/9.

7. We can assume that the circle has its centre at $(0, 0)$ and a radius of 1. Call the three points A, B and C, and let a, b, and c denote the length of the counterclockwise arc from $(1, 0)$ to A, B and C, respectively. Rotating the circle if necessary, we can also assume that $a = \frac{\pi}{3}$. Since b and c are chosen at random from $[0, 2\pi)$, the ordered pair (b, c) is chosen at random from a square with area $4\pi^2$ in the bc - plane. The condition of the problem is met if and only if $0 < b < \frac{2\pi}{3}, 0 < c < \frac{2\pi}{3}$ and $|b-c| < \frac{\pi}{3}$



This last inequality is equivalent to $b - \frac{\pi}{3} < c < b + \frac{\pi}{3}$

The graph of the common solution to these inequalities is the shaded region shown. The area of this region is

$\left(\frac{6}{8}\right)\left(\frac{2\pi}{3}\right)^2 = \frac{\pi^2}{3}$ so, the required probability is

$\frac{\pi^2/3}{4\pi^2} = \frac{1}{12}$.

8. Of the numbers less than 1000 – 499 of them are divisible by two, 333 are divisible by 3, and 199 are divisible by 5. There are 166 multiples of 6, 99 multiples of 10, and 66 multiples of 15. And there are 33 numbers that are divisible by 30. So by the Inclusion - Exclusion Principle there are

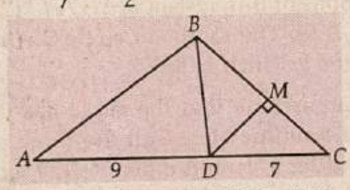
$499 + 333 + 199 - 166 - 99 - 66 + 33 = 733$

numbers that are divisible by at least one of 2, 3 or 5. Of the remaining $999 - 733 = 266$ numbers, 165 are primes other than 2, 3, or 5. Note that 1 is neither prime nor composite. This leaves exactly 100 prime looking numbers.

9. **(d)** : By the angle - bisector theorem, $\frac{AB}{BC}=\frac{9}{7}$.

Let $AB = 9x$ and $BC = 7x$, let $m\angle ABD = m\angle CBD = \theta$, and let M be the midpoint of BC. Since M is on the perpendicular bisector of BC, we have $BD = DC = 7$.

Then $\cos\theta=\frac{7x/2}{7}=\frac{x}{2}$



Using the law of cosine to ΔABD yields

$$9^2 = (9x^2) + (7)^2 - 2(9x)(7)\left(\frac{x}{2}\right)$$

From which $x = 4/3$ and $AB = 12$. Apply Heron's formula to obtain the area of triangle ABD as $\sqrt{14.2.5.7} = 14\sqrt{5}$.

10. **(d)** : Let D be the intersection of the horizontal line through A and the vertical line through B. In right triangle ABC, we have $BC = 3$ and $AB = 5$, so $AC = 4$, Let x be the radius of the third circle, and D be the centre. Let E and F be the points of intersection of the horizontal line through D with the vertical lines through B and A, respectively, as shown.

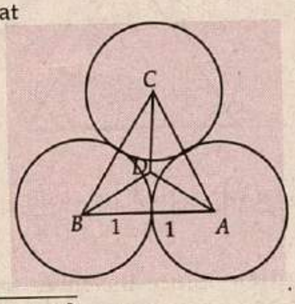
In ΔBED we have $BD = 4 + x$ and $BE = 4 - x$, so $DE^2 = (4 + x)^2 - (4 - x)^2 = 16x$.

and $DE = 4\sqrt{x}$. In ΔADF we have $AD = 1 + x$ and $AF = 1 + x$, so $FD^2 = (1 + x)^2 - (1 - x)^2 = 4x$

and $FD = 2\sqrt{x}$. Hence, $4 = AC = FD + DE$

$= 2\sqrt{x} + 4\sqrt{x} = 6\sqrt{x}$ and $\sqrt{x}=\frac{2}{3}$, which implies $x=\frac{4}{9}$.

11. **(b)** : Let A, B, C and E be the centres of the three small spheres and the large sphere, respectively. Then ΔABC is equilateral with side length 2. If D is the intersection of the medians of ΔABC, then E is directly above D. Because $AE = 3$ and $AD = \frac{2\sqrt{3}}{3}$ it follows that



$$DE=\sqrt{3^2-\left(\frac{2\sqrt{3}}{3}\right)^2}=\frac{\sqrt{69}}{3}$$

Because D is 1 unit above the plane and the top of the larger sphere is 2 units above E. The distance from the plane to the top of the larger sphere is $3+\frac{\sqrt{69}}{3}$.

12. **1st Solution** : Suppose that the triangle has vertices $A(a, a^2)$, $B(b, b^2)$ and $C(c, c^2)$. The slope of line segment AB is $\frac{b^2-a^2}{b-a}=b+a$

so the slopes of the three sides of the triangle have a sum

$$(b+a)+(c+b)+(a+c)=2.\frac{m}{n}$$

The slope of one side is $2 = \tan\theta$, for some angle θ, and the two remaining sides have slopes.

$$\tan\left(\theta\pm\frac{\pi}{3}\right)=\frac{\tan\theta\pm\tan(\pi/3)}{1\mp\tan\theta\tan(\pi/3)}=\frac{2\pm\sqrt{3}}{1\mp2\sqrt{3}}=-\frac{8\pm5\sqrt{3}}{11}$$

Therefore

$$\frac{m}{n}=\frac{1}{2}\left(2-\frac{8+5\sqrt{3}}{11}-\frac{8-5\sqrt{3}}{11}\right)=\frac{3}{11} \text{ and } m+n=14$$

Such a triangle exists. The x - coordinates of its vertices are $\frac{(11\pm5\sqrt{3})}{11}$ and $\frac{-19}{11}$.

2nd Solution :

Define the vertices as in the first solution, with the added stipulations that $a < b$ and AB has slope 2.

Then $2=\frac{b^2-a^2}{b-a}=b+a$, so $a=1-k$ and $b=1+k$

For some $k > 0$. If D is the midpoint of AB, then

$$D=\left[1,\frac{(1-k)^2+(1+k)^2}{2}\right]=(1,1+k^2)$$

The slope of the altitude CD is $-1/2$, so

$1 - c = 2(c^2 - 1 - k^2)$

$$CD^2=(1-c)^2+(c^2-1-k^2)^2=\frac{5}{4}(1-c)^2$$

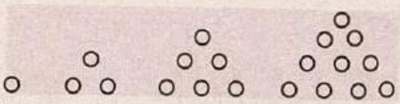
Because ΔABC is equilateral, we also have

$$CD^2=\frac{3}{4}AB^2=\frac{3}{4}(2k)^2+(4k)^2=15k^2$$

Hence $\frac{5}{4}(1-c)^2=15k^2, \text{so } k^2=\frac{(1-c)^2}{12}$

Substitution into the equation $1-c=2(c^2-1-k^2)$ yields $c=1$ or $c=-19/11$, Because $c<1$, it follows that $a+b+c=2-\frac{19}{11}=\frac{3}{11}=\frac{m}{n}$, so $m+n=14$.

13. The numbers 1, 3, 6, 10, 15,....... are called triangular numbers because they can be visualized as below



It is easily seen that

$1=1$

$3=1+2$

$6=1+2+3$ etc

n^{th} triangular number is $1+2+3......+n=\frac{n(n+1)}{2}$

we have $\frac{m(m+1)}{2}-\frac{n(n+1)}{2}=2006$

$\Rightarrow m(m+1)-n(n+1)=4012$

$\Rightarrow (m-n)(m+n+1)=4012=2^2\times 17\times 59$

$\therefore (m-n)(m+n+1)=1\times 4012$ or 4×1003 or 17×236 or 59×68.

$\Rightarrow (m, n)=(2006, 2005), (503, 499), (126, 109), (63, 4)$.

14. $x, |x+1|, |x-1|$ are in A.P.

When $-1\le x\le 1$ we have $x, x+1, 1-x$ are in AP.

$(2x+1)=1-x+x$

$x+1=\frac{1}{2}\ \Rightarrow x=-\frac{1}{2}$

The first three terms are $-\frac{1}{2},\frac{1}{2},\frac{3}{2}$. Common difference is 1.

$$t_{2007}=-\frac{1}{2}+(2006).1=2006-\frac{1}{2}=\frac{4012-1}{2}=\frac{4011}{2}$$

when $x<-1$

$x, -(x+1), 1-x$ are in A.P.

$-2x-2=1-x+x\ \Rightarrow -3=2x\Rightarrow\ x=-\frac{3}{2}$

The first three terms are $-\frac{3}{2},\frac{1}{2},\frac{5}{2}$

$$t_{2007}=-\frac{3}{2}+(2006)(2)$$

$\therefore$ (Here 2 is the common difference.)

$$=4012-\frac{3}{2}=\frac{8024-3}{4}=\frac{8021}{2}.$$

15. **(d)** : Given $m^2+(m+1)^2=n^4+(n+1)^4$

$m^2+m^2+2m+1=n^4+n^4+4n^3+6n^2+4n+1$

$\Rightarrow 2m^2+2m=2n^4+4n^3+6n^2+4n$

$\Rightarrow m^2+m=n^4+2n^3+3n^2+2n$

$m^2+m+1=n^4+2n^3+3n^2+2n+1$

$=(n^2+n+1)^2$

RHS is a perfect square. Let us prove that LHS cannot be a perfect square for any integer m.

One way of proving it is to establish that m^2+m+1 lies between two successive squares

$m^2<m^2+m+1<m^2+2m+1$

(*i.e.,*) $m^2<m^2+m+1<(m+1)^2$

$\therefore m^2+m+1$ cannot be a square.

Therefore these cannot be integers m, n such that the equation holds.

16. **(a)** : $\frac{n^3-1}{5}=\frac{(n-1)(n^2+n+1)}{5}$. This must be a prime number.

Since it must be an integer first. $(n-1)(n^2+n+1)$ must be divisible by 5.

Let $n-1=5k$ (*i.e.,*) $n=5k+1$

$n^2+n+1=(5k+1)^2+(5k+1)+1$

$=25k^2+10k+1+5k+1+1$

$=25k^2+15k+2$

When $k=1, n-1=5, n^2+n+1=43$

$\therefore \frac{n^2-1}{5}=\frac{5\times 43}{5}=43$ which is a prime.

Then $\frac{n^2-1}{5}=k(n^2+n+1)$ becomes composite (Note the presence of k)

$\therefore n=6$ is the only value.

17. **(c)** : As per the given condition $a=bc$, $b=ca$, $c=ab$

$\Rightarrow abc=(abc)^2\Rightarrow (abc)^2-(abc)=0$

$(abc)[abc-1]=0$

Since a, b, c are non-zero, have $abc=1$.

$\Rightarrow bc=\frac{1}{a}$ and $bc=a$

$\Rightarrow a=\frac{1}{a}\Rightarrow a=\pm 1, b=\pm 1, c=\pm 1.$

The triplets are (1, 1, 1), (1, –1, –1), (–1, 1, –1), (–1, –1, 1). But (–1, –1, –1) will not satisfy.

18. **(d)** : $\frac{14x+5}{9}$ will be an integer if $(14x+5)$ is a multiple of 9. When a number is a multiple of 9, then it is a multiple of 3 also.

$14x+5=12x+2x+5=(12x+3)+(2x+2)$

$(12x+3)$ is a multiple of 3. $\therefore (2x+2)$ must be a multiple of 3.

$\frac{17x-5}{12}$ is an integer means $17x - 5$ is a multiple of 12 or a multiple of 3 and 4.
$17x - 5 = (15x - 6) + (2x + 1)$
$15x - 6$ is a multiple of 3 $\Rightarrow (2x + 1)$ must be a multiple of 3.
$2x + 1$ and $2x + 2$ must be multiple of 3.
But $2x + 1$ and $2x + 2$ are consecutive integers. They cannot be multiples of 3 simultaneously.

There are no integer x for which $\frac{14x+5}{9}$ and $\frac{14x-5}{12}$ are both integers.

19. The equation is $x^2 + 2y^2 = x^2y^2 - 2000$
The first two terms can be written as $x^2(y^2 - 1)$
But the third term is not cooperating to take anything common.
If $a(y^2 - 1)$ factor is present there is some hope.
We can achieve this by adding 2 on both sides.
$x^2y^2 - x^2 - 2y^2 + 2 = 2002$
$x^2(y^2 - 1) - 2(y^2 - 1) = 2002$
$\Rightarrow (x^2 - 2)(y^2 - 1) = 2002$
We want integers x, y.
$(x^2 - 2)(y^2 - 1) = 1 \times 2002$ or 2×1001 or 7×286 or 11×182 or 13×154 or 14×143 or 22×91 or 26×77, we get $x = \pm 4$, $y = \pm 12$.
The solutions are $(4, 12), (-4, 12), (4, -12), (-4, -12)$.

20. (a) : x_1, x_2 are the roots of $x^2 + px - \frac{1}{2p^2} = 0$

$$x_1 + x_2 = -p,\ x_1x_2 = -\frac{1}{2p^2}$$

Let us now find $x_1^4 + x_2^4$ in terms of p.

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = p^2 + \frac{1}{p^2}$$

$$x_1^4 + x_2^4 = (x_1^2 + x_2^2)^2 - 2x_1^2x_2^2$$

$$= \left(p^2 + \frac{1}{p^2}\right)^2 - 2\frac{1}{4p^4} = p^4 + \frac{1}{p^4} + 2 - \frac{1}{2p^4}$$

$$= p^4 + \frac{1}{2p^4} + 2 \qquad \text{.....(1)}$$

Now A.M. $\geq$ G.M. inequality applied to $p^4, \frac{1}{2p^4}$ gives

$$\frac{p^4 + \frac{1}{2p^4}}{2} \geq \sqrt{p^4 \frac{1}{2p^4}} \Rightarrow p^4 + \frac{1}{2p^4} \geq 2\sqrt{\frac{1}{2}}$$

$$\Rightarrow p^4 + \frac{1}{2p^4} \geq \sqrt{2} \text{ from (1)} \Rightarrow x_1^4 + x_2^4 \geq 2 + \sqrt{2}$$

21. (b) : We observe that when $n = 1$,
$n^3 - 8n^2 + 20n - 13 = 1 - 8 + 20 - 13 = 0$
$\therefore$ $(n - 1)$ is a factor of $n^3 - 8n^2 + 20n - 13$
Factorizing completely we get $(n - 1)(n^3 - 7n + 13)$ and if it is a prime, 1 must be a factor and the other must be a prime.
Let $n - 1 = 1 \Rightarrow n = 2 \Rightarrow$. The other factor is $4 - 14 + 13 = 3$, a prime.
When $n = 2$, $n^3 - 8n^2 + 20n - 13$ is a prime
Let $n^2 - 7n + 13 = 1 \Rightarrow n^2 - 7n + 12 = 0$
$\Rightarrow (n - 3)(n - 4) = 0$
$n = 3 \Rightarrow n^3 - 8n^2 + 20n - 13 = (n - 1)(n^2 - 7n + 13)$
$= 2 \times 1 = 2$ prime
$n = 4 \Rightarrow n^3 - 8n^2 + 20n - 13 = (n - 1)(n^2 - 7n + 13)$
$= 3 \times 1 = 3$ prime
For $n = 2, 3, 4$ the expression $(n^3 - 8n^2 + 20n - 13)$ is a prime.

22. (b) : From the given equations solve for a, b, c.
$a = m^2 - 3m + 1$, $b = m^2 + 3m - 1$, $c = -m^2 + 3m + 1$.
we have
$m^2 - 3m + 1 \leq m^2 + 3m - 1 \leq -m^2 + 3m + 1$

$$\Rightarrow \frac{1}{3} \leq m \leq 1$$

When $n = 1/3$ we get $a = b < c$
When $m = 1$ we get $a < b = c$.

23. As n is odd. Let $n = 2k - 1$.
$S = n^3 + 3n^2 - n - 3$
$= (2k - 1)^3 + 3(2k - 1)^2 - (2k - 1) - 3$
$= 8k^3 - 1 - 6k(2k - 1) + 3(4k^2 - 4k + 1) - 2k + 1 - 3$
$= 8k^3 - 1 - 12k^2 + 6k - 12k^2 + 3 - 2k + 1 - 3$
$= 8k^3 - 8k$
$= 8k(k^2 - 1) = 8k(k - 1)(k + 1)$
$= 8k(k - 1)k(k + 1)$
$k - 1$, k, $k + 1$ are consecutive natural numbers.
$\therefore$ one of them is divisible by 3 and atleast one of them is even so divisible by 2.
$\therefore$ It is divisible by 6.
It is of the form 6λ where λ is an integer.
$\therefore S = 8 \times 6\lambda = 48\lambda$.
S is divisible by 48.

24. (a) : Observe that, since RHS is positive LHS must be positive $\Rightarrow y < x$.
We have $x^3 - y^3 - xy = 61$.
$\Rightarrow (x - y)(x^2 + xy + y^2) - xy = 61$
$(x - y)(x^2 + xy + y^2) + x^2 + y^2 - xy - x^2 - y^2 = 61$
$\Rightarrow (x - y - 1)(x^2 + xy + y^2) + x^2 + y^2 = 61$ .....(1)
Now, $y < x \Rightarrow (x - y - 1)(x^2 + xy + y^2) \geq 0$.
$\Rightarrow x^2 + y^2 \Rightarrow 61 \Rightarrow x \leq 6$, $y \leq 5$ with $y < x$, $x = 6$, $y = 5$ is a solution.

25. (c) : Given $ax^2 = by^2 = cz^2$, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

Let us consider $\sqrt[3]{ax^2 + by^2 + cz^2}$

$$= \sqrt[3]{\frac{ax^3}{x} + \frac{by^3}{y} + \frac{cz^3}{z}} = \sqrt[3]{\frac{ax^3}{x} + \frac{by^3}{y} + \frac{cz^3}{z}}$$

$$= \sqrt[3]{ax^3\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)} = x(a)^{1/3}$$

Thus if we take $K = \sqrt[3]{ax^2+by^2+cz^2}$

$K = xa^{1/3}$

Similarly $K = ya^{1/3}$, $K = zc^{1/3}$

$$\Rightarrow \frac{1}{x} = \frac{1}{K}a^{1/3}, \frac{1}{y} = \frac{1}{K}b^{1/3}, \frac{1}{z} = \frac{1}{k}c^{1/3}$$

Given $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1.$

$$\Rightarrow \frac{1}{K}(a^{1/3}+b^{1/3}+c^{1/3}) = 1 \Rightarrow K = \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}$$

$$\therefore \sqrt[3]{ax^2+by^2+cz^2} = \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}.$$

26. $a > 0, b > 0$ and $4\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right) - 20\left(\frac{a}{b}+\frac{b}{a}\right) + 33 \le 0$

$$4\left(\left(\frac{a}{b}+\frac{b}{a}\right)^2 - 2\right) - 20\left(\frac{a}{b}+\frac{b}{a}\right) + 33 \le 0$$

$$\Rightarrow 4\left(\frac{a}{b}+\frac{b}{a}\right)^2 - 20\left(\frac{a}{b}+\frac{b}{a}\right) + 25 \le 0$$

$$\Rightarrow \left(2\left(\frac{a}{b}+\frac{b}{a}\right) - 5\right)^2 \le 0$$

The square of a real number cannot be negative.

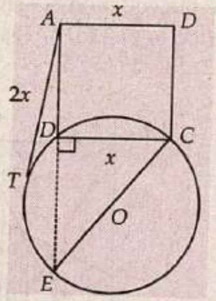
$$\Rightarrow 2\left(\frac{a}{b}+\frac{b}{a}\right) - 5 = 0 \quad \Rightarrow 2\left(\frac{a}{b}\right) + \frac{2}{\left(\frac{a}{b}\right)} - 5 = 0$$

$$\Rightarrow 2\left(\frac{a}{b}\right)^2 - 5\left(\frac{a}{b}\right) + 2 = 0 \quad \Rightarrow \left(\frac{2a}{b} - 1\right)\left(\frac{a}{b} - 2\right) = 0$$

$$\Rightarrow \frac{2a}{b} = 1 \text{ or } \frac{a}{b} = 2$$

$\Rightarrow 2a = b$ or $a = 2b$.

27. (a) :Let the side of the square be x.



Given $AT = 2x$.

We have $AD.AE = AT^2$

$x(x + DE) = 4x^2$

$x + DE = 4x$

$DE = 3x$.

$CE = 10$

$\Rightarrow x^2 + 9x^2 = 100$

$\Rightarrow 10x^2 = 100 \Rightarrow x^2 = 10$

$\Rightarrow x = \sqrt{10}$ cm.

28. (a) :The given equation is $x^4 - x^3 - 6x^2 - 2x + 9 = 0$. Rewriting the equation we get

$x^4 - 6x^2 + 9 - x^3 - 2x = 0.$

$(x^2 - 3)^2 - x(x^2 + 2) = 0$

If x is negative $(x^2 - 3)^2 - x(x^2 + 2)$ is positive. It cannot be zero.

The equation cannot have negative roots.

29. (a) :$x^3 + 113 = y^3$

$\Rightarrow y^3 - x^3 = 11^3 \Rightarrow (y - x)(y^2 + yx + x^2) = 11^3$

Case I :Let $y - x = 1 \Rightarrow y = 1 + x$ and $y^2 + x^2 + yx = 11^3$

i.e., $y = 1 + x$ and $y^2 + x^2 + yx = 1331$

$\Rightarrow (1 + x)^2 + x^2 + x(1 + x) = 1331$

$\Rightarrow 3x^2 + 3x + 1 = 1331$

When LHS is divided by 3 the remainder is 1 when RHS is divided by 3 the remainder is 2.

There cannot be integers x, y such that $y - x = 1$ and $y^2 + x^2 + yx = 113$

Case II :$y - x = 11$ and $y^2 + x^2 + xy = 121$

$\Rightarrow y = x + 11$

$\therefore (x + 11)^2 + x^2 + x(x + 11) = 121$

$3x^2 + 33x + 121 = 121$

$\Rightarrow 3x^2 + 33x = 0 \Rightarrow 3x(x + 11) = 0$

$x = 0, x = 11$

When $x = 0$, $y = 11$ and $x = -11$, $y = 0$

Case III :$y - x = 11^2$, $x^2 + y^2 + xy = 11$

$y = 121 + x \Rightarrow x^2 + (121 + x)^2 + x(121 + x) = 11$

$\Rightarrow 3x^2 + 36x + 121^2 = 11$

The discriminant < 0 no real values for n.

Case IV :$y - x + 11^2$, $x^2 + y^2 + xy = 1$

$y = 1331 = x$, $x^2 + y^2 + xy = 1$

If we substitute y we find a quadratic in x whose discriminant < 0.

Therefore, only solutions are (0, 11) and (–11, 0).

30. (d) : **Case I** :When x is a whole number.

Then $[x] = x$. The equation reduces to $|x - 2x| = 4$.

$|-x| = 4 \Rightarrow x = \pm 4$

Case II :Let x be not a whole number.

Let $x = n + f$ where $0 < f < 1$.

$[x] = n$

Given equation becomes $|n - 2(n + f)| = 4$

$\Rightarrow |-n - 2f| = 4$

$\Rightarrow n + 2f = \pm 4$ or $n = \pm 4 - 2f$

n is a whole number $\Rightarrow$ RHS must be a whole number

$$\Rightarrow f = \frac{1}{2}$$

$\therefore n = \pm 4 - 1$ *i.e.*, $n = -5$ or $n = 3$

$$\therefore x = n + f = -5 + \frac{1}{2} \text{ or } 3 + \frac{1}{2} \text{ i.e., } x = -\frac{9}{2} \text{ or } \frac{7}{2}$$

The solutions are 4, –4, $\frac{-9}{2}, \frac{7}{2}$.

PART - B

1. we have to find the positive integers x, y, z such that

$2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4 = 576.$

Let $E = 2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4$

$= 4x^2y^2 - [x^4 + y^4 - z^4 + 2x^2y^2 - 2y^2z^2 - 2z^2x^2]$
$= 4x^2y^2 - [x^2 + y^2 - z^2]^2$
$= (2xy)^2 - (x^2 + y^2 - z^2)^2$
$= (2xy + x^2 + y^2 - z^2)(2xy + z^2 - x^2 - y^2)$
$= [(x + y)^2 - z^2][z^2 - (x - y)^2]$
$= (x + y + z)(x + y - z)(z + x - y)(z - x + y)$

$\therefore$ The equation becomes
$(x + y + z)(x + y - z)(x - y + z)(-x + y + z) = 576$

x, y, z are positive integers. We find that $x + y + z$ $= (x + y - z) + 2z$

$\Rightarrow$ All the factors are of same parity.
$\Rightarrow$ All of them must be even.

Let $x + y + z = 2a$, $x + y - z = 2b$, $x - y + z = 2c$, $-x + y + z = 2d$ $\Rightarrow abcd = 36$

Without loss of generality assume $x \geq y \geq z$
$\Rightarrow a > b \geq c \geq d.$

We observe $a = b + c + d$. $36 = 6 \times 3 \times 2 \times 1$. This factorization is unique.
$\Rightarrow a = 6, b = 3, c = 2, d = 1$
$x = 5, y = 4, z = 3$

Exploiting the symmetry of the equation, x, y, z can be cyclically changed.

2. Let Δ be the area of the triangle ABC

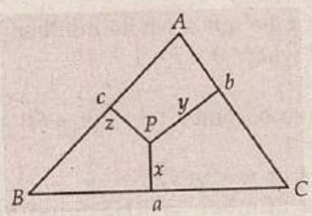
Δ = Area of ΔPBC + Area of ΔPCA + Area of ΔPAB

$$\Delta = \frac{1}{2}ax + \frac{1}{2}by + \frac{1}{2}cz$$

$\Rightarrow ax + by + cz = 2\Delta$

Apply A.M. – G.M. inequality to ax, by, cz

$$\frac{ax + by + cz}{3} \geq (axbycz)^{1/3}$$



$$\Rightarrow axbycz \leq \left(\frac{2\Delta}{3}\right)^3$$

Since a, b, c are constants, xyz attains its maximum when $axbycz$ is maximum and it happens when $ax = by = cz = k$(say)

$$\Rightarrow x = \frac{k}{a}, y = \frac{k}{b}, z = \frac{k}{c}$$

$$x : y : z = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$$

$ax = by = cz \Rightarrow$ Area of ΔPBC = Area of ΔPCA = Area of ΔPAB.

$\Rightarrow P$ is the centroid of the triangle ABC.

3. $(a - b)^4 = [(a - b)^2]^2 = (a^2 + b^2 - 2ab)^2$

$= a^4 + b^4 + 4a^2b^2 + 2a^2b^2 - 4ab^3 - 4a^3b$
$= a^4 + b^4 + 6a^2b^2 - 4ab^3 - 4a^3b$ .....(1)

Given $a^2 + b^2 + (a - b)^2 = c^2 + d^2 + (c - d)^2$
$\Rightarrow 2(a^2 + b^2 - ab) = 2(c^2 + d^2 - cd)$
$a^4 + b^4 + a^2b^2 + 2a^2b^2 - 2ab^3 - 2a^3b$
$= c^4 + d^4 + c^2d^2 + 2c^2d^2 - 2cd^3 - 2c^3d$
$\Rightarrow a^4 + b^4 + 3a^2b^2 - 2ab^3 - 2a^3b$
$= a^4 + b^4 + 3a^2b^2 - 2ab^3 - 2c^3d$
$\Rightarrow 2a^4 - 2b^4 + 6a^2b^2 - 4ab^3 - 4c^3d$
$= 2c^4 + 2d^4 + 6c^2d^2 - 4cd^3 - 4c^3d$
$\Rightarrow a^4 + b^4 + (a^4 + b^4 + 6a^2b^2 - 4ab^3 - 4a^3b)$
$\Rightarrow c^4 + d^4 + (c^4 + d^4 + 6c^2d^2 - 4cd^3 - 4dc^3)$
$\Rightarrow a^4 + b^4 + (a - b)^4 = c^4 + d^4 + (c - d)^4$ (using) (1)

4. We have to find positive integers x, y, z such that $2^x + 2^y + 2^z = 2336$.

Let us first express 2336 in powers of 2 if possible.

$2336 = 2^3 \times 73$
$\therefore 2^x + 2^y + 2^z = 2^5 \times 73.$

$$\Rightarrow \frac{2^x}{2^5} + \frac{2^y}{2^5} + \frac{2^z}{2^5} = 73$$

$\Rightarrow 2^{x-5} + 2^{y-5} + 2^{z-5} = 73$

RHS is odd. LHS being powers of 2 is even $\therefore$ One of the terms must be odd. This is possible only when 2^{x-5} or 2^{y-5} or 2^{z-5} is 1

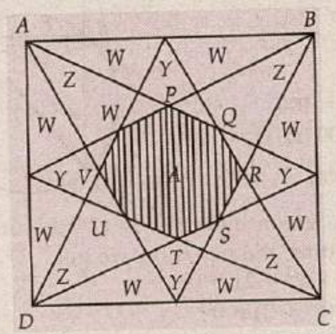
Let $2^{x-5} = 1 \Rightarrow x - 5 = 0 \Rightarrow x = 5$
$1 + 2^{y-5} + 2^{z-5} = 73 \Rightarrow 2^{y-5} + 2^{z-5} = 72$
$2^{y-5} + 2^{z-5} = 8 \times 9 = 2^3 \times 9$
$\Rightarrow 2^{y-8} + 2^{z-8} = 9$

Again RHS is odd, LHS is even. Thus one of the two terms is odd.

Let $2^{y-8} - 1 \Rightarrow y - 8 = 0 \Rightarrow y = 8$
$\therefore 2^{y-8} = 8 = 2^3$
$\Rightarrow z - 8 = 3 \Rightarrow z = 11$

Since the equation is symmetric in x, y, z the solution set is (x, y, z) = (5, 8, 11),(8, 5, 11), (11, 5, 8), (5, 11, 8), (8, 11, 5), (11, 8, 5).

5.



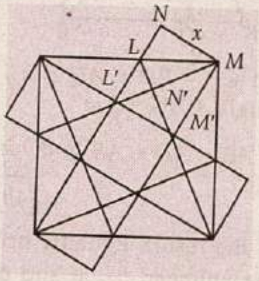
The octagon is $PQRSTUVW$. Let A be the area of the octagon.

The figure is highly symmetric. It consists of four types of figures.

Around the octagon these are 8 congruent triangles each of area X (say).

Around that there are two types of 4 kinds of areas Y, Z each.

Finally 8 congruent right angled triangles of area W each.



If the triangle LMN is reflected on LM we get another congruent triangled $LN'M$.

The figure $N'MM'L'$ is a square clearly.

Such squares, if-we recognize as in the diagram, there are 5 of side a (say). $\therefore\ 5a^2 = 1$ which is the area of the original square $ABCD$.

$$\therefore a^2 = \frac{1}{5}$$

From the figure we have

$$A + 4X = \frac{1}{5}$$

$$W = \frac{1}{4}\cdot\frac{1}{5} = \frac{1}{20}$$

$$Y + 2W = \frac{1}{8} \Rightarrow Y = \frac{1}{40}$$

$$X + Y + W = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{12} \Rightarrow X = \frac{1}{120}$$

$$\Rightarrow A = \frac{1}{5} - \frac{1}{30} = \frac{1}{6}$$

6. Let ABC be the triangle in which a square has to be inscribed. The condition is that one side of the square is along a side (say BC). Construct the square $BCDE$ on BC on the opposite side of A.

Join AE, AD so that they respectively cut BC at P and Q.

Erect perpendiculars at P, Q to meet respectively AB, AC at R and S.

Clearly $PQRS$ is a rectangle. We claim that it is the required square.

We have to prove $PQ = PR$.

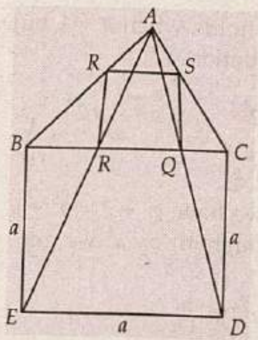
Consider the triangle AED, PQ is parallel to ED.

Using Basic proportionality theorem we get

$$\frac{PQ}{ED} = \frac{AP}{AE} = \frac{PQ}{a}$$

where a is the side of square

$$\Rightarrow PQ = \frac{AP}{AE}a \qquad .....(1)$$



Consider the triangle AEB, PR is parallel to BE

$$\Rightarrow \frac{PR}{EB} = \frac{AP}{AE} = \frac{PR}{a} \quad \Rightarrow PR = \frac{AP}{AE}a \qquad .....(2)$$

From (1), (2) we get $PQ = PR$. $PQRS$ is a square.

7. **(a)** : For both parts, the proof will be by induction on n. (i) For $n = 1$, $3^2 | 2^3 + 1$ and so the statement is true for $n = 1$.

Suppose $3^{k+1} | 2^{3k} + 1$ for some k. We have to show that $3^{k+2} | 2^{3k+1} + 1$.

$3^{k+1} | 2^{3^k} + 1 \Rightarrow (3^{k+1})^3 | (2^{3k} + 1)^3$

i.e. $3^{3k+3} | 2^{3^{k+1}} + 1 + 3.2^{3^k}(2^{3^k} + 1)$.

Since $3k + 3 > k + 2$, the above expression implies that $3^{k+2} | 2^{3k+1} + 1 + 3.2^{3k}(2^{3k} + 1)$

But $3^{k+1} | 2^{3^k}+1$ by the induction hypothesis and hence $3^{k+2} | 3(2^{3^k}+1)$. Hence $3^{k+2} | 2^{3^{k+1}} + 1$ and we are through.

(ii) Here the induction hypothesis is that for any n, 3^{n+2} does not divide $2^{3^n} + 1$.

For $n = 1$ it is true that 3^3 does not divide $2^3 + 1$.

Suppose that 3^{k+2} does not divide $2^{3^k} + 1$ for some k.

Proceeding as in (i) we get that

$3^{k+3} | 2^{3^{k+1}} + 1 + 3.2^{3^k}(2^{3^k} + 1)$.

Now if 3^{k+3} divides $2^{3^{k+1}} + 1$, then 3^{k+3} divides $3.2^{3^k}(2^{3^k} + 1)$ which in turn implies that 3^{k+2} divides $2^{3^k} + 1$ which is a contradiction to the induction hypothesis. Therefore 3^{k+3} does not divide $2^{3^{k+1}} + 1$ and that finishes the proof.

8. Consider $a^6 + 1$

$a^6 + 1 = (a^2)^3 + 1 = (a^2 + 1)(a^4 + 1)$.

We are given the value of $a^5 - a^3 + a$ as 2.

$$a^6+1=\left(\frac{a^2+1}{a}\right)a\left(a^4-a^2+1\right)=\left(\frac{a^2+1}{a}\right)\left(a^5-a^3+a\right)$$

$$=\left(a+\frac{1}{a}\right)^2 \quad \text{But } \frac{a+\frac{1}{a}}{2}\geq\sqrt{a.\frac{1}{a}}\Rightarrow a+\frac{1}{a}\geq 2$$

The equality holds when $a = 1$ but $a = 1$ does not satisfy the equation.

$a^5 - a^3 + a = 2 \qquad \therefore\ a+\frac{1}{a}>2 \qquad \therefore \qquad a^6 + 1 > 4$

$a^6 > 3 \Rightarrow 3 < a^6$

To prove $a^6 < 4$.

From given we have $a^3 + 2 = a^5 + a$.

Dividing throughout by a^3 we get

$$1+\frac{2}{a^3}=a^2+\frac{1}{a^2}$$

$$\text{But } \frac{a^2+\frac{1}{a^2}}{2}>\sqrt{a^2\frac{1}{a^2}}\Rightarrow a^2+\frac{1}{a^2}>2 \quad \therefore\ 1+\frac{2}{a^2}>2$$

$\frac{2}{a^3}>1 \Rightarrow a^3 < 2 \Rightarrow a^6 < 4 \ \therefore\ 3 < a^6 < 4.$

9. Let α, β, γ be the roots of the given cubic $x^3 - ax + b = 0$, where $a > 0$ and $b > 0$. We have then

$$\left.\begin{aligned}\alpha+\beta+\gamma &= 0\\ \alpha\beta+\beta\gamma+\gamma\alpha &= -a\\ \alpha\beta\gamma &= -b\end{aligned}\right\} \qquad \ldots (1)$$

From the last of these equations, we see that either all the roots are negative or two are positive and one negative. However the second equation in (1) shows that all three cannot be negative. So two of α, β, γ are positive and the remaining root is negative. The first equation in (1) implies that the negative root is numerically larger than the other two positive roots. Hence we may assume that that $\gamma < 0 < \alpha \leq \beta$ where $|\alpha| \leq |\beta| \leq |\gamma|$. We have

$b - a\alpha = -\alpha\beta\gamma + \alpha(\alpha\beta + \beta\gamma + \gamma\alpha) = \alpha^2(\beta + \gamma) = -\alpha^3 < 0.$

Since a is positive, we get $\frac{b}{a} < \alpha$ proving the first inequality.

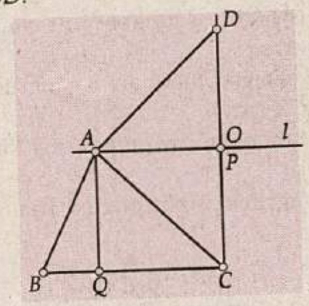
Again, we have

$$\begin{aligned}3b - 2a\alpha &= -3\alpha\beta\gamma + 2\alpha(\alpha\beta + \beta\gamma + \gamma\alpha)\\ &= -\alpha\beta\gamma + 2\alpha^2\beta + 2\alpha^2\gamma = \alpha[2\alpha(\beta+\gamma) - \beta\gamma]\\ &= \alpha[-2(\beta+\gamma)^2 - \beta\gamma] \qquad (\text{since } \alpha = -(\beta+\gamma))\\ &= -\alpha(2\beta+\gamma)(\beta+2\gamma) = -\alpha(\beta-\alpha)(\gamma-\alpha)\end{aligned}$$

Observe that $-\alpha < 0$, $\beta \geq \alpha$, $\gamma - \alpha < 0$. Hence $3b - 2a\alpha$ is non-negative. This proves the second inequality,

$$\alpha \leq \frac{3b}{2a}.$$

10. Draw a line l parallel to BC through A and reflect AC in this line to get AD. Let CD intersect l in P. Join BD.



Observe that $CP = PD = AQ = h_a$, AQ being the altitude through A. We have

$$b + c = AC + AB = AD + AB \geq BD = \sqrt{CD^2 + CB^2}$$
$$= \sqrt{4h_a^2 + a^2}.$$

which yields the result. Equality occurs if and only if B, A, D are collinear, *i.e.*, if and only if $AD = AB$ (as AP is parallel to BC and bisects DC) and this is equivalent to $AC = BC$.

Alternatively, the given inequality is equivalent to

$$(b+c)^2 - a^2 \geq 4h_a^2 = \frac{16\Delta^2}{a^2},$$

where Δ is the area of the triangle ABC. Using the identity $16\Delta^2 = \left[(b+c)^2 - a^2\right]\left[a^2 - (b-c)^2\right]$

we see that the inequality to be proved is $a^2 - (b-c)^2 \leq a^2$ (here we use $a < b + c$) which is true. Observe that equality holds if and only if $b = c$. ■

Form IV

1.	Place of Publication	: New Delhi
2.	Periodicity of its publication	: Monthly
3.	Printer's and Publisher's Name	: Mahabir Singh
	Nationality	: Indian
	Address	: Mathematics Today, 406, Taj Apartment, New Delhi - 110029.
4.	Editor's Name	: Anil Ahlawat
	Nationality	: Indian
	Address	: Mathematics Today, 19, National Media Centre, Gurgaon Haryana - 122002.
5.	Name and address of individuals who own the newspapers and partners or shareholders holding more than one percent of the total capital	: Mahabir Singh 406, Taj Apartment New Delhi

I, Mahabir Singh, here by declare that particulars given above are true to the best of my knowledge and belief:

Mahabir Singh
Publisher

(a) 2/5 (b) 4/5 (c) 6/5 (d) 8/5

16. The sum of squares of deviation is least when measured from
(a) median (b) mean
(c) mode (d) none of these

17. The mean of the binomial distribution

$${}^{10}C_x\left(\frac{2}{5}\right)^x\left(\frac{3}{5}\right)^{10-x}\ ;\ x=0,1,2,\ldots,10 \text{ is}$$

(a) 4 (b) 5 (c) 6 (d) 0

18. If $\sin\beta=\frac{1}{5}\sin(2\alpha+\beta)$, then $\frac{\tan(\alpha+\beta)}{\tan\alpha}=$
(a) 5/3 (b) 2/3 (c) 3/2 (d) 3/5

19. The shortest distance of the point (9, –12) from the circle $x^2+y^2=16$ is
(a) 7 units (b) 11 units
(c) 15 units (d) 4 units

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20. The equation of the plane perpendicular to the line $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z+1}{2}$ and passing through the point (2, 3, 1) is
(a) $\vec{r}\cdot(\hat{i}+\hat{j}+2\hat{k})=1$ (b) $\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=1$
(c) $\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=7$ (d) none of these

21. If vectors $\overrightarrow{AB}=-3\hat{i}+4\hat{k}$ and $\overrightarrow{AC}=5\hat{i}-2\hat{j}+4\hat{k}$ are the sides of a ΔABC, then the length of the median through A is
(a) $\sqrt{14}$ (b) $\sqrt{18}$ (c) $\sqrt{29}$ (d) 5

22. If the sides of a triangle are in the ratio 2 : 3 : 4, then the triangle is
(a) right-angled (b) acute-angled
(c) obtuse-angled (d) none of these

23. If $f(x)$ and $g(x)$ are differentiable function in [0, 1] such that $f(0)=2$, $f(1)=6$, $g(0)=0$, $g(1)=2$, then there exists $0<c<1$ such that
(a) $f'(c)=g'(c)$ (b) $f'(c)=-g'(c)$
(c) $f'(c)=2g'(c)$ (d) $2f'(c)=g'(c)$

24. The area bounded by the curve $x^{2/3}+y^{2/3}=1$ is
(a) $\frac{3\pi}{2}$ (b) $\frac{3\pi}{4}$ (c) $\frac{3\pi}{8}$ (d) $\frac{3\pi}{16}$

25. Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$ where $\theta+\phi=\pi/2$, be two points on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h, k) is the point of intersection of normals at P and Q, then k is
(a) $\frac{a^2+b^2}{a}$ (b) $-\frac{(a^2+b^2)}{a}$
(c) $\frac{a^2+b^2}{b}$ (d) $-\frac{(a^2+b^2)}{b}$

Directions : This section contains 5 questions numbered 26 to 30. Each of these questions contains two statements, statement-1 (assertion) and statement-2 (reason). Each question has four choices (a), (b), (c) and (d) out of which only one is correct. Mark your responses from the following options.
(a) Statement-1 and statement-2 are true and statement-2 is the correct explanation of statement-1.
(b) Statement-1 and statement-2 are true but statement-2 is not a correct explanation of statement-1.
(c) Statement-1 is true but statement-2 is false.
(d) Statement-1 is false but statement-2 is true.

26. **Statement-1 :** If $f(x)=x-x^2+1$ and $g(x)=\max\{f(t):0\le t\le x\}$, then $\int_0^1 g(x)=\frac{29}{24}$.
Statement-2 : $f(x)$ is increasing in (0, 1/2) and decreasing in (1/2, 1).

27. **Statement-1 :** The function
$f(x)=(x^3+3x-14)(x^2+3x-10)$
has a local extremum at $x=2$.
Statement-2 : $f(x)$ is continuous and differentiable and $f'(2)=0$.

28. **Statement-1 :** If $a>0$ and $b^2-4ac<0$ then the value of the integral $\int\frac{dx}{ax^2+bx+c}$ will be of the type $\mu\tan^{-1}\left(\frac{x+A}{B}\right)+C$ where A, B, C, μ are constants.
Statement-2 : If $a>0$, $b^2-4ac<0$, then ax^2+bx+c can be written as sum of two squares.

29. **Statement-1 :** $(b^2-ac)^2+(c^2-bd)^2+(ad-bc)^2=0$ then a, b, c, d are in G.P.
Statement-2 : $x^2+y^2+z^2=0$, $x=y=z=0$.

30. **Statement-1 :** If $a, b, c \in R$ and equations $ax^2+bx+c=0$ and $x^2+5x+7=0$ has a common root then $\frac{a+c}{b}=\frac{7}{5}$.
Statement-2 : If both roots of $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$ are identical then $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$, where a_1, b_1, c_1 and a_2, b_2, $c_2 \in R$.

ANSWER KEYS

1. (b)	2. (b)	3. (a)	4. (b)	5. (a)
6. (c)	7. (c)	8. (b)	9. (b)	10. (b)
11. (a)	12. (c)	13. (b)	14. (c)	15. (d)
16. (b)	17. (a)	18. (c)	19. (b)	20. (b)
21. (b)	22. (c)	23. (c)	24. (c)	25. (d)
26. (a)	27. (b)	28. (a)	29. (c)	30. (d)

MOCK TEST PAPER

ISI 2011

Indian Statistical Institute

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PART - A

SECTION - I

Straight Objective Type

1102121

1. Let $\alpha = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$, where $i^2 = -1$. The fourth degree polynomial which has α, α^3, α^7 and α^9 as its roots is
(a) $x^4 + x^3 + x^2 + x + 1 = 0$
(b) $x^4 - x^3 + x^2 - x + 1 = 0$
(c) $x^4 - x^3 - x^2 + x + 1 = 0$
(d) $x^4 + x^3 + x^2 - x - 1 = 0$

2. The number of solutions of the equation $\sin^3 x + 1 = 2\sqrt[3]{2\sin x - 1}$ on the interval $[0, 2\pi]$ is
(a) 1 (b) 3 (c) 5 (d) 6

3. The vertices of a parallelogram are $(3^{11} - 1, 3^{12} + 2)$, $(3^{11}, 3^{12})$, $(3^{11} + 1, 3^{12} - 3)$. There are three other points that can be the fourth vertex of the parallelogram. The sum of the abscissas of those three other points are
(a) 3^{12} (b) $3^{12} + 2$ (c) 3^{11} (d) $3^{11} + 2$

4. Suppose the diagonals PR and QS of a convex quadrilateral $PQRS$ intersect at M and the areas of triangle MPQ and MRS are 16 cm^2 and 25 cm^2 respectively. Let the minimum possible area of the quadrilateral $PQRS$ be A(in cm^2). Then the sum of the squares of the digits in A is
(a) 37 (b) 39 (c) 68 (d) 65

5. There is a point P on the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ such that its distance to the right directrix is the average of its distance to the two foci. Let the x-coordinate of P be $\frac{m}{n}$ with m and n being integers, $(n > 0)$ having no common factor except 1. Then $n - m$ equals
(a) 59 (b) 69 (c) -59 (d) -69

6. The number of subsets $\{a, b, c\}$ of $\{-3, -2, -1, 0, 1, 2, 3\}$ such that line $ax + by + c = 0$ makes an acute angle with the positive x-axis is
(a) 45 (b) 36 (c) 43 (d) 1

7. Six distinct integers are picked at random from $\{1, 2, 3, ..., 10\}$. What is the probability that, among those selected, the second smallest is 3 ?
(a) $\frac{1}{60}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

8. $ABCD$ is a square of side 1 unit. A circle passes through vertices A, B of the square and the remaining two vertices of the square lie outside the circle. The length of the tangent drawn to the circle from vertex D is 2 units. The radius of the circle is
(a) $\sqrt{5}$ (b) $\frac{1}{2}\sqrt{10}$ (c) $\frac{1}{3}\sqrt{12}$ (d) $\sqrt{8}$

9. The equation of circum-circle of a ΔABC is $x^2 + y^2 + 3x + y - 6 = 0$. If $A = (1, -2)$, $B = (-3, 2)$ and the vertex C varies then the locus of orthocentre of ΔABC is a
(a) straight line (b) circle
(c) parabola (d) ellipse

10. The locus of the point of intersection of the tangents at the extremities of the chord of the ellipse $x^2 + 2y^2 = 6$ which touches the ellipse $x^2 + 4y^2 = 4$ is
(a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = 6$
(c) $x^2 + y^2 = 9$ (d) $x^2 + y^2 = 2$

11. The period of the function $f(x) = \sin 3x\cos[3x] - \cos 3x\sin[3x]$, where [.] denotes the greatest integer function is
(a) 6 (b) 3 (c) 1/3 (d) 1/6

12. Let λ and μ be real numbers and $f(x) = \lambda\sin x + \mu\sqrt[3]{x} + 4$. If $f(\log_{10}(\log_3 10)) = 5$ then the value of $f(\log_{10}(\log_3 3))$ is
(a) -5 (b) -3
(c) 3
(d) dependent on λ and μ

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13. The function whose graph is the reflection about the line $x + y = 0$ of the inverse function $f^{-1}(x)$ of a function $f(x)$ is

(a) $-f(x)$ (b) $-f(-x)$ (c) $-f^{-1}(x)$ (d) $-f^{-1}(-x)$

14. Let the range of the function $f : R \to R$ $f(x) = |x-1| + |x-a| + |x| + |x+1| + |x+2a-21|$, '$a$' being a real parameter, be given by $[\alpha, \infty)$, then the number of integer values of 'a' for which there is exactly one $x_0 \in R$ such that $f(x_0) = \alpha$, is

(a) 7 (b) 8 (c) 9 (d) 10

15. If (α, β) is a point on the circle whose centre is on the x-axis and which touches the line $x + y = 0$ at $(2, -2)$, then the greatest value of α is

(a) $4-\sqrt{2}$ (b) 6

(c) $4+2\sqrt{2}$ (d) $4+\sqrt{2}$

16. If in a ΔABC, $a = 6$, $b = 3$ and $\cos(A-B) = \frac{4}{5}$ then

(a) $\angle C = \frac{\pi}{4}$ (b) $\angle A = \sin^{-1}\frac{2}{\sqrt{5}}$

(c) area $(\Delta ABC) = 9$ (d) $\angle C = \frac{\pi}{3}$

17. A man starts from the points $P(-3, 4)$ and reaches points $Q(0, 1)$ touching x-axis at R such that $PR + RQ$ is minimum, then the point R is

(a) $\left(\frac{3}{5}, 0\right)$ (b) $\left(-\frac{3}{5}, 0\right)$

(c) $\left(-\frac{2}{5}, 0\right)$ (d) $(-2, 0)$

18. The period of the function

$f(x) = 4\sin^4\left(\frac{4x-3\pi}{6\pi^2}\right) + 2\cos\left(\frac{4x-3\pi}{3\pi^2}\right)$ is

(a) $\frac{3\pi^2}{4}$ (b) $\frac{3\pi^3}{4}$ (c) $\frac{4\pi^2}{3}$ (d) $\frac{4\pi^3}{3}$

19. How many roots does the following equation possess $3^{|x|}\{(|2 - |x||)\} = 1$

(a) 1 (b) 2 (c) 3 (d) 4

SECTION - II

Multiple Correct Choice Type

20. Let a quadratic polynomial f satisfy $f(0) = 3$ and $\int_0^3 f(x)g(x)dx = 0$ for every polynomial g of degree 1 or 0. Which of the following statements is correct?

(a) The number of possible polynomials f is exactly two

(b) The number of possible polynomials f is exactly one

(c) $f(3)$ equals 3

(d) $f(3)$ equals 3 or -3, and both these values are possible

21. Let f be a function with two continuous derivatives and $f(0) = 0$. Define a function g by $g(x) = \begin{cases} \frac{f(x)}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

Then which of the following statements is correct?

(a) g has a continuous first derivative

(b) g has a first derivative

(c) g is continuous but g fails to have a derivative

(d) g has a first derivative but the first derivative is not continuous

22. Suppose the quadratic equation whose roots are squares of those of $x^2 + ax + b = 0$, $a, b \in R$ is identical with the given equation, then which of the following is correct?

(a) The number of ordered pairs (a, b) is 4.

(b) The number of different possible values that $a + b$ can take is 3

(c) The number of different possible values that ab can take is 3

(d) The number of different possible values that $a + b$ can take is 2.

23. Let a function f satisfy $f(x)f'(-x) = f(-x)f'(x)$, $x \in R$ and $f(0) = 3$, which of the following statements is correct?

(a) The value of $f(x)f(-x)$ for all real x is 9

(b) The value of $f(x)f(-x)$ for all real x is -9

(c) $\int_{-51}^{51} \frac{1}{3+f(x)}dx$ equals 17

(d) $\int_{-51}^{51} \frac{1}{3+f(x)}dx$ equals 34

24. A real valued function f satisfies $f(10 + x) = f(10 - x)$ and $f(20 - x) = -f(20 + x)$, for all $x \in R$ which of the following statements is true?

(a) f is an even function (b) f is an odd function

(c) f is a periodic function

(d) f is a non-periodic function

PART - B

Short Answer Type

25. A rhombus has half the area of the square with the same side length. Find the ratio of the longer diagonal to that of the shorter one.

26. Given the base and vertical angle of a triangle, find the locus of its orthocentre and incentre.

27. Eliminate θ and φ from the equations $a\sin^2\theta + b\cos^2\theta = a\cos^2\varphi + b\sin^2\varphi = 1, a\tan\theta = b\tan\varphi$

28. Solve the equation $\cos 3x\cos^3 x + \sin 3x \sin^3 x = 0$.

29. Solve the equation
$\sin 2x + \cos 2x + \sin x + \cos x + 1 = 0$.

30. Find all real numbers x such that
$x=\left(1-\frac{1}{x}\right)^{1/2}+\left(x-\frac{1}{x}\right)^{1/2}$.

31. Let n be a natural number such that $n \geq 2$. Show that $\frac{1}{n+1}\left(1+\frac{1}{3}+...+\frac{1}{2n-1}\right)>\frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)$

32. Determine all the positive integers $n \geq 3$, such that 2^{2000} is divisible by $1+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}$.

33. Find all solutions in real numbers x, y, z to the system of equations $x+y+(z^2-8z+14)\sqrt{x+y-2}=1$ and $2x+5y+\sqrt{xy+z}=3$.

34. Find all pairwise relatively prime integers l, m, n such that $(l+m+n)\left(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}\right)$ is an integer.

35. Let R^+ be the set of positive real numbers. Prove that there does not exist a function $f: R^+ \to R^+$ such that $(f(x))^2 \geq f(x+y)(f(x)+y)$ for every $x, y \in R^+$.

36. Find all integers (a, b, c, x, y, z) such that $a+b+c = xyz$, $x+y+z = abc$ and $a \geq b \geq c \geq 1, x \geq y \geq z \geq 1$.

37. Points D, E lie on side AB of the triangle ABC and satisfy $\frac{AD}{DB}\cdot\frac{AE}{EB}=\left(\frac{AC}{CB}\right)^2$.
Prove that $\angle ACD = \angle BCE$.

38. Find all real numbers x such that $x[x[x[x]]] = 88$.

SOLUTIONS

1. **(b)** : Use $(x-1)(x-\alpha)(x-\alpha^2)\ ....\ (x-\alpha^9) = x^{10}-1$ and $(x-1)(x-\alpha^2)\ ......\ (x-\alpha^8) = x^5-1$

2. **(b)** : $\sin^3 x+1=2\sqrt[3]{2\sin x-1}$

$\Rightarrow \frac{\sin^3 x+1}{2}=\sqrt[3]{2\sin x-1}$

Put $\sin x = t \Rightarrow \frac{t^3+1}{2}=\sqrt[3]{2t-1}$

Let $f(t)=\sqrt[3]{2t-1} \Rightarrow y^3 = 2t-1 \Rightarrow t=\frac{y^3+1}{2}$

The solutions to the original equation is given by

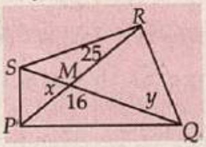
$\frac{t^3+1}{2}=t$

(Recall the concept of inverse functions)

We get $t=1, \frac{-(1+\sqrt{5})}{2}, \frac{\sqrt{5}-1}{2}$ of which only two lies in $[-1, 1]$

3. **(a)**

4. **(d)** : We have $xy = 25 \times 16$



$\frac{x+y}{2} \geq \sqrt{25\times 16}$, $x+y \geq 40$

Then the minimum area = 40 + 41 = 81

5. **(b)** : It turns out that P has to be on the left branch. x-coordinate is found to be $-64/5$.

6. **(c)** : The slope of line is $-a/b$. Assume that $a > 0$ and $b < 0$ so that $a \neq b$ Also $a \neq c \neq b$.
If $c = 0$ there are 3 ways to choose a and 3 ways to choose b. Note that line $x-y=0$, $2x-2y=0$ and $3x-3y=0$ are the same. Then the number of lines $= 3^2-2 = 7$. If $c \neq 0$, there are 3 ways to choose a, 3 ways to choose b and 4 ways to choose c. Then the number of lines in this case $= 3^2 \times 4 = 36$. The total number of lines = 7 + 36 = 43.

7. **(c)** : There are $^{10}C_6$ ways to pick = 210. There are $(^2C_1)(^7C_4) = 70$ favourable ways. So the probability is $\frac{70}{210}=\frac{1}{3}$.

8. **(b)** : Let $A = (0, 1)$, $B = (0, 0)$, $C = (1, 0)$, $D = (1, 1)$. Family of circles passing through A, B is $x^2+y^2-y+\lambda x = 0$, $\sqrt{1+\lambda}=2 \Rightarrow \lambda = 3$.

9. **(b)** : Equation of circum-circle is

$\left(x+\frac{3}{2}\right)^2+\left(y+\frac{1}{2}\right)^2=\frac{17}{2}$

$C=\left(\frac{-3}{2}+\sqrt{\frac{17}{2}}\cos\theta, \frac{-1}{2}+\sqrt{\frac{17}{2}}\sin\theta\right)$

Circum centre of ΔABC is $\left(\frac{-3}{2},\frac{-1}{2}\right)$

Centroid can be obtained.
In a triangle centroid, circumcentre and orthocentre are collinear.

10. **(c)** : We can write $x^2+4y^2 = 4$ as

$\frac{x^2}{4}+\frac{y^2}{1}=1$ .....(1)

Equation of the tangent to the ellipse (1) is

$\frac{x}{2}\cos\theta+y\sin\theta=1$ .....(2)

Equation of the ellipse $x^2+2y^2 = 6$ can be written as $\frac{x^2}{6}+\frac{y^2}{3}=1$ .....(3)

Suppose (2) meets the ellipse (3) at P and Q and the tangent at P and Q to the ellipse (3) intersect at then (2) is the chord of contact of (h, k) with respect to the ellipse (3) and thus its equation is

$$\frac{hx}{6}+\frac{ky}{3}=1 \qquad .....(4)$$

Since (2) and (4) represents the same line

$$\frac{\frac{h}{6}}{\frac{(\cos\theta)}{2}}=\frac{\frac{k}{3}}{\sin\theta}=1. \Rightarrow h = 3\cos\theta,\ k = 3\sin\theta$$

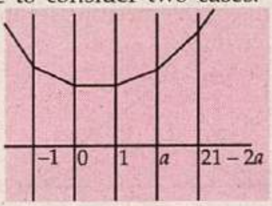
And the locus of (h, k) is $x^2 + y^2 = 9$.

11. (c) : $f(x) = \sin 3\{x\}$, where $\{.\}$ is a fractional part function.

12. (c) : Observe that $f(x) + f(-x) = 8$
Also $\log_{10}(\log_{10} 3) = -\log_{10} \log_3 10$

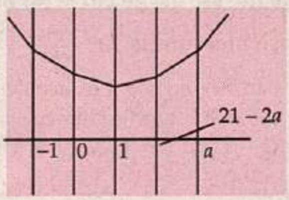
13. (b) : The graph of f and f^{-1} are symmetric about the line $x = y$ while the graph of $f^{-1}(x)$ and the desired functions are symmetric about $x + y = 0$. Then the graph of $f(x)$ and the desired function are symmetric about the origin. Then the function must be $-f(-x)$.

14. (a) : Let $f(x)=|x-a_1|+|x-a_2|+...+|x-a_n|, a_1 < a_2 < ... < a_n$. If n is odd, then the function attains its minimum value at only one point. If n is even ($n = 2m$), the graph of the function becomes horizontal on the interval $[a_m, a_{m+1}]$ and then we require that points $-1, 0, 1, a, 21 - 2a$ must be all distinct and both a and $21 - 2a$ must lie to the right of 1 on x-axis. Of the numbers 'a' and '$21 - 2a$', anyone can be greater. Then we have to consider two cases.



This gives $1 < a < 7$

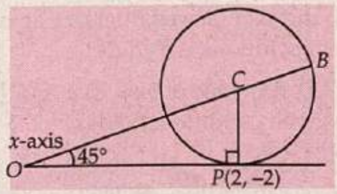
or,



This gives $7 < a < 10$.

$\therefore$ The number of integral values is 7.

15. (c) : If $(a, 0)$ is the centre C and P is $(2, -2)$, then $\angle COP = 45°$, since the equation of OP is $x + y = 0$.



$\therefore OP = 2\sqrt{2} = CP$. Hence $OC = 4$

Greatest value of $\alpha = 4+2\sqrt{2}$.

16. (c) : $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{1}{3}\cot\frac{C}{2}$

$$\therefore \quad \frac{4}{5}=\cos(A-B)=\frac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)}=\frac{1-\frac{1}{9}\cot^2\frac{C}{2}}{1+\frac{1}{9}\cot^2\frac{C}{2}}$$

$$\therefore \quad \tan^2\frac{C}{2}=1 \Rightarrow C=\frac{\pi}{2}$$

$$\therefore \quad ar(\Delta ABC)=\frac{1}{2}ab=\frac{1}{2}6.3=9$$

Also, $\sin A=\frac{6}{\sqrt{3^2+6^2}}$.

17. (b) : Let $R = (\alpha, 0)$. For $PR + PQ$ to be minimum it should be the path of light and thus we have $\Delta APR \sim \Delta BQR$

$$\Rightarrow \quad \frac{AR}{RB}=\frac{PA}{QB} \Rightarrow \frac{\alpha+3}{0-\alpha}=\frac{4}{1} \Rightarrow \alpha=-\frac{3}{5}$$

18. (b) : $\because\ f(x)=\left\{2\sin^2\left(\frac{4x-3\pi}{6\pi^2}\right)\right\}^2+2\cos\left(\frac{4x-3\pi}{3\pi^2}\right)$

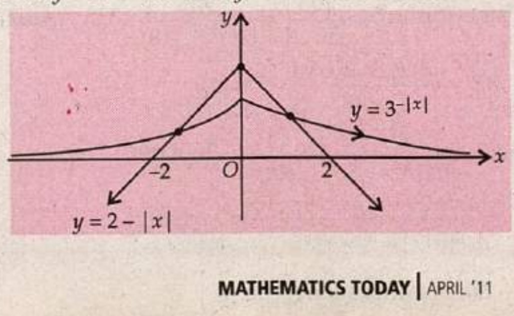
$$=\left\{1-\cos\left(\frac{4x-3\pi}{3\pi^2}\right)\right\}^2+2\cos\left(\frac{4x-3\pi}{3\pi^2}\right)$$

$$=1+\cos^2\left(\frac{4x-3\pi}{3\pi^2}\right)$$

$$=\frac{1}{2}\left\{2+1+\cos\left(\frac{8x-6\pi}{3\pi^2}\right)\right\}=\frac{3}{2}+\frac{1}{2}\cos\left(\frac{8x-6\pi}{3\pi^2}\right)$$

$$\therefore \quad \text{Period of } f(x)=\frac{2\pi}{\left(\frac{8}{3\pi^2}\right)}=\frac{6\pi^3}{8}=\frac{3\pi^3}{4}.$$

19. (b) : Here, $3^{|x|}\{(|2-|x|)\}=1 \Rightarrow 2-|x|=3^{-|x|}$. In order to determine the number of roots, it is sufficient to find the points of intersection of the curves $y = 2 - |x|$ and $y = 3^{-|x|}$, shown as



We observe the two curves intersect at two points

$\therefore$ two real solutions $\in (-2, 2)$

20. (a, c) : Let $f(x) = ax^2 + bx + 3$, a and b are to be found. The given condition yields $a = 2$, $b = -6$. Then $f(x) = 2x^2 - 6x + 3$

21. (a, b) : One can easily establish from the definition of g that $g'(0) = 1/2f''(0)$

Continuity of g' at 0 is also easy to check.

22. (a, b, c, d) : The four equations that satisfy the criteria are

	Roots
$x^2 = 0$	0, 0
$x^2 - x = 0$	0, 1
$x^2 - 2x + 1 = 0$	1, 1
$x^2 + x + 1 = 0$	ω, ω^2

23. (a, c) : $f(x)f'(-x) = f(-x)f'(x)$

$\Rightarrow f(x)f'(-x) - f(-x)f'(x) = 0 \Rightarrow \frac{d}{dx}\{f(-x)\,f(x)\} = 0$

Then $f(x)f(-x) = \text{constant} = f(0)f(0) = 9$

Let $I = \int_{-51}^{51} \frac{dx}{3+f(x)}$, $I = \int_{-51}^{51} \frac{dx}{3+f(-x)}$

Adding $2I = \int_{-51}^{51} \frac{6+f(x)+f(-x)}{9+3\{f(x)+f(-x)\}+f(x)f(-x)}dx$

$$= \frac{1}{3}\int_{-51}^{51} dx = \frac{2\times 51}{3}. \therefore I = 17.$$

24. (b, c) : Change x to $10 - x$ to obtain $f(20 - x) - f(x)$. We have $f(20 - x) = -f(20 + x)$
$\Rightarrow f(x) = -f(20 + x)$. Now change x to $20 + x$
$f(20 + x) = -f(40 + x)$, $-f(x) = -f(40 + x)$
$f(x) = f(40 + x)$, so f is periodic.
Again $f(-x) = -f(20 - x) = -f(x)$. Thus f is odd.

25. If a is the side of the rhombus, then area of the rhombus is $\frac{1}{2}a^2\sin 2\theta \times 2$.

But by hypothesis, this area is equal to $\frac{1}{2}a^2$ *i.e.*, $\frac{1}{2}a^2 = a^2\sin 2\theta \Rightarrow \sin 2\theta = \frac{1}{2}$

$\Rightarrow 2\theta = 30°$ or $150° \Rightarrow \theta = 15°$ or $75°$

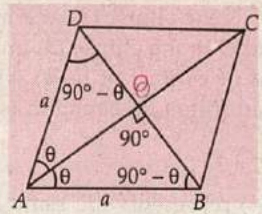
[If the acute angle of the rhombus is 30°, the other angle which is obtuse is 150°]

By sine formula, $\frac{BD}{\sin 2\theta} = \frac{AB}{\sin(90° - \theta)}$ (In ΔABD)

$\Rightarrow BD = \frac{a \times 2\sin\theta\cos\theta}{\cos\theta} = 2a\sin\theta$

Again $\frac{AC}{\sin(180° - 2\theta)} = \frac{a}{\sin\theta}$ (In ΔABC)

$AC = \frac{a\sin 2\theta}{\sin\theta} = \frac{2a\sin\theta\cos\theta}{\sin\theta} = 2a\cos\theta$



$AC : BD = \cos\theta : \sin\theta$

[If $\theta = 15°$, then $AC > BD$ and if $\theta = 75°$, $BD > AC$]

$AC : BD = \cos 15° : \sin 15° = \sin 75° : \sin 15°$

$= \sin(45° + 30°) : \sin(45° - 30°)$

$= \sin 45°\cos 30° + \cos 45°\sin 30° : \sin 45°\cos 30° - \cos 45°\sin 30°$

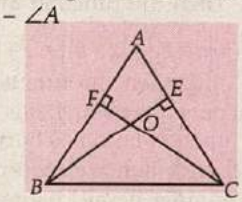
$= \frac{1}{2}(\sqrt{3}+1) : \frac{1}{2}(\sqrt{3}-1) = (\sqrt{3}+1) : (\sqrt{3}-1)$

or $\frac{AC}{BD} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = (2+\sqrt{3})$.

26. Let ABC be a triangle on the given base BC and having its vertical angle is a given angle.

Let BE and CF be the altitudes from B and C respectively meeting at O. O is the orthocentre.

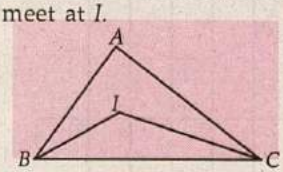
$\angle FOE = 180° - \angle A$



(as O, E, A and F are concyclic)

So the locus of O is the circular arc on BC which contains an angle whose measure is $180° - A$.

To find the locus of the incentre, let the bisectors of $\angle B$ and $\angle C$ meet at I.



$\angle BIC = 180° - \frac{1}{2}(B + C) = 180° - \frac{1}{2}(180° - A) = 90° + \frac{A}{2}$

So, the locus is the arc of the circle on BC containing an angle whose measure is $90° + \frac{A}{2}$.

27. We have $a\sin^2\theta + b\cos^2\theta = 1$, $a\cos^2\varphi + b\sin^2\varphi = 1$

Hence $a\tan^2\theta + b = 1 + \tan^2\theta$, $b\tan^2\varphi + a = 1 + \tan^2\varphi$

Consequently $(a - 1)\tan^2\theta = 1 - b$, $(b - 1)\tan^2\varphi = 1 - a$,

$\frac{\tan^2\theta}{\tan^2\varphi} = \left(\frac{1-b}{1-a}\right)^2$

On the other hand, $\frac{\tan^2\theta}{\tan^2\varphi} = \frac{b^2}{a^2}$.

From the last two equalities, we get (assuming that a is not equal to b) $a + b - 2ab = 0$.

28. Since $\cos 3x = \cos^3 x - 3\sin^2 x\cos x$, $\sin 3x = -\sin^3 x + 3\sin x\cos^2 x$ the equation takes the form $(\cos^3 x - 3\sin^2 x\cos x)\cos^3 x + (-\sin^3 x + 3\sin x\cos^2 x)\sin^3 x = 0$, $\cos^6 x - 3\cos^4 x\sin^2 x + 3\sin^4 x\cos^2 x - \sin^6 x = 0$ or $(\cos^2 x - \sin^2 x)^3 = 0$, $\cos 2x = 0$.

29. Since $\sin 2x + 1 = (\sin x + \cos x)^2$, we have $(\sin x + \cos x)^2 + (\sin x + \cos x) + \cos^2 x - \sin^2 x = 0$
Hence $(\sin x + \cos x)(1 + 2\cos x) = 0$ or, $\cos x(1 + \tan x)(1 + 2\cos x) = 0$ and so $\tan x = -1$ and $\cos x = -\frac{1}{2}$ are the required solutions of our equation.

30. **1st solution :** $x = \frac{(1+\sqrt{5})}{2}$ is the only solution. Isolating $\sqrt{\frac{1-1}{x}}$ and squaring, we obtain $x^2 - 2\sqrt{x^3 - x} + x - \frac{1}{x} = \frac{1-1}{x}$.

Isolating $2\sqrt{x^3 - x}$ and squaring again, we obtain $x^4 + 2x^3 - x^2 - 2x + 1 = 4x^3 - 4x$, $x^4 - 2x^3 - x^2 + 2x + 1 = 0$, $(x-2)(x-1)x(x+1) + 1 = 0$.

This equation is symmetric about $x = \frac{1}{2}$, so we make the substitution $u = x - \frac{1}{2}, x = u + \frac{1}{2}$ to get $u^4 - \frac{5}{2}u^2 + \frac{25}{16} = \left(\frac{u^2-5}{4}\right)^2 = 0$.

Therefore, $u = \pm\frac{\sqrt{5}}{2}$ and $x = \frac{(1\pm\sqrt{5})}{2}$. Checking these in our original equation, only $x = \frac{(1+\sqrt{5})}{2}$ is a valid solution.

2nd solution : From the original equation, we have $x > 0$ and $1 > \frac{1}{x}$, thus $x > 1$. Now isolating $\sqrt{\frac{1-1}{x}}$ and squaring as in the first solution, we obtain $(x^2-1) - 2(\sqrt{x(x^2-1)} + x) = 0 \Rightarrow (\sqrt{x^2-1} - \sqrt{x})^2 = 0$.

31. We prove that

$$n\left(1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right) > (n+1)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)$$

by induction. For $n = 2$, $\frac{8}{3} > \frac{9}{4}$. Assume the claim is true for $n = k \geq 2$, *i.e.*, we have

$$k\left[1+\frac{1}{3}+\ldots.+\frac{1}{2n-1}\right] > k+1\left(\frac{1}{2}+\frac{1}{4}+\ldots.+\frac{1}{2k}\right)$$

Now let $n = k + 1$.

Note that $\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right)+\frac{k+1}{2k+1}$

$$=\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right)+\frac{1}{2}+\frac{k+1}{2k+1}$$

$$>\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)+\frac{1}{2}+\frac{k+1}{2k+1}$$

$$>\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)+\frac{k+1}{2k+2}+\frac{1}{2k+1}$$

$$>\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)+\frac{k+2}{2k+2}.$$

Induction is complete as

$$(k+1)\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}+\frac{1}{2k+1}\right)$$

$$=k\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right)+\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right)+\frac{k+1}{2k+1}$$

$$>k\left(1+\frac{1}{3}+\ldots+\frac{1}{2k-1}\right)+\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}\right)+\frac{k+2}{2k+2}$$

$$>(k+2)\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k+2}\right).$$

32. The solutions are $n = 3, 7, 23$. Since 2 is a prime, $1+\binom{n}{1}+\binom{n}{2}+\binom{n}{3} = 2^k$ for some positive integer $k \leq 2000$. We have

$$1+\binom{n}{1}+\binom{n}{2}+\binom{n}{3} = (n+1)\frac{(n^2-n+6)}{6},$$

i.e., $(n+1)(n^2-n+6)=3\times 2^{k+1}$. Let $m=n+1$, then $m\geq 4$ and $m(m^2-3m+8)=3\times 2^{k+1}$. We consider the following two cases.

(a) $m=2^s$. Since $m\geq 4$, $s\geq 2$. We have $2^{2s}-3\times 2^s+8=m^2-3m+8=3\times 2^t$ for some positive integer t. If $s\geq 4$, then $8\equiv 3\times 2^t \pmod{16} \Rightarrow 2^t=8$ $\Rightarrow m^2-3m+8=24 \Rightarrow m(m-3)=16$, which is impossible. Thus either $s=3$, $m=8$, $t=4$, $n=7$, or $s=2$, $m=4$, $t=2$, $n=3$.

33. The only answer is $(x, y, z)=(4, -1, 4)$.
Let $u=x+y-2$ and $v=z^2-8z+14=(z-4)^2\geq -2$. Rewriting the first equation, we have $u+1=-v\sqrt{u}\leq 2\sqrt{u}$, which implies that $(\sqrt{u}-1)^2\leq 0$. So, $u=1$, $u=-2$, *i.e.*, $z=4$, $x+y=3$. We substitute $x=3-y$ into the second equation, isolate the square root, and square both sides to obtain $9y^2+18y+9=y(3-y)+4$. Solving the quadratic equation, we have $y=-\frac{1}{2}, -1$. Only $y=-1$ leads to a valid solution $(4, -1, 4)$ to the original system.

34. (1, 1, 1), (1, 1, 2), (1, 2, 3) and all permutations. Bringing the second term to a common denominator, we get $lmn\mid(l+m+n)(lm+mn+nl)$ and, in particular, $l\mid(l+m+n)(lm+mn+nl) \Rightarrow l\mid(m+n)mn$ $\Rightarrow l\mid m+n$; likewise $m\mid n+l$ and $n\mid l+m$. Now assume WLOG that $l\geq m\geq n$. Then $\frac{(m+n)}{l}=1$ or 2; if the latter we have $l=m=n=1$. In the case $m+n=l$, we get $m\geq\frac{l}{2}\Rightarrow\frac{l}{m}\leq 2$ and $\frac{(n+l)}{m}\leq 3$. Moreover $l\geq m$ so this ratio cannot be 1; hence $n+l=2m$ giving the (1, 2, 3) solution or $n+l=3m$ giving the (1, 1, 2) solution.

35. 1st solution : We proceed indirectly; suppose that f exists. Then, $f(x+y)\leq\frac{f(x)^2}{f(x)+y}<f(x)$.
So, f is a strictly decreasing function. (In particular, this means that f is one-to-one.)
But since $f(x)$ is always positive, f must have some fixed point; suppose that $f(a)=a$ for some a. Now we can plug a in for x and obtain : $f(a+y)\leq\frac{a^2}{(a+y)}$.
This means that for some b, all $x\geq b$ will have $f(x)<1$. Observe that if we plug $f(x)$ in for y in the given equation, then we have $f(x)\geq 2f(x+f(x))$. But now if we plug b in for x here, we find
$1>f(b)\geq 2f(b+f(b))\geq 2f(b+1)$.
Hence $f(b+1)<\frac{1}{2}$. Plugging in $b+1$ yields
$\frac{1}{2}>f(b+1)\geq 2f(b+1+f(b+1))>2f\left(b+1+\frac{1}{2}\right)$.
Hence $f\left(b+1+\frac{1}{2}\right)<\frac{1}{4}$.
Proceeding in this manner, we find that
$f\left(b+\sum_{i=0}^{n-1}\frac{1}{2^i}\right)<\frac{1}{2^n}$.
Therefore, for each quantity $\varepsilon\in(0, 1)$, we can find a $\delta\in(0, 2)$ such that $f(b+x)<\varepsilon$ for all $x<2-\delta$. This means that $f(x)$ does not exist for $x\geq b+2$, and so we are done.

2nd solution : This is another way of presenting the idea in the first solution. Again, we proceed indirectly; suppose that f exists. As in the first solution, we can prove that f is a strictly decreasing function. For $(x, y)=(x, f(x))$, we have
$$f(x+f(x))\leq\frac{f(x)^2}{f(x)+f(x)}=\frac{f(x)}{2}.$$
Let $x_0\in R^+$ and $f(x_0)=a$. For $(x, y)=(x_0, f(x_0))$, we have $\frac{a}{2}=\frac{f(x_0)}{2}\geq f(x_0+f(x_0))=f(x_0+a)$.
Let $x_1=x_0+a$. Then $f(x_1)\leq\frac{a}{2}$. For $(x, y)=(x_1, f(x_1))$, we have $\frac{a}{4}\geq\frac{f(x_1)}{2}\geq f(x_1+f(x_1))\geq f\left(x_1+\frac{a}{2}\right)$.
Proceeding in this manner, we find that, as $n\to\infty$
$$f(x_0+3a)<f\left(x_0+\sum_{i=0}^{n}\frac{a}{2^n}\right)<\frac{a}{2^{n+1}}\to 0, \text{ a contradiction.}$$
So our assumption is false and there is no such function.

36. 1st solution : First we claim that at least one of bc and yz has its value less than 3. If $bc=3$, $c=1$, $a+b+c<3a=abc$; if $bc>3$, then $abc>3a\geq a+b+c$. Thus for $bc\geq 3$, we have $abc>a+b+c$ and $3x\geq x+y+z=abc>a+b+c=xyz\Rightarrow 3>yz$
This proves our claim. *WLOG*, suppose that $yz=1$ or 2.
If $yz=1$, then $y=z=1$.
We have $abc=x+y+z=x+2=xyz+2=a+b+c+2$.
If $c\geq 2$, then $bc\geq 4$ and $4a\leq abc-a+b+c+2\leq 4a$; thus $a=b=c=2$. We obtain the solutions (2, 2, 2, 6, 1, 1) and (6, 1, 1, 2, 2, 2). If $c=1$, then $ab=a+b+3$. If $b\geq 3$, then $3a\leq ab=a+b+3\leq 3a$ $\Rightarrow a=b=3$.
We obtain the solutions (3, 3, 1, 7, 1, 1) and (7, 1, 1, 3, 3, 1). If $b=2$, we have $a=5$ and obtain the solutions (5, 2, 1, 8, 1, 1) and (8, 1, 1, 5, 2, 1). If $b=1$, we have $a=a+4$, which is impossible.
If $yz=2$, then $y=2$, $z=1$. We have $2abc=2(x+y+z)=2x+6=xyz+6=a+b+c+6\leq 3a+6$.
If $c\geq 2$, then $8a\leq 2abc\leq 3a+6\Rightarrow 5a<6$, which

contradicts the fact that $a \geq c$. Thus $c = 1$, and $2ab = a + b + 7$. If $b \geq 3$, $6a \leq 2ab = a + b + 7 \Rightarrow a \leq \frac{b}{5} + \frac{7}{5}$, which contradicts the fact that $a \geq b$. If $b = 2$, then $4a = 2ab = a + 9$ and $a = 3$. We obtain the solution (3, 2, 1, 3, 2, 1). If $b = 1$, we have $a = 8$, repeating the solution (8, 1, 1, 5, 2, 1).

2nd solution : Let $A = (ab - 1)(c - 1)$, $B = (a - 1)(b - 1)$, $X = (xy - 1)(z - 1)$, $Y = (x - 1)(y - 1)$.

Thus A, B, X, Y are non-negative integers such that $A + B + X + Y = 4$.

Clearly, neither of c and z can be greater than 2; that would force either A or Y be greater than 4, and contradict the fact that $A + B + X + Y = 4$. If $c = 2$, we have $a, b \geq 2$ and $A \geq 3$, $B \geq 1$. Thus $A = 3$, $B = 1$, $X = Y = 0$. This yields the solution (2, 2, 2, 6, 1, 1). Similarly, if $z = 2$, we have (6, 1, 1, 2, 2, 2) as a solution.

Now we suppose that $c = z = 1$. We have $A = X = 0$ and $B + Y = 4$. WLOG, suppose that $Y \leq B$, (*i.e.*, $Y = 0, 1, 2$.)

If $Y = 0$, we have $B = (a - 1)(b - 1) = 4$. This leads to the solutions (5, 2, 1, 8, 1, 1) and (3, 3, 1, 7, 1, 1). By symmetry, we also have the solutions (8, 1, 1, 5, 2, 1) and (7, 1, 1, 3, 3, 1).

If $Y = 1$, then $x = y = 2$ and $B = (a - 1)(b - 1) = 3$ $\Rightarrow a = 4$, $b = 2$, but $a + b + c = 7 \neq xyz$.

If $Y = 2$, then $(x - 1)(y - 1) = (a - 1)(b - 1) = 2$ $\Rightarrow a = x = 3$, $b = y = 2$. We obtain (3, 2, 1, 3, 2, 1) as our last solution.

37. 1st solution :Applying the law of sines translates the problem into $\frac{AD}{DB} \cdot \frac{AE}{EB} = \left(\frac{\sin B}{\sin A}\right)^2$.

Let $\theta = \angle ACD$, $\phi = \angle ECB$, $\alpha = \angle DCE$. Again, application of the law of sines yields that $\frac{CD}{\sin A} = \frac{AD}{\sin\theta}$ and $\frac{DE}{\sin B} = \frac{BE}{\sin\phi}$.

Dividing these two equations, $\frac{CD\sin B}{CE\sin A} = \frac{AD\sin\phi}{AE\sin\theta}$

$$\Leftrightarrow \frac{CD\sin B}{CE\sin A} = \frac{BD\sin^2 B\sin\phi}{AE\sin^2 A\sin\theta} \Leftrightarrow \frac{CD}{CE} = \frac{BD\sin B\sin\phi}{AE\sin A\sin\theta}$$

$$\Leftrightarrow \frac{CD \cdot AE\sin A}{CE \cdot DB\sin B} = \frac{\sin\phi}{\sin\theta}.$$

But $\frac{CD}{BD} = \frac{\sin B}{\sin(\phi + \alpha)}$ and $\frac{AE}{CE} = \frac{\sin(\phi + \alpha)}{\sin A}$

Thus, $\frac{\sin A\sin B\sin(\theta + \alpha)}{\sin B\sin(\phi + \alpha)\sin A} = \frac{\sin\phi}{\sin\theta}$

$$\Leftrightarrow \sin(\theta + \alpha)\sin\theta = \sin\phi\sin(\phi + \alpha)$$

$$\Leftrightarrow \cos(2\theta + \alpha) - \cos\alpha = \cos(2\phi + \alpha) - \cos\alpha$$

$$\Leftrightarrow \cos(2\theta + \alpha) = \cos(2\phi + \alpha)$$

Since $(2\theta + \alpha) + (2\phi + \alpha) = 2\angle C < 360°$, $2\theta + \alpha = 2\phi + \alpha$ and hence $\angle ACD = \theta = \phi = \angle BCE$.

2nd solution :Extend AC and let F and G be points on the ray AC such that $BF \parallel CE$ and $BG \parallel CD$. We have $\angle CGB = \angle ACD$ and $\angle CBF = \angle BCE$. We claim that $\angle CBF = \angle CGB$, which implies that $\angle ACD = \angle BCE$.

We rewrite our condition as $\frac{AC^2 \cdot BD \cdot BE}{AD \cdot AE} = BC^2$.

38. Let $f(x) = x[x[x[x]]]$.

Lemma : Let a and b be real numbers. If a and b have the same sign and $|a| > |b| \geq 1$, then $|f(a)| > |f(b)|$.

Proof : We notice that $\|a\| \geq \|b\| \geq 1$. Multiplying this by $|a| > |b| \geq 1$, we have $|a[a]| > |b[b]| \geq 1$. Notice that $a[a]$ and $a[a[a]]$ have the same signs as $b[b]$ and $b[b[b]]$ respectively. In a similar manner, $|a[a[a]]| > |b[b[b]]| \geq 1$, $\|a[a[a]]\| \geq \|b[b[b]]\| \geq 1$, and $|f(a)| > |f(b)|$, as claimed.

We have $f(x) = 0$ for $|x| < 1$, $f(1) = f(-1) = 1$. Suppose that $f(x) = 88$. So $|x| > 1$, and we consider the following two cases :

(a) $x \geq 1$. It is easy to check that $f\left(\frac{22}{7}\right) = 88$. From the lemma, we know that $f(x)$ is increasing for $x > 1$. So $x = \frac{22}{7}$ is the unique solution on this interval.

(b) $x \geq -1$. From the lemma, $f(x)$ is decreasing for $x < 1$. Since $|f(-3)| = 81 < f(x) = 88 < \left|f\left(-\frac{112}{37}\right)\right| = 112$, $-3 > x > -\frac{112}{37}$ and $[x[x[x]]] = -37$.

But then $x = -\frac{88}{37} > -3$ a contradiction. So there is no solution on this interval.

Therefore, $x = \frac{22}{7}$ is the only solution.

Note : $\frac{22}{7}$ and $-\frac{112}{37}$ are found by finding $[x]$, $[x[x]]$ and $[x[x[x]]]$ in that order. For example, for $x \geq 1$, $f(3) < 88 < f(4)$ so $3 < x < 4$. Then $[x] = 3$ and $[x[x[3x]]] = 88$.

Then $f(3) < 88 < f\left(\frac{10}{3}\right)$ so $\left[x[x]\right] = 9$ and so on.

OLYMPIAD CORNER

Challenging problems for Olympiads, IIT-JEE and other contests.

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ALGEBRA

1. Determine the number of real solutions a of the equation $\left[\frac{a}{2}\right]+\left[\frac{a}{3}\right]+\left[\frac{a}{5}\right]=a.$

2. Let a, b, c be positive real numbers. Show that there is a triangle with sides a, b, c if and only if there exist real numbers x, y, z such that

$$\frac{y}{z}+\frac{z}{y}=\frac{a}{x},\ \frac{z}{x}+\frac{x}{z}=\frac{b}{y},\ \frac{x}{y}+\frac{y}{x}=\frac{c}{z}.$$

3. Let x_1, x_2, x_3, x_4 be positive real numbers such that $x_1x_2x_3x_4 = 1$.

Prove that $\sum_{i=1}^{4} x_i^3 \geq \max\left\{\sum_{i=1}^{4} x_i, \sum_{i=1}^{4}\frac{1}{x_i}\right\}$.

4. Let D be a point inside an acute triangle ABC such that $DA\cdot DB\cdot AB + DB\cdot DC\cdot BC + DC\cdot DA\cdot CA = AB\cdot BC\cdot CA$. Determine the geometric position of D.

5. Suppose a, b are natural numbers such that $p=\frac{b}{4}\sqrt{\frac{2a-b}{2a+b}}$ is a prime number. What is the maximum possible value of p?

NUMBER THEORY

6. Find the smallest integer n such that among any n integers, there exist 18 integers whose sum is divisible by 18.

7. **(a)** For which positive integers n do there exist positive integers x, y such that $\text{lcm}(x, y) = n!$, $gcd(x, y) = 1998$?

(b) For which n is the number of such pairs x, y with $x \leq y$ less than 1998?

8. Find all positive integers n that have exactly 16 positive integral divisors $d_1, d_2, ..., d_{16}$ such that $1 = d_1 < d_2 < .... < d_{16} = n$, $d_6 = 18$ and $d_9 - d_8 = 17$.

9. Solve the following equation in natural numbers : $x^2 + y^2 = 1997(x - y)$.

10. Prove that the equation $x^2y^2 = z^2(z^2 - x^2 - y^2)$ has no solutions in positive integers.

TRIGONOMETRY

11. Given a circle of radius 1 unit and AB is a chord of the circle with length 1 unit. If C is any point on the major segment, show that $AC^2 + BC^2 \leq 2(2+\sqrt{3})$.

12. Find $x, y, z \in R$ satisfying

$$\frac{4\sqrt{x^2+1}}{x}=\frac{5\sqrt{y^2+1}}{y}=\frac{6\sqrt{z^2+1}}{z} \text{ and } xyz = x + y + z.$$

13. Let $\cos^2\theta=\frac{\cos\alpha}{\cos\beta}, \cos^2\varphi=\frac{\cos\gamma}{\cos\beta}, \frac{\tan\theta}{\tan\varphi}=\frac{\tan\alpha}{\tan\gamma}$.

Prove that $\tan^2\frac{\alpha}{2}.\tan^2\frac{\gamma}{2}=\tan^2\frac{\beta}{2}$.

14. Eliminate θ from the equations $\cos(\alpha - 3\theta) = m\cos^3\theta$, $\sin(\alpha - 3\theta) = m\sin^3\theta$.

15. Let ABC be a non-obtuse triangle such that $AB > AC$ and $\angle B = 45°$. Let O and I denote the circumcentre and incentre of ΔABC respectively. Suppose that $\sqrt{2}OI = AB - AC$. Determine all the possible values of $\sin\angle BAC$.

16. Let $ABCD$ be a quadrilateral with $\angle BAC = 40°$ and $\angle ABC = 60°$. Let D and E be the points lying on the sides AC and AB, respectively, such that $\angle CBD = 40°$ and $\angle BCE = 70°$. Let BD and CE meet at F. Show that the line AF is perpendicular to the line BC.

GEOMETRY

17. Points K, L, M, N lie on the edges AB, BC, CD, DA of a (not necessarily right) parallelopiped $ABCDA_1B_1C_1D_1$. Prove that the centres of the circumscribed spheres of the tetrahedra A_1AKN, B_1BKL, C_1CLM, D_1DMN are the vertices of a parallelogram.

18. A triangle ABC has positive integer sides, $\angle A = 2\angle B$ and $\angle C > 90°$. Find the minimum length of the perimeter of ABC.

19. ABC is a triangle. The bisectors of $\angle B$ and $\angle C$

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meet AC and AB at D and E respectively & BD and CE intersect at O. If $OD = OE$, prove that either $\angle BAC = 60°$ or the triangle is isosceles.

20. In acute triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that the triangle ABC is equilateral.

21. Let D, E, F be points on the sides BC, CA, AB, respectively of triangle ABC. Let P, Q, R be the second intersections of AD, BE, CF, respectively, with the circumcircle of ABC. Show that $\frac{AD}{PD}+\frac{BE}{QE}+\frac{CF}{RF}\ge 9$ and determine when equality occurs.

22. Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, AB as their respective bases. Prove that the lines through A, B, C perpendicular to the lines EF, FD, DE, respectively, are concurrent.

MISCELLANEOUS

23. The set $S=\left\{\frac{1}{r}: r=1, 2, 3,...\right\}$ of reciprocals of the positive integers contains arithmetic progressions of various lengths. For instance, $\frac{1}{20}, \frac{1}{8}, \frac{1}{5}$ is such a progression, of length 3 and common difference $\frac{3}{40}$. Moreover, this is a maximal progression in S of length 3 since it cannot be extended to the left or right within S. $\left(-\frac{1}{40} \text{ and } \frac{11}{40} \text{ not being members of } S\right)$

(a) Find a maximal progression in S of length 1996.

(b) Is there a maximal progression in S of length 1997 ?

24. Two players take turns drawing a card at random from a deck of four cards labelled 1, 2, 3, 4. The game stops as soon as the sum of the numbers that have appeared since the start of the game is divisible by 3, and the player who drew the last card is the winner. What is the probability that the player who goes first wins?

25. Let m, n be natural numbers such that $A=\frac{(m+3)^n+1}{3m}$ is an integer. Prove that A is odd.

26. A sequence of real numbers x_n is defined recursively as follows : x_0, x_1 are arbitrary positive real numbers, and $x_{n+2}=\frac{1+x_{n+1}}{x_n}, n=0,1,2,...$ Find x_{1998}.

27. Prove that if $\cos\theta = \cos\alpha\cos\beta$, $\cos\varphi = \cos\alpha_1\cos\beta$, $\tan\frac{\theta}{2}\tan\frac{\varphi}{2}=\tan\frac{\beta}{2}$, then $\sin^2\beta=\left(\frac{1}{\cos\alpha}-1\right)\left(\frac{1}{\cos\alpha_1}-1\right)$.

28. Find the number of polynomials of degree 5 with distinct coefficients from the set {1, 2, ..., 9} that are divisible by x^2-x+1.

SOLUTIONS

1. There are 30 solutions. Since $\left[\frac{a}{2}\right], \left[\frac{a}{3}\right]$, and $\left[\frac{a}{5}\right]$ are integers, so it a. Now write $a = 30p + q$ for integers p and q, $0 \le q < 30$.

Then $\left[\frac{a}{2}\right]+\left[\frac{a}{3}\right]+\left[\frac{a}{5}\right]=a$

$\Leftrightarrow 31p+\left[\frac{q}{2}\right]+\left[\frac{q}{3}\right]+\left[\frac{q}{5}\right]=30p+q$

$\Leftrightarrow p=q-\left[\frac{q}{2}\right]-\left[\frac{q}{3}\right]-\left[\frac{q}{5}\right].$

Thus, for each value of q, there is exactly one value of p (and one value of a) satisfying the equation. Since q can equal any of thirty values, there are exactly 30 solutions, as claimed.

2. If x, y, z exist, then some two of them share the same sign: say, x and y. Then $z=\frac{c}{\left(\frac{x}{y}+\frac{y}{x}\right)}>0$. Thus $a+b-c=\frac{2xy}{z}, b+c-a=\frac{2yz}{x}$, and $c+a-b=\frac{2zx}{y}$ are all positive, so a, b, c form a triangle.
Conversely, if there is a triangle with sides a, b, c, then let $u = b + c - a$, $v = c + a - b$, $w = a + b - c$; by the triangle inequality, these are all positive. If there did exist satisfactory x, y, z, then from above $u=\frac{2yz}{x}, v=\frac{2zx}{y}, w=\frac{2xy}{z}$. Solving these equations gives $x=\sqrt{\frac{vw}{2}}, y=\sqrt{\frac{wu}{2}}, z=\sqrt{\frac{uv}{2}}$, and these values indeed satisfy the equations.

3. Let $A=\sum x_i^3$ and $A_i=A-x_i^3$, so that $A=\frac{1}{3}\sum A_i$. We claim that $A\ge\sum\frac{1}{x_i}$ and $A\ge\sum x_i$. From A.M.G.M.

$\frac{1}{3}A_1\ge\sqrt{x_2^3x_3^3x_4^3}=\frac{1}{x_1}.$

Combining the analogous inequalities gives $A\ge\sum\frac{1}{x_i}$, as claimed.
Also, by the power mean inequality,

$\frac{1}{4}A\ge\left(\frac{\sum x_i}{4}\right)^3\ge\left(\frac{\sum x_i}{4}\right)\left(\frac{\sum x_i}{4}\right)^2\ge\frac{\sum x_i}{4},$

since $\sum x_i\ge 4$ by A.M.G.M. So $A\ge\sum x_i$, as claimed.

4. Lemma : Let D be a point inside an acute triangle ABC. We have

$$DA\cdot DB\cdot AB + DB\cdot DC\cdot BC + DC\cdot DA\cdot CA \geq AB\cdot BC\cdot CA; \quad ....(4)$$

equality holds if and only if D is the orthocentre of ABC.

It is clear that the lemma contains our main result. We are going to prove the lemma in two ways.

1st solution : Let E and F be points such that $BCDE$ and $BCAF$ are both parallelograms. Thus $EDAF$ is also a parallelogram. We have $AF = ED = BC$, $EF = AD$, $EB = CD$, $BF = AC$.

Applying Ptolemy's theorem to quadrilaterals $ABEF$ and $AEBD$, we have $AB\cdot AD + BC\cdot CD = AB\cdot EF + AF\cdot BE \geq AE\cdot BF = AE\cdot AC$; $BD\cdot AE + AD\cdot CD = BD\cdot AE + AD\cdot BE \geq AB\cdot ED = AB\cdot BC$.

Now we have $DA\cdot DB\cdot AB + DB\cdot DC\cdot BC + DC\cdot DA\cdot CA = DB(AB\cdot AD + BC\cdot CD) + DC\cdot DA\cdot CA \geq DB\cdot AE\cdot AC + DC\cdot DA\cdot CA \geq AC(BD\cdot AE + AD\cdot CD) \geq AC\cdot AB\cdot BC$.

Equality holds if and only if both $ABEF$ and $AEBD$ are cyclic, which implies that $AFEBD$ and $AFED$ are cyclic. Since $AFED$ is a parallelogram, $AFED$ is a rectangle and $AD \perp ED$. Since $BCDE$ is parallelogram, we have $ED \parallel BC$ and $AD \perp BC$. Since $AEBD$ is cyclic, $\angle ABE = \angle ADE$, which implies that $BE \perp AB$. Since $BCDE$ is a parallelogram, we have $CD \parallel EB$ and $CD \perp AB$. Thus D is the orthocentre of ABC.

2nd solution : Let D be the origin of the complex plane and let the complex coordinates of A, B, C be u, v, w respectively. We rewrite as

$$|uv(u-v)| + |vw(v-w)| + |wu(w-u)| \geq |(u-v)(v-w)(w-u)|. \quad (1)$$

But it is easy to check that $uv(u-v) + uw(v-w) + wu(w-u) = -(u-v)(v-w)(w-u)$, (2)

which implies (2) and thus (1). Now we only need to determine when the equality holds. (3)

Let $z_1 = \dfrac{uv}{(u-w)(v-w)}, z_2 = \dfrac{vw}{(v-u)(w-u)}$,

$z_3 = \dfrac{wu}{(w-v)(u-v)}$.

We can rewrite (2) and (3) as

$|z_1| + |z_2| + |z_3| \geq 1$, $z_1 + z_2 + z_3 = 1$.

Equality holds if and only if z_1, z_2, z_3 are all positive real numbers.

Suppose that z_1, z_2, z_3 are all positive real numbers.

Since $-\dfrac{z_1z_2}{z_1} = \left(\dfrac{w}{u-v}\right)^2, -\dfrac{z_3z_1}{z_2} = \left(\dfrac{u}{v-w}\right)^2$,

we know $\dfrac{u}{(v-w)}$ and $\dfrac{v}{(w-u)}$ are pure imaginary numbers; thus $AD \perp BC$ and $BD \perp AC$ and D is the orthocentre of ABC.

Suppose that D is the orthocentre of the triangle ABC. Since the triangle is acute, D is inside the triangle. Therefore there are some positive numbers r_1, r_2, r_3 such that $\dfrac{u}{v-w} = -r_1 i$, $\dfrac{v}{w-u} = -r_2 i$, $\dfrac{w}{u-v} = -r_3 i$.

Thus z_1, z_2, z_3 are all positive real numbers.

From the above, we know that the equality in (1) holds if and only if D is the orthocentre of ABC.

5. The largest p is 5. Note that b is even, so we may write $b = 2c$.

Now $p = \dfrac{c}{2}\sqrt{\dfrac{a-c}{a+c}}$ or $\dfrac{4p^2}{c^2} = \dfrac{a-c}{a+c}$.

Write $\dfrac{2p}{c} = \dfrac{m}{n}$ in lowest terms. If $k = gcd(a-c, a+c)$, then $a - c = km^2$, $a + c = km^2$, $a + c = kn^2$,

so $2c = k(n^2 - m^2)$ and $4pn = km(n^2 - m^2)$.

In case m, n are both odd, then $8 \mid m^2 - n^2$ and so p is even, that is, $p = 2$. On the other hand, if m and n are not both odd, $n^2 - m^2$ is odd, so k must be even. Write $k = 2r$, so that we have $2pn = rm(n^2 - m^2)$. Now n is coprime to m and to $n^2 - m^2$, so n divides r. Writing $r = ns$, we have $2p = s(n-m)(n+m)m$.

Suppose $p > 2$. Since $m + n$ and $n - m$ are odd, $m + n = p$ and $n - m = 1$. Thus s, m are each at most 2. This leaves only the possibilities $(m, n) = (1, 2)$ or $(2, 3)$. In either case, $p \leq 5$, and fortunately this can be achieved with $m = 2$, $n = 3$, $s = 1$, $r = 3$, $k = 6$, $c = 15$, $b = 30$, and $a = 39$.

6. The minimum is $n = 35$, the 34-element set of 17 zeroes and 17 ones shows that $n \geq 35$, it remains to show that among 35 integers, there are 18 whose sum is divisible by 18. In fact, one can show that for n, among $2n - 1$ integers there are n whose sum is divisible by n.

We show this claim by induction on n, it's clear for $n = 1$. If n is composite, say $n = pq$, we can assemble sets of p integers whose sum is divisible by p as long as at least $2p - 1$ numbers remain, this gives $2q - 1$ sets, and again by the induction hypothesis, some q of these have sum divisible by q.

Now suppose $n = p$ is prime. The number x is divisible by p if and only if $x^{p-1} \neq 1 \pmod p$. Thus if the claim is false, then the sum of $(a_1 + ... + a_p)^{p-1}$ over all subsets $\{a_1, ..., a_p\}$ of the given numbers is congruent to $\dbinom{2p-1}{p-1} \equiv 1 \pmod p$. On the other hand, the sum of $a_1^{e_1} .... a_p^{e_p}$ for $e_1 + ... + e_p \leq p - 1$ is always divisible by p : if $k \leq p - 1$ of the e_i are non-zero, then each product is repeated $\dbinom{2p-1-k}{p-k}$ times, and the latter

is a multiple of p. This contradiction shows that the claim holds in this case. (Note : to solve the original problem, of course it sufficies to prove the cases $p = 2, 3$ directly).

7. **(a)** Let $x = 1998a$, $y = 1998b$. So a, b are positive integers such that $a < b$, $\gcd(a, b) = 1$. We have $\text{lcm}(x, y) = 1998ab = 2.3^3.37ab = n!$. Thus $n \geq 37$ and it is easy to see that this condition is also sufficient.

(b) The answers are $n = 37, 38, 39, 40$. We only need to consider positive integers $n \geq 37$. For $37 \leq n < 41$, let $k = ab = \dfrac{n!}{1998}$. Since $\gcd(a, b) = 1$, any prime factor of k that occurs in a cannot occur in b, and vice-versa. There are 11 prime factors of k, namely 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31. For each of those prime factors, one must decide only whether it occurs in a or in b. These 11 decisions can be made in a total of $2^{11} = 2048$ ways. However, only half of these ways will satisfy the condition $a < b$. Thus there will be a total of 1024 such pairs of (x, y) for $n = 37, 38, 39, 40$. Since 41 is a prime, we can see by a similar argument that there will be at least 2048 such pairs of (x, y) for $n \geq 41$.

8. Let integer $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ with $p_1, \ldots, p_m$ distinct primes. Then n has $(a_1 + 1)(a_2 + 1) \ldots (a_n + 1)$ divisors. Since $18 = 2 \cdot 3^2$, it has 6 factors : 1, 2, 3, 6, 9, 18. Since d has 16 divisors, we know that $d = 2 \cdot 3^3 p$ or $d = 2 \cdot 3^7$. If $b = 2.3^7$, $d_8 = 54$, $d_9 = 81$ and $d_9 - d_8 \neq 17$. Thus $d = 2 \cdot 3^3 \cdot p$ for some prime $p > 18$. If $p < 27$, then $d_7 = p$, $d_8 = 27$, $d_9 = 2p = 27 + 17 = 44 \Rightarrow p = 22$, a contradiction. Thus $p > 27$. If $p < 54$, $d_7 = 27$, $d_8 = p$, $d_9 = 54 = d_8 + 17 \Rightarrow p = 37$. If $p > 54$, then $d_7 = 27$, $d_8 = 54$, $d_9 = d_8 + 17 = 71$. We obtain two solutions for the problem : $2 \cdot 3^3 \cdot 37 = 1998$ and $2 \cdot 3^3 \cdot 71 = 3834$.

9. The solutions are $(x, y) = (170, 145)$ or $(1827, 145)$.

We have $x^2 + y^2 = 1997(x - y)$

$$2(x^2 + y^2) = 2 \times 1997(x - y)$$
$$x^2 + y^2 + (x^2 + y^2 - 2 \times 1997(x - y)) = 0$$
$$(x + y)^2 + ((x + y)^2 - 2 \times 1997(x - y)) = 0$$
$$(x + y)^2 + (1997 - x + y)^2 = 1997^2.$$

Since x and y are positive integers, $0 < x + y < 1997$ and $0 < 1997 - x + y < 1997$. Thus the problem reduces to solving $a^2 + b^2 = 1997^2$ in positive integers. Since 1997 is a prime, $gcd(a, b) = 1$. By Pythagorean substitution, there are positive integers $m > n$ such that $gcd(m, n) = 1$ and $1997 = m^2 + n^2$, $a = 2mn$, $b = m^2 - n^2$. Since $m^2, n^2 \equiv 0, 1, -1 \pmod 5$ and $1997 \equiv 2 \pmod 5$, $m, n \equiv \pm 1 \pmod 5$. Since $m^2, n^2 \equiv 0, 1 \pmod 3$ and $1997 \equiv 2 \pmod 3$, $m > n$, $m, n \equiv \pm 1 \pmod 3$. Therefore $m, n \equiv 1, 4, 11, 14 \pmod{15}$. Since $m > n$, $\dfrac{1997}{2} \leq m^2 \leq 1997$.

Thus we only need to consider $m = 34, 41, 44$. The only solution is $(m, n) = (34, 29)$. Thus $(a, b) = (1972, 315)$, which leads to our final solutions.

10. We begin by proving the following lemma.

Lemma-1 : The equation $s^4 - t^4 = u^4$ .....(1)

has no solutions in positive integers.

Proof : We proceed indirectly; suppose that there exists a non-empty set S of integers such that if $a \in S$ then there exist natural numbers b and c such that $a^4 - b^4 = c^2$. By the Well Ordering Principle, there exists a minimum element of S; suppose that in our equation $a^4 - b^4 = c^2$, a is equal to the minimum. Thus $gcd(a, b, c) = gcd(a, b) = gcd(b, c) = gcd(c, a) = 1$.

We consider the following cases :

(a) c is even. Then a and b are odd and $a^2 + b^2 \equiv 2 \pmod 4$. Since $gcd(a, b) = 1$, $4 \mid c^2$, and $(a^2 + b^2)(a^2 - b^2) = c^2$, $gcd(a^2 + b^2, a^2 - b^2) = 2$.

Let $x = \sqrt{\dfrac{(a^2 + b^2)}{2}}$ and $y = \sqrt{\dfrac{(a^2 - b^2)}{2}}$.

Then x and y are both integers, and we have $x^4 - y^4 = (ab)^2$. Since $a > b$, $x < a$; therefore this violates our assumption that our choice of a yields a minimal solution.

(b) c is odd. Then $c^2 \equiv 1 \pmod 4$. Since $a^4, b^4 \equiv 0$ or $1 \pmod 4$, a is odd and b is even. We rewrite our equation : $(a^2)^2 + (b^2)^2 + c^2$. We use Pythagorean substitution. There exist positive integers r and s such that $gcd(r, s) = 1$, $a^2 = r + s^2$, $b^2 = 2rs$, $c = r^2 - s^2$. We are only going to use the first identities. (Both are symmetric with respect to r and s.) Since $2 \mid b$, $4 \mid b^2$ and exactly one of r and s is even. We denote by r' the even one and by s' the odd one. Then there exist positive integers x, y for which $\dfrac{r'}{2} = x^2$ and $s' = y^2$. Thus $a^2 = 4x^4 + y^4$. By Pythagorean substitution, there are positive integers m, n such that $gcd(m, n) = 1$ and $a = m^2 + n^2$, $2x^2 = 2mn$, $y^2 = m^2 - n^2$. Since $x^2 = mn$ and $gcd(m, n) = 1$, $m = p^2$ and $n = q^2$. So $y^2 = m^2 - n^2 = p^4 - q^4$. Thus (p, q, y) is a new solution of (1) with $p = \sqrt{m} \leq x = \sqrt{\dfrac{r'}{2}} < r' \leq a$, and so again we violate our extremal assumption.

Therefore, our assumption is false and there is no solution for (1) in positive integers.

Now we apply the lemma to our problem. Solving the given equation for z^2, we find that the discriminant of the resulting quadratic is $x^4 + 6x^2y^2 + y^4$. Proceed indirectly again; suppose that $x^4 + 6x^2y^2 + y^4$ is a perfect square. Let $a = x^2 + y^2$ and $b = 2xy$. But now $a^2 + b^2$ and $a^2 - b^2$ are both perfect square, and $a^4 - b^4$ must

therefore be a perfect square as well! By our lemma, this is impossible, and we are finally done.

11. To solve this problem we use the sine formula from trigonometry. In the diagram, $\angle ACB = \frac{1}{2}\angle AOB = \frac{60°}{2} = 30°$

If $\angle CAB = \theta$ then $\angle ABC = (150° - \theta)$

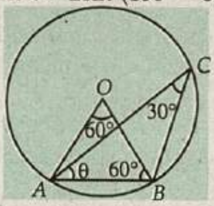
By sine rule $\frac{AB}{\sin 30°} = \frac{BC}{\sin\theta} = \frac{AC}{\sin(150° - \theta)}$

$\Rightarrow AC = 2 \times AB\sin(150° - \theta) = 2\sin(150° - \theta)$

and $BC = 2AB\sin\theta = 2\sin\theta$

$AC^2 + BC^2 = 4\sin^2\theta + 4\sin^2(150° - \theta)$

$= 2[2\sin^2\theta + 2\sin^2(150° - \theta)]$



$\because\ 2\sin^2 A = (1 - \cos 2A)$

Therefore $2[2\sin^2\theta + 2\sin^2(150° - \theta)]$

$= 2[2 - \cos 2\theta + \cos(300° - 2\theta)]$

$= 2[2 - (\cos 2\theta + \cos(300° - 2\theta))]$

$= 2\left[2 - \left(2\cos\frac{300°}{2}.\cos(150° - 2\theta)\right)\right]$

$\left[\because \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$

$= 2[2 - 2\cos 150°.\cos(150° - 2\theta)]$

$= 2[2 + \sqrt{3}.\cos(150° - 2\theta)]$

$\le 2(2 + \sqrt{3}) \because \cos(150° - 2\theta) \le 1$

Again $AC^2 + BC^2$ is a maximum when $\cos(150° - 2\theta)$ takes the maximum, value, *i.e.*, when $150° - 2\theta = 0$ $\Rightarrow \theta = 75°$

Then $\angle ABC = 75°$ therefore $\angle BAC = \theta = 75°$

Thus C takes the position at the mid-point of the major segment and $AC = BC$.

12. Let $x = \tan\alpha, y = \tan\beta, z = \tan\gamma, \frac{-\pi}{2} < \alpha, \beta, \gamma < \frac{+\pi}{2}$

$$\frac{4\sqrt{(\tan^2\alpha + 1)}}{\tan\alpha} = \frac{5\sqrt{(\tan^2\beta + 1)}}{\tan\beta} = \frac{6\sqrt{(\tan^2\gamma + 1)}}{\tan\gamma}$$

$$\Rightarrow \frac{4}{\sin\alpha} = \frac{5}{\sin\beta} = \frac{6}{\sin\gamma}$$

Again $\tan\alpha\tan\beta\tan\gamma = \tan\alpha + \tan\beta + \tan\gamma$

$\Rightarrow\ \tan\alpha(\tan\beta\tan\gamma - 1) = (\tan\beta + \tan\gamma)$

$$\Rightarrow -\tan\alpha = \frac{(\tan\beta + \tan\gamma)}{1 - \tan\beta\tan\gamma} = \tan(\beta + \gamma)$$

$\Rightarrow\ \tan(k\pi - \alpha) = \tan(\beta + \gamma) \Rightarrow \alpha + \beta + \gamma = k\pi$

Taking $k = 1$, we get $\alpha + \beta + \gamma = \pi$ which implies that there exists a Δ whose angles are α, β and γ and whose sides opposite to these angles are proportional to 4, 5 and 6 respectively.

Let the sides of such Δ be $4k$, $5k$ and $6k$.

s = semiperimeter of the $\Delta = \frac{15k}{2}$

$$\tan\frac{\alpha}{2} = \sqrt{\frac{(s-5k)(s-6k)}{s(s-4k)}} = \sqrt{\frac{\frac{5k}{2}\times\frac{3k}{2}}{\frac{15k}{2}\times\frac{7k}{2}}} = \sqrt{\frac{1}{7}}$$

$$x = \tan\alpha = \frac{2t}{1-t^2} = \frac{2\sqrt{\frac{1}{7}}}{1-\frac{1}{7}} = \frac{\sqrt{7}}{3}$$

Similarly, $y = \tan\beta = \frac{5\sqrt{7}}{9}$ and $z = \tan\gamma = 3\sqrt{7}$

$$\left[\tan\frac{\beta}{2} = \sqrt{\frac{(s-4k)(s-6k)}{s(s-5k)}} \text{ and } \tan\frac{\gamma}{2} = \sqrt{\frac{(s-4k)(s-5k)}{s(s-6k)}}\right]$$

where α, β, γ are measures of the angles A, B and C of ΔABC.

13. We have $1 + \tan^2\theta = \frac{\cos\beta}{\cos\alpha}, 1 + \tan^2\varphi = \frac{\cos\beta}{\cos\gamma}$

Hence $\frac{\tan^2\theta}{\tan^2\varphi} = \frac{\cos\beta - \cos\alpha}{\cos\alpha}.\frac{\cos\gamma}{\cos\beta - \cos\gamma}$

On the other hand, it is given that $\frac{\tan^2\theta}{\tan^2\varphi} = \frac{\tan^2\alpha}{\tan^2\gamma}$.

Therefore, we have $\frac{\cos\beta - \cos\alpha}{\cos\beta - \cos\gamma}.\frac{\cos\gamma}{\cos\alpha} = \frac{\tan^2\alpha}{\tan^2\gamma}$

From this equality we get

$$\cos\beta = \frac{\cos^2\alpha\sin^2\gamma - \cos^2\gamma\sin^2\alpha}{\cos\alpha\sin^2\gamma - \sin^2\alpha\cos\gamma} = \frac{\sin^2\gamma - \sin^2\alpha}{\cos\alpha\sin^2\gamma - \sin^2\alpha\cos\gamma}$$

But $\tan^2\frac{\beta}{2} = \frac{1 - \cos\beta}{1 + \cos\beta}$

$$= \frac{\cos\alpha\sin^2\gamma - \sin^2\alpha\cos\gamma - \sin^2\gamma + \sin^2\alpha}{\cos\alpha\sin^2\gamma - \sin^2\alpha\cos\gamma + \sin^2\gamma - \sin^2\alpha}$$

$$= \frac{\sin^2\alpha(1 - \cos\gamma) - \sin^2\gamma(1 - \cos\alpha)}{\sin^2\gamma(1 + \cos\alpha) - \sin^2\alpha(1 + \cos\gamma)}$$

$$= \frac{8\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\sin^2\frac{\gamma}{2} - 8\sin^2\frac{\gamma}{2}\cos^2\frac{\gamma}{2}\sin^2\frac{\alpha}{2}}{8\sin^2\frac{\gamma}{2}\cos^2\frac{\gamma}{2}\cos^2\frac{\alpha}{2} - 8\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\cos^2\frac{\gamma}{2}}$$

$$= \frac{\sin^2\frac{\alpha}{2}\sin^2\frac{\gamma}{2}\left(\cos^2\frac{\alpha}{2} - \cos^2\frac{\gamma}{2}\right)}{\cos^2\frac{\alpha}{2}\cos^2\frac{\gamma}{2}\left(\sin^2\frac{\gamma}{2} - \sin^2\frac{\alpha}{2}\right)} = \tan^2\frac{\alpha}{2}.\tan^2\frac{\gamma}{2}$$

Since $\cos^2\frac{\alpha}{2}-\cos^2\frac{\gamma}{2}=\sin^2\frac{\gamma}{2}-\sin^2\frac{\alpha}{2}$.

14. Expanding the given equalities, we get $\cos\alpha\cos3\theta+\sin\alpha\sin3\theta=m\cos^3\theta$, $\sin\alpha\cos3\theta-\cos\alpha\sin3\theta=m\sin^3\theta$. Multiplying the first equality by $\cos3\theta$, the second by $-\sin3\theta$ and adding them term by term, we find $\cos\alpha=m\{\cos^3\theta\cos3\theta-\sin^3\theta\sin3\theta\}$

But it is known that $\cos3\theta=4\cos^3\theta-3\cos\theta$,

$\sin3\theta=3\sin\theta-4\sin^3\theta$.

Consequently $\cos^3\theta\cos3\theta-\sin^3\theta\sin3\theta$

$=4(\cos^6\theta+\sin^6\theta)-3(\sin^4\theta+\cos^4\theta)$.

But squaring the original equality and adding, we get

$$\cos^6\theta+\sin^6\theta=\frac{1}{m^2}.$$

Compute $\cos^4\theta+\sin^4\theta$, we have

$\cos^6\theta+\sin^6\theta=(\cos^2\theta+\sin^2\theta)(\cos^4\theta+\sin^4\theta-\cos^2\theta\sin^2\theta)$

$=\cos^4\theta+\sin^4\theta-\cos^2\theta\sin^2\theta$

Therefore $\frac{1}{m^2}=(\cos^2\theta+\sin^2\theta)^2-3\sin^2\cos^2\theta$,

$$3\sin^2\theta\cos^2\theta=1-\frac{1}{m^2}$$

$$\sin^4\theta+\cos^4\theta=1-2\sin^2\theta\cos^2\theta$$

$$=1-\frac{2}{3}\left(1-\frac{1}{m^2}\right)=\frac{1}{3}\left(1+\frac{2}{m^2}\right).$$

Thus $\cos\alpha=m\{4(\cos^6\theta+\sin^6\theta)-3(\sin^4\theta+\cos^4\theta)\}$

$=m\left\{\frac{4}{m^2}-1-\frac{2}{m^2}\right\}=\frac{2-m^2}{m}$, *i.e.*, $m^2+m\cos\alpha=2$.

15. 1st solution : Let $a=BC$, $b=CA$, $c=AB$, $\alpha=\angle CAB$, $\beta=\angle ABC$, $\gamma=\angle BCA$, and let R and r be the circumradius and inradius of ABC, respectively. Applying law of sines to ABC, we have

$a=2R\sin\alpha$, $b=2R\sin\beta$, $c=2R\sin\gamma$.

Since $\beta=45°$, $\sin\beta=\frac{\sqrt{2}}{2}$, $\tan\left(\frac{\beta}{2}\right)=(\sqrt{2}-1)$,

$$\sin\gamma=\sin(135°-\alpha)=\frac{\sqrt{2}(\sin\alpha+\cos\alpha)}{2}. \quad ..... (1)$$

Thus $r=\frac{(c+a-b)}{2}\tan\left(\frac{\beta}{2}\right)=R(\sqrt{2}-1)(\sin\alpha+\sin\gamma-\sin\beta)$.

From Euler's formula $OI^2=R(R-2r)$, we have

$$OI^2=R^2(1-2(\sin\alpha+\sin\gamma-\sin\beta)(\sqrt{2}-1)) \quad ..... (2)$$

Since $\sqrt{2}OI=AB-AC$.

$$OI^2=(c-2(\sin\alpha+\sin\gamma-\sin\beta)(\sqrt{2}-1)). \quad ..... (3)$$

From (1) and (2), we obtain

$$2(\sin\gamma-\sin\beta)^2=(1-2(\sin\alpha+\sin\gamma-\sin\beta)(\sqrt{2}-1))$$

$$\Leftrightarrow 1-2(\sin\gamma-\sin\beta)^2=(\sin\alpha+\sin\gamma-\sin\beta)(\sqrt{2}-1)$$

$$\Leftrightarrow 1-2\sin^2\gamma+2\sqrt{2}\sin\gamma-1$$

$$=2(\sin\alpha+\sin\gamma)(\sqrt{2}-1)-(2-\sqrt{2})$$

$$\Leftrightarrow -(\sin\alpha+\cos\alpha)+2(\sin\alpha+\cos\alpha)$$

$$=(2\sqrt{2}-2)\sin\alpha+(2-\sqrt{2})(\sin\alpha+\cos\alpha)-(2-\sqrt{2})$$

$$\Leftrightarrow -1-2\sin\alpha\cos\alpha=(\sqrt{2}-2)\sin\alpha-\sqrt{2}\cos\alpha-(2-\sqrt{2})$$

$$\Leftrightarrow 2\sin\alpha\cos\alpha-(2-\sqrt{2})\sin\alpha-\sqrt{2}\cos\alpha+(\sqrt{2}-1)=0$$

$$\Leftrightarrow(\sqrt{2}\sin\alpha-1)(\sqrt{2}\cos\alpha-\sqrt{2}+1)=0$$

Thus $\sin\alpha=\frac{\sqrt{2}}{2}$ or $\alpha=1-\frac{\sqrt{2}}{2}$,

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\frac{\sqrt{4\sqrt{2}-2}}{2}.$$

2nd solution : Let I_c, I_a, I_b be the feet of perpendiculars from I to AB, BC, CA respectively. Let D be the foot of the perpendicular from O to BC. Thus OD is the perpendicular bisector of BC and $BD=CD$. From equal tangents, we have $AI_c=AI_b$, $BI_a=BI_c$, $CI_a=CI_b$. We have

$$\sqrt{2}OI=c-b=BI_a-I_aC.$$

$$=(AI_c+I_cB)-(AI_b+I_bC)=I_cB-I_bC$$

Since $c>b$, D is on BI_a. We have $BI_a=BD+DI_a$, $I_aC=CD-DI_a$. So $\sqrt{2}OI=2DI_a$, *i.e.*, $OI=\sqrt{2}DI_a$. Thus line OI and line DI_a form a 45° angle, which implies that either $OI\perp AB$ or $OI \parallel AB$.

(a) $OI\perp AB$. Then OI is the perpendicular bisector of AB. Thus $AC=BC$, $\alpha=\beta=45°$, and $\sin\alpha=\frac{\sqrt{2}}{2}$.

(b) $OI \parallel AB$. Let E be the foot of the perpendicular from O to AB. So $\angle AOE=\angle C=\gamma$, $R\cos\angle AOE=R\cos\gamma=OE=II_c=r$.

Since $r=4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$, we have

$$\cos\gamma=4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=2\sin\frac{\beta}{2}\left(2\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\right)$$

$$=2\sin\frac{\beta}{2}\left(-\cos\left(\frac{\alpha+\gamma}{2}\right)+\cos\left(\frac{\alpha-\gamma}{2}\right)\right)$$

$$=2\sin\frac{\beta}{2}\left(-\sin\frac{\beta}{2}+\cos\left(\frac{\alpha-\gamma}{2}\right)\right)$$

$$=-2\sin^2\frac{\beta}{2}+2\sin\frac{\beta}{2}\cos\left(\frac{\alpha-\gamma}{2}\right)$$

$$=\cos\beta-1+\sin\left(\frac{\alpha+\beta-\gamma}{2}\right)+\sin\left(\frac{\beta+\gamma-\alpha}{2}\right)$$

$=\cos\beta-1+\sin(90°-\gamma)+\sin(90°-\alpha)$

$=\cos\beta-1+\cos\gamma+\cos\alpha$, which implies that

$$\cos\alpha=1-\cos\beta=1-\frac{\sqrt{2}}{2}$$

and $\sin\alpha=\sqrt{1-\cos^2\alpha}=\frac{\sqrt{4\sqrt{2}-2}}{2}$.

16. Note that $\angle ABD = 20°$, $\angle BCA = 80°$ and $\angle ACE = 10°$. Let G be the foot of the altitude from A to BC.
Then $\angle BAG = 90° - \angle ABC = 30°$ and
$\angle CAG = 90° - \angle BCA = 10°$.

Now, $$\frac{\sin\angle BAG \sin\angle ACE \sin\angle CBD}{\sin\angle CAG \sin\angle BCE \sin\angle ABD} = \frac{\sin 30° \sin 10° \sin 40°}{\sin 10° \sin 70° \sin 20°}$$
$$= \frac{\left(\frac{1}{2}\right)(\sin 10°)(2\sin 20° \cos 20°)}{\sin 10° \cos 20° \sin 20°} = 1.$$

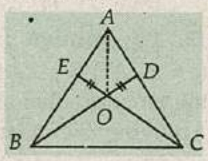
Then by the trigonometric form of Ceva's Theorem, AG, BD, and CE are concurrent. Therefore, F lies on AG so AF is perpendicular to the line BC, as desired.

17. Introduce coordinates with $ABCD$ parallel to $z = 0$. Let E, F, G, H be the circumcentres of triangles AKN, BKL, CLM, DMN and let W, X, Y, Z be the circumcentres of tetrahdra A_1AKN, B_1BKL, C_1CLM, D_1DMN. Also for each point Q we have labelled, let Q_1, Q_2, Q_3 denote the x, y, z-coordinates of Q.
We first show that $EFGH$ is a parallelogram, by solving that the midpoints of EG and FH coincide. It sufficies to show this for the projections of the segments in two different directions (*e.g.*, introduce coordinates along those directions). But this is evident for the projections onto AB, as E and F project onto the midpoints of AK and BK, respectively, so the segment between them has length $AB/2$, as does the corresponding segment on CD. Likewise, the claim is evident for the projections onto CD.
We now have $E_1 + G_1 = F_1 + H_1$ and $E_2 + G_2 = F_2 + H_2$. Also, since W and E are equidistant from AKN, WE is perpendicular to AKN and thus to the plane $z = 0$. Thus $W_1 = E_1$ and $W_2 = E_2$, and likewise for X, Y, Z. Thus $W_1 + Y_1 = X_1 + Z_1$ and $W_2 + Y_2 = X_2 + Z_2$. All that remains is to show $W_3 + Y_3 = X_3 + Z_3$. Notice that W and X both lie on the plane perpendicular to ABB_1A_1 and passing through the midpoints of AA_1 and BB_1. Thus $W_3 = aW_1 + bW_2 + c$ and $X_3 = aX_1 + bX_2 + c$ for some constants a, b, c. Similarly, Y and Z both lie on the plane perpendicular to CDD_1C_1 and passing through the midpoints of CC_1 and DD_1. Since DCC_1D_1 is parallel and congruent to ABB_1A, we have $Y_3 = aY_1 + bY_2 + d$ and $Z_3 = aZ_1 + bZ_2 + d$ for d another constant, but a and b the same constants as above. Therefore $W_3 + Y_3 = X_3 + Z_3$, completing the proof that $WXYZ$ is a parallelogram.

18. 1st solution : Let $BC = a$, $CA = b$, $AB = c$. We have $A = 2B$ and $C = 180° - 3B$. By the law of sines, $\frac{b}{\sin B} = \frac{a}{\sin A} = \frac{c}{\sin C}$.
Since $\sin A = \sin 2B = 2\sin B \cos B$, $\sin C = \sin 3B = 3\sin B - 4\sin^3 B$, we have $a = 2b\cos B$, $c = b(3 - 4\sin^2 B) = b(4\cos^2 B - 1)$ and hence $a^2 = b(b + c)$. Since we are looking for a triangle of smallest perimeter, we may assume that $gcd(a, b, c) = 1$. In fact, $gcd(b, c) = 1$, since any common factor of b and c would be a factor of a as well. We notice that since a perfect square a^2 is being expressed as the product of two relatively prime integers b and c, it must be the case that both b and $b + c$ are perfect squares. Thus, for some integers m and m, with $gcd(m, m) = 1$, we have $b = m^2$, $b + c = n^2$, $a = mn$, $2\cos B = \frac{n}{m} = \frac{a}{b}$. Since $C > 90°$, we have $0 < B < 30°$ and $\sqrt{3} < 2\cos B = \frac{n}{m} < 2$.
It is easy to check that $(m, n) = (4, 7)$ is the smallest pair that generates a triangle $(a, b, c) = (28, 16, 33)$ that meets all the conditions.

2nd solution: We use same notations as those in the first solution. Let the angle bisector of $\angle CAB$ meet BC at D. Since $\angle BAD = \angle ABD$, we let $AD = BD = x$. We have $\angle ACD = B$, $\angle ACB = \angle ACD$, so triangles ABC and DAC are similar. We have $\frac{x}{c} = \frac{b}{a} = \frac{a - x}{b}$ which leads to $ax = bc$, $b^2 = a^2 - ax \Rightarrow a^2 = b(b + c)$, and the rest is the same.

19. Join AO. In $\angle AOD$,
$$m\angle OAD = \frac{A}{2}, m\angle ODA = m\angle BDA = C + \frac{B}{2}$$
(exterior $\angle$ = sum of the remote interior $\angle$s)



$$\angle AOD = 180^o - \frac{A}{2} - \frac{B}{2} - C$$
$$= 180^o - \frac{1}{2}(180° - C) - C = 90° - \frac{1}{2}C$$
Similarly in ΔAOE,
$$\angle OAE = \frac{A}{2}, m\angle OEA = m\angle CEA = B + \frac{C}{2}$$
(exterior $\angle$ = sum of the remote interior $\angle$s)
and $\angle EOA = 180^o - \frac{A}{2} - \frac{C}{2} - B$
$$= 180° - \frac{1}{2}(180° - B) - B = 90° - \frac{1}{2}B$$
Using sine formula for the two triangles ADO and AEO, we get $$\frac{OD}{\sin\angle OAD} = \frac{AO}{\sin\angle ADO} \Rightarrow \frac{OD}{\sin\frac{A}{2}} = \frac{AO}{\sin\left(C + \frac{B}{2}\right)}$$
$$\Rightarrow OD = \frac{AO\sin\frac{A}{2}}{\sin\left(C + \frac{B}{2}\right)} \quad \text{........(1)}$$
Again $$\frac{OE}{\sin\angle OAE} = \frac{AO}{\sin\angle OEA}$$

$$\Rightarrow \frac{OE}{\sin\frac{A}{2}} = \frac{AO}{\sin\left(B+\frac{C}{2}\right)} \Rightarrow OE = \frac{OA\sin\frac{A}{2}}{\sin\left(B+\frac{C}{2}\right)} \quad \text{........(2)}$$

But $OD = OE$ (given). From (1) and (2), we get $\sin\left(C+\frac{B}{2}\right) = \sin\left(B+\frac{C}{2}\right)$

$$\Rightarrow C+\frac{B}{2} = B+\frac{C}{2} \quad \text{....(3)}$$

$$\text{or } C+\frac{B}{2} = 180^o - \left(B+\frac{C}{2}\right) \quad \text{....(4)}$$

$\Rightarrow \frac{C}{2} = \frac{B}{2}$ from (3)

$\Rightarrow \angle B = \angle C \Rightarrow \frac{3}{2}(B+C) = 180^o \Rightarrow B + C = 120°$ from (4)

$\therefore \angle A = 180° - 120° = 60°$.

20. Let $\angle ACF = \angle CBM = A$, let $CM = AM = m$. Then $MB = CF = 2m\cos A$. By the Law of Sines, $\frac{CM}{\sin\angle CBM} = \frac{MB}{\sin\angle MCB}$,
and so $\sin\angle MBC = 2\cos A\sin A = \sin 2A$.
This leaves two possibilities. If $\angle MCB + 2A = 180°$, then $\angle CMB = A = \angle MBC$.
Then $CB = MC$ and $MB = 2MC\sin A$. Also $MB = CF = AC\cos A = 2MC\cos A$. Therefore $\sin A = \cos A$ so $A \geq 45°$, $\angle MCB \geq 90°$, a contradiction.
Thus we conclude $\angle MCB = 2A$, so $\angle ACF = \angle BCF$.
Therefore triangle ACF is congruent to BCF and $AC = BC$. Now triangle ACF is congruent to CBM, so $\angle CAF = \angle BCM$. Therefore $BC = AB$, so ABC is equilateral.

21. Fix ABC and note that $\frac{AD}{PD} = \frac{d(A, BC)}{d(P, BC)}$, which has a constant numerator and so is minimized when the denominator is maximized, which occurs when P is the midpoint of the arc BC; and analogously for Q and R. Hence it suffices to prove the result when rays AD, BE, CF are angle bisectors. We have $\angle PBD = \frac{\angle BAC}{2} = \angle PAB$ and so triangles PBD, PAB are similar and $\frac{PA}{PD} = \frac{PA}{PB}\cdot\frac{PB}{PD} = \left(\frac{PA}{PB}\right)^2 = \left(\frac{AB}{BD}\right)^2$.

But using the angle bisector theorem, $\frac{AB}{BD} = \frac{(b+c)}{a}$ and likewise $\frac{BC}{CE} = \frac{(c+a)}{b}, \frac{CA}{AF} = \frac{(a+b)}{c}$; now either expanding, regrouping, and using A.M. G.M. or, more elegantly, using RMS-A.M. and A.M. G.M. as shown below, gives $\sum\frac{PA}{PD} = \sum\left(\frac{AB}{BD}\right)^2 \geq \frac{1}{3}\left(\sum\frac{AB}{BD}\right)^2$ $= \frac{1}{3}\left(\sum\frac{b+c}{a}\right)^2 \geq 12$ and subtracting 3 from sides gives our result. Equality requires that AD, BE, CF be angle bisectors and (because of the A.M. G.M. step) that ABC be equilateral.

22. First solution : We first show that for any four points W, X, Y, Z in the plane, the lines WX and YZ are perpendicular if and only if

$$WY^2 - WZ^2 = XY^2 - XZ^2 \quad \text{.......(*)}$$

To prove this, introduce Cartesian coordinates such that $W = (0, 0)$, $X = (1, 0)$, $Y = (x_1, y_1)$, and $Z = (x_2, y_2)$. Then (*) becomes $X_1^2 + Y_1^2 - X_2^2 - Y_2^2 = (x_1 - 1)^2 + y_1^2 - (x_2 - 1)^2 - y_2^2$, which upon cancellation yields $x_1 = x_2$. This is true if and only if line YZ is perpendicular to the x-axis WX.
If P is the intersection of the perpendicular from B and C to lines FD and DE, respectively, then the fact noted above yields $PF^2 - PD^2 = BF^2 - BD^2$ and $PD^2 - PE^2 = CD^2 - CE^2$.
From the given isosceles triangles, we have $BF = AF$, $BD = CD$, and $CE = AE$. $\therefore$ $PF^2 - PE^2 = AF^2 - AE^2$.
Hence line PA is also perpendicular to line EF, which completes the proof.

Second Solution : Let C_1 be the circle with centre D and radius BD, C_2 the circle with centre E and radius CE, and C_3 the circle of centre F and radius AF. The line through A and perpendicular to EF is the radical axis of circles C_2 and C_3, the line through B and perpendicular to DF is the radical axis of circles C_1 and C_3, and the line through C and perpendicular to DE is the radical axis of circles C_1 and C_2. The result follows because three radical axes meet at the radical centre of the three circles.

Third solution : Let A', B', C' be points on EF, DF, DE respectively, with $AA' \perp EF$, $BB' \perp DF$ and $CC' \perp DE$. In addition, let D', E', F' be points on BC, AC, AB, respectively, with $DD' \perp BC$, $EE' \perp AC$, and $FF' \perp AB$. Because DD', EE', and FF' are the perpendicular bisectors of the sides of triangle ABC, these three lines are concurrent, meeeting at the circumcentre of triangle ABC. Thus, by the trigonometric form of Cevas Theorem applied in triangle DEF.

23. There is a maximal progression of length n, for all $n > 1$. Dirichlet's theorem implies that there is a prime number p of the form $1 + dn$ for some positive integer d. Now consider the progression $\frac{1}{(p-1)!}, \frac{1+d}{(p-1)!}, \ldots, \frac{1+(n-1)d}{(p-1)!}$.

Since the numerators divide the denominators, each fraction is the reciprocals of an integer, but this is not the case for $\frac{(1+nd)}{(p-1)!}=\frac{p}{(p-1)!}$ since p is prime. Therefore this sequence is a maximal progression. (To solve (a), simply take $p = 1997$).

24. 1st solution : Let P be the desired probability. For positive integers n, let b_n be the probability that the n-th card has been drawn and the sum of the numbers on the first n cards is 1 modulo 3; let c_n be the probability that the n-th card has been drawn and the sum of the numbers on the first n cards is 2 modulo 3; and let a_n be the probability that the game ends immediately after the n-th card has been drawn.

We notice that $a_1=\frac{1}{4}, b_1=\frac{1}{2}, c_1=\frac{1}{4}, b_2=\frac{3}{16}$, and we have the following relations:

$$b_{n+1}=\frac{b_n}{4}+\frac{c_n}{4},\ c_{n+1}=\frac{b_n}{2}+\frac{c_n}{4},\ a_{n+1}=\frac{b_n}{4}+\frac{c_n}{2}.$$

Subtracting the first two equations from each other, we obtain that $c_{n+1}=\frac{b_n}{4+b_{n+1}}$. Subtracting this back to the first equation, we have $16b_{n+2}=8b_{n+1}+b_n$. Solving the characteristic equation $16x^2 = 6x + 1$ we have

$$b_n=\left(\frac{3\sqrt{2}-2}{4}\right)\left(\frac{\sqrt{2}+1}{4}\right)^n-\left(\frac{3\sqrt{2}+2}{4}\right)\left(\frac{1-\sqrt{2}}{4}\right)^n.$$

We have

$$a_{n+2}=\frac{b_{n+1}}{4}+\frac{c_{n+1}}{2}=\frac{b_{n+1}}{4}+\frac{\left(\frac{b_n}{4}+b_{n+1}\right)}{2}=\frac{3b_n}{4}+\frac{b_n}{8}$$

To solve our problem, we only need to calculate four convergent infinite series. In fact,

$$p = a_1 + a_3 + \dots + a_{2n-1} + \dots$$

$$=\frac{1}{4}+\frac{3(b_2+b_4+\dots)}{4}+\frac{b_1+b_3+\dots}{4}=\frac{1}{4}+\frac{21}{92}+\frac{2}{23}=\frac{13}{23}.$$

2nd solution : Let a_1, a_2 be the probabilities that player one will win given that when he starts a turn, the sum is 1 or 2(mod 3), respectively (in other words, as if the sum were 1 or 2 at the beginning of the game.). Let b_1 and b_2 be the probabilities that player one will win given that when his opponent starts a turn, the sum is 1 or 2(mod 3), respectively. We wish to find $P=\frac{1}{4}+\frac{1}{2}b_1+\frac{1}{4}b_2$.

Now, we have the following relations:

$$a_1=\frac{1+b_1+2b_2}{4},\ a_2=\frac{2+b_1+b_2}{4},$$

$$b_1=\frac{a_1+2a_2}{4},\ b_2=\frac{a_1+a_2}{4}.$$

We multiply the first two equations by 4 and the last two equations by 16, then substitute the first two equations into the last two equations to get two equations in b_1 and b_2. These equations have solution $b_1=\frac{11}{23}, b_2=\frac{7}{23}$, which implies that $P=\frac{13}{23}$.

25. 1st solution : Let p be a prime and 'a' a positive integer. Let $\left(\frac{a}{p}\right)$ be the Legendre symbol, *i.e.*, $\left(\frac{a}{p}\right)$ will have value 1 if a is a quadratic residue module p, -1 if a is a quadratic non-residue modulo p, and 0 if $\frac{p}{a}$. For odd primes p and q, we have LQR (Law of Quadratic Reciprocity):

$$\left(-\frac{1}{p}\right)=(-1)^{\frac{(p-1)}{2}},\ \left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{\frac{(p-1)(q-1)}{4}}.$$

If m is odd, then $(m + 3)^n + 1$ is odd and A is odd. Now we suppose that m is even. Since A is an integer, $0 \equiv (m+3)^n+1 \equiv m^n+1 \pmod 3$, so $n = 2k+1$ is odd and $m \equiv -1 \pmod 3$. We consider the following cases.

(a) $m = 8m'$ for some positive integer m'. Then $(m + 3)^n + 1 \equiv 3^{2k+1} + 1 \equiv 4 \pmod 8$ and $3m \equiv 0 \pmod 8$. So A is not an integer.

(b) $m = 2m'$ for some odd positive integer m', *i.e.*, $m \equiv 2 \pmod 4$. Then $(m + 3)^n + 1 \equiv (2 + 3)^n + 1 \equiv 2 \pmod 4$ and $3m \equiv 2 \pmod 4$. So A is odd.

(c) $m = 4m'$ for some odd positive integer m'. Since $m \equiv -1 \pmod 3$, there exists an odd prime p such that $p \equiv -1 \pmod 3$ and $p \mid m$. Since A is an integer, $0 \equiv (m + 3)^n + 1 \equiv 3^{2k+1} \equiv 1 \pmod m$ and $3^{2k+1} \equiv -1 \pmod p$. Let a be a primitive root module p; let b be a positive integer such that $3 \equiv a^b \pmod p$. Thus $a^{(2k+1)b} \equiv -1 \pmod p$. Note that $\left(\frac{p}{3}\right)=\left(\frac{-1}{3}\right)=-1$. We consider the following cases :

(i) $p \equiv 1 \pmod 4$. From LQR, $\left(\frac{-1}{p}\right)=1$, so $a^{2c} \equiv -1 \equiv a^{(2k+1)b} \pmod p$ for some positive integer c. Therefore b is even and $\left(\frac{3}{p}\right)=1$. Again, from LQR, we have

$$-1=\left(\frac{3}{p}\right)\left(\frac{p}{3}\right)=(-1)^{\frac{(3-1)(p-1)}{4}}=1, \text{ a contradiction.}$$

(ii) $p \equiv 3 \pmod 4$. From LQR, $\left(\frac{-1}{p}\right)=-1$, so $a^{2c+1} \equiv -1 \equiv a^{(2k+1)b} \pmod p$ for some positive integer c. Therefore b is even and $\left(\frac{3}{p}\right)=-1$. Again, from LQR,

we have $1=\left(\frac{3}{p}\right)\left(\frac{p}{3}\right)=(-1)^{\frac{(3-1)(p-1)}{4}}=-1,$

a contradiction.

Thus for $m = 4m'$ and m' is odd, A is not an integer.

From the above, we see that if A is an integer, A is odd.

2nd solution : We prove by contradiction. Assume, on the contrary, that A is even. Then m is even. Since A is an integer, $0 \equiv (m+3)^n + 1 \equiv m^n + 1 \pmod 3$ yields $n = 2k+1$ and $m = 3t+2$. Let $m = 2^l m_1$, where $l \geq 1$ and m_1 is odd. In fact $l > 1$, as otherwise $m \equiv 2 \pmod 4$,

$(m+3)^n+1 \equiv (2+3)^n+1 \equiv 2 \pmod 4$ and $A=\frac{(m+3)^n+1}{2m'}$ is odd.

Since A is an integer, we have

$0 \equiv (m+3)^n + 1 \equiv 3^{2k+1} + 1 \pmod m$ .....(i)

From (i), we have $2^l \mid (3^{2k+1}+1) \mid$. But $3^{2k+1}+1 = 9^k \times 1 \equiv 4 \pmod 8$, so $l = 2$, $m = 4m_1$, and $m_1 \equiv m = 2 \pmod 3$, From (i), we also have $m_1 \mid (3^{2k+1} + 1)$, which implies that $m_1 \mid a^2 + 3$, where $a = 3^{k+1}$. Since $gcd(m_1, 2) = gcd(m_1, 3) = 1$ and $m_1 \equiv 2 \pmod 3$, $m_1 = 6s + 5$. Since $gcd(m_1, a) = gcd(m_1, 3) = 1$. Thue's lemma implies that there exist integers x and y such that $1 \leq x, |y| \leq \left[\sqrt{m_1}\right]$ and $m_1 \mid ax + y$. Since $m_1 \equiv 2 \pmod 3$, m_1 is not a perfect square and $1 \leq x, |y| < \sqrt{m_1}$. Now

$m_1 \mid a^2+3 \Leftrightarrow a^2+3 \equiv 0 \pmod{m_1}$

$m_1 \mid ax+y \Leftrightarrow ax \equiv -y \pmod{m_1}$

imply that $3x^2 + y^2 \equiv 0 \pmod{m_1}$, *i.e.*, $3x^2 + y^2 = km_1$. But $3x^2 + y^2 < 4m_1$ gives $k = 1, 2, 3$.

(a) If $k=1$, then $3x^2+y^2=m_1=6s+5$ yields $y^2 \equiv 2 \pmod 3$, which is impossible.

(b) If $k = 2$, then $3x^2 + y^2 = 2m_1 = 12s + 10$ yields $3x^2 + y^2 \equiv 2 \pmod 4$, which is impossible.

(c) If $k = 3$, then $3x^2 + y^2 = 3m_1$ yields $y = 3y_1$ and $x^2 + 3y_1^2 = m_1$.

We are back in the first case, which is impossible. Thus our assumption is wrong and A is odd.

26. We have $x_2=\frac{1+x_1}{x_0}$, $x_3=\frac{x_0+x_1+1}{x_0x_1}$, $x_4=\frac{1+x_0}{x_1}$

$x_5 = x_0$, and $x_6 = x_1$. Therefore, x_k periodically repeats every 5 terms and $x_{1998}=x_3=\frac{x_0+x_1+1}{x_0x_1}$.

27. Put $\tan\frac{\theta}{2}=x, \tan\frac{\varphi}{2}=y$

Then $\cos\theta=\frac{1-x^2}{1+x^2}=\cos\alpha\cos\beta, \cos\varphi=\frac{1-y^2}{1+y^2}$

$\cos\varphi=\frac{1-y^2}{1+y^2}=\cos\alpha_1.\cos\beta$

Further $x^2=\frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta}, y^2=\frac{1+\cos\alpha_1\cos\beta}{1+\cos\alpha_1\cos\beta}$,

therefore

$$\tan^2\frac{\beta}{2}=x^2y^2=\frac{(1-\cos\alpha\cos\beta)(1-\cos\alpha_1\cos\beta)}{(1+\cos\alpha\cos\beta)(1+\cos\alpha_1\cos\beta)}.$$

Add unity to both members of the equality. We find

$$\frac{2}{1+\cos\beta}=\frac{2\left(1+\cos\alpha\cos\alpha_1\cos^2\beta\right)}{(1+\cos\alpha\cos\beta)(1+\cos\alpha_1\cos\beta)}$$

Assuming $\cos\beta \neq 0$, we obtain

$\cos\alpha + \cos\alpha_1 = 1 + \cos\alpha\cos\alpha_1 \cos^2\beta$,

$\cos\alpha + \cos\alpha_1 = 1 + \cos\alpha \cos\alpha_1(1 - \sin^2\beta)$

$\cos\alpha+\cos\alpha_1\sin^2\beta=1+\cos\alpha\cos\alpha_1-\cos\alpha-\cos\alpha_1$

$= (1 - \cos\alpha)(1 - \cos\alpha_1)$ and consequently, indeed

$$\sin^2\beta=\left(\frac{1}{\cos\alpha}-1\right)\left(\frac{1}{\cos\alpha_1}-1\right).$$

28. Let the 5th degree equation be $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$. The roots of $x^2 - x + 1$ are the non-real roots of $x^3 + 1$, namely $e^{\pi i/3}$ and $e^{5\pi i/3}$. Therefore the 5th degree equation is divisible by $x^2 - x + 1$ iff $ae^{5\pi i/3}+b^{4\pi i/3}+ce^{\pi i}+de^{2\pi i/3}+ee^{\pi i/3}+f=0.$

In other words, so $i\sin 60(-a-b+d+e)=0$, or $a-d=e-b$, and $\frac{a}{2}-\frac{b}{2}-c-\frac{d}{2}+\frac{e}{2}+f=0$, on $e+2f+a=b+2c+d$ or 9 since $a-d=e-b, a-d=c-f=e-b$. It follows that exactly $\frac{1}{12}$ of the polynomials will have coefficients $p+k, q, r+k, p, q+k, r$ for $k > 0$ and $p \leq q \leq r$.

For a given k, there are $\binom{9-k}{3}$ values of p, q, r such that $r+k \leq 9$. However, the coefficients must be distinct, so we must subtract those with 2 of p, q, r differing by k. There are $9-2k$ ways to select two numbers differing by k, and $7-k$ ways to select the remaining number. However, we have counted those of the form $x, x+d, x+2d$ twice, and there are $9-3k$ of these.

Therefore, for a given k, there are

$\binom{9-k}{3}-(9-2k)(7-k)+9-3k$ polynomials. Adding, we have $(1 + 4 + 10 + 20 + 35 + 56) - (42 + 25 + 12 + 3) + (3 + 6) = 53$ polynomials of the prescribed form, and $53 \cdot 12 = 636$ polynomials in total.

MOCK TEST PAPER 2011

West Bengal-JEE

(Full Syllabus)

Exam on 22nd May 2011

This paper is useful for AIEEE | IIT-JEE and other Engineering Entrance Exams.

SECTION-1

This section contains 80 multiple choice questions numbered 1 to 80. Each question has four choices of which one is correct. Each question carries +1 mark for correct answer and –1/3 for wrong answer.

1. The solution of the differential equation

$$xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0 \text{ is}$$

(a) $y = x\tan\left(\frac{x^2+y^2+c}{2}\right)$

(b) $x = y\tan\left(\frac{x^2+y^2+c}{2}\right)$

(c) $x = y\tan\left(\frac{c-x^2-y^2}{2}\right)$

(d) none of these

2. If 1, ω, ω^2 are cube roots of unity and if $\begin{bmatrix}1+\omega & 2\omega\\ -2\omega & -b\end{bmatrix} + \begin{bmatrix}a & -\omega\\ 3\omega & 2\end{bmatrix} = \begin{bmatrix}0 & \omega\\ \omega & 1\end{bmatrix}$, then $a^2 + b^2$ is equal to

(a) $1 + \omega^2$ (b) $\omega^2 - 1$ (c) $1 + \omega$ (d) $(1 + \omega)^2$

3. If $A + B = 45°$ then $(\cot A - 1)(\cot B - 1)$ is equal to

(a) 1 (b) 1/2 (c) –1 (d) 2

4. In triangle ABC if $\angle A = 60°$, $a = 5$, $b = 4$ then c is a root of the equation

(a) $c^2 - 5c - 9 = 0$ (b) $c^2 - 4c - 9 = 0$

(c) $c^2 - 10c + 25 = 0$ (d) $c^2 - 5c - 49 = 0$

5. Let f and g be two continuous and differentiable functions satisfying $f(x + y) = f(x) + f(y)$ $\forall\ x, y \in R$. Also $f(x) = x^2 g(x)$. Then $|f(15) - f(-15)|$ is

(a) 30 (b) –30

(c) 0

(d) cannot be determined

6. $2\left[\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + ...\right]$ is

(a) $\log\frac{m}{n}$ (b) $\log\frac{n}{m}$

(c) $\log mn$ (d) none of these

7. $\int \frac{x + (\cos^{-1}3x)^2}{\sqrt{1-9x^2}}\,dx =$

(a) $c - \left(\frac{1}{9}\sqrt{1-9x^2} + (\cos^{-1}3x)^3\right)$

(b) $c + \left(\frac{1}{9}\sqrt{1-9x^2} + (\cos^{-1}3x)^3\right)$

(c) $c - \left(\sqrt{1-9x^2} + (\cos^{-1}3x)^3\right)$

(d) none of these

8. Let r be a relation from R (set of real numbers) to R defined by $r = \{(a, b) : a, b \in R$ and $a - b + \sqrt{3}$ is a rational number$\}$. The relation r is

(a) an equivalence relation

(b) reflexive only (c) symmetric only

(d) transitive only

9. If P_m stands for ${}^mP_m$ then $1 + 1\cdot P_1 + 2\cdot P_2 + 3\cdot P_3 + ... + n\cdot P_n$ is equal to

(a) $n!$ (b) $(n + 3)!$

(c) $(n + 2)!$ (d) $(n + 1)!$

10. If $\int(\log x)^2 dx = x[f(x)]^2 + Ax[f(x) - 1] + C$, then

(a) $f(x) = \log x$, $A = 2$

(b) $f(x) = \log x$, $A = -2$

(c) $f(x) = -\log x$, $A = 2$

(d) $f(x) = -\log x$, $A = -2$

11. The value of $\begin{vmatrix}1 & \log_x y & \log_x z\\ \log_y x & 1 & \log_y z\\ \log_z x & \log_z y & 1\end{vmatrix} =$

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79. $I_n = \int_0^{\pi/4} \tan^n x\, dx,\ n > 2,$ then $I_n + I_{n-2}$ forms

(a) A.P. (b) G.P.
(c) H.P. (d) A.G.P.

80. If $f(x)$ is continuous function such that $f(x) \geq 0$ $\forall x \in [2, 10]$ and $\int_4^8 f(x)dx = 0$, then $f(0) =$

(a) 2 (b) -2
(c) 0 (d) 1

SECTION-2

This section contains 10 descriptive questions numbered 1 to 10.

1. If each term of a series in A.P. be multiplied by 3, would the series so obtained be again in A.P.? Give reasons for your answer.

2. Prove that the product of n geometric means between a and b is $(ab)^{n/2}$.

3. Show that $\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\left(\cos\frac{2\pi}{10} + i\sin\frac{2\pi}{10}\right)$ $\left(\cos\frac{3\pi}{10} + i\sin\frac{3\pi}{10}\right)\left(\cos\frac{4\pi}{10} + i\sin\frac{4\pi}{10}\right) = -1$

4. If there is no real roots of the equation $x^2 - (k + 2)x + (k + 2) = 0$, then prove that $-2 < k < 2$.

5. Find the rank of the word 'LAND' when its letters are arranged as in a dictionary.

6. Find the coefficient of x^2 in $\log_e(1 + x + x^2 + x^3 + ... + \infty)$

7. If the system of equations $x - ky - z = 0$, $kx - y - z = 0$, $x + y - z = 0$ has a non-zero solution, then find the possible values of k.

8. Find the value of a and b, such that the function $f(x)$ defined by

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x, & \text{when } 0 \leq x \leq \frac{\pi}{4} \\ 2x\cot x + b, & \text{when } \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \\ a\cos 2x - b\sin x, & \text{when } \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

is continuous for all values of x in $0 \leq x \leq \pi$.

9. Prove that the tangent to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) are at right angles.

10. Solve : $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$ with $y = 1$, $\frac{dy}{dx} = 0$ at $x = 0$.

Answer keys

SECTION-1

1. (c)	2. (c)	3. (d)	4. (b)	5. (c)
6. (a)	7. (a)	8. (b)	9. (d)	10. (a)
11. (c)	12. (b)	13. (b)	14. (a)	15. (c)
16. (d)	17. (a)	18. (b)	19. (b)	20. (d)
21. (a)	22. (d)	23. (a)	24. (d)	25. (c)
26. (a)	27. (b)	28. (b)	29. (d)	30. (a)
31. (d)	32. (c)	33. (c)	34. (b)	35. (b)
36. (a)	37. (c)	38. (a)	39. (d)	40. (c)
41. (d)	42. (b)	43. (c)	44. (b)	45. (d)
46. (d)	47. (c)	48. (a)	49. (b)	50. (b)
51. (c)	52. (b)	53. (d)	54. (d)	55. (c)
56. (c)	57. (c)	58. (b)	59. (c)	60. (b)
61. (c)	62. (b)	63. (d)	64. (d)	65. (d)
66. (b)	67. (b)	68. (c)	69. (b)	70. (a)
71. (b)	72. (a)	73. (d)	74. (a)	75. (c)
76. (a)	77. (b)	78. (a)	79. (c)	80. (c)

SECTION-2

5. 14 **6.** 1/2 **7.** 1, –1 **8.** $a = \frac{\pi}{6}$, $b = -\frac{\pi}{12}$
10. $y = (1 + 4x)e^{-4x}$

CHINESE Olympiad Problems

PAPER I

1. Prove that $\sin 3\theta = 4\sin\theta\sin\left(\frac{\pi}{3}+\theta\right)\sin\left(\frac{2\pi}{3}+\theta\right)$.

2. Determine the equation of a hyperbola whose vertices are at a distance 2 apart, and whose asymptotes have equations $x + y = 0$ and $x - y = 0$.

3. Find the smallest circle which covers triangle ABC if $\angle A$ is obtuse.

4. Prove that two intersecting chords of a circle cannot bisect each other unless both are diameters.

5. Solve the system of equations $x - y + z = 1$, $y - z + u = 2$, $z - u + v = 3$, $u - v + x = 4$ and $v - x + y = 5$.

6. Solve the equation $5x^2 + x - x\sqrt{5x^2 - 1} - 2 = 0$.

7. Prove that a line l on a plane Π is perpendicular to a line m not in Π if and only if it is perpendicular to the projection of m onto Π.

8. In triangle ABC, $\angle A$, $\angle B$ and $\angle C$ form an arithmetic progression and $\frac{1}{BC}, \frac{1}{CA}$ and $\frac{1}{AB}$ also form an arithmetic progression. Determine $\angle A$, $\angle B$ and $\angle C$.

9. Find the equations of all circles which passes through the point (3, 1) and tangent to the lines $x + 2y + 3 = 0$ and $x + 2y - 7 = 0$.

10. In triangle ABC, $BC > CA > AB$. Determine which is the largest among the three squares whose vertices are all on the perimeter of triangle ABC.

PAPER II

1. Let $f(x) = x^2 - 6x + 5$. Mark on a diagram the set of points (x, y) for which $f(x) + f(y) \geq 0$ and $f(x) - f(y) \geq 0$.

2. Consider the following proposition : "In a quadrilateral, if one pair of opposite angles are equal to each other, and one pair of opposite sides are equal to each other, then the quadrilateral is a parallelogram." If this is true, give a proof. If this is false, give a counter example.

3. Let $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$.
   (a) Prove that $\sec^2\alpha + \text{cosec}^2\alpha\ \text{cosec}^2\beta\ \sec^2\beta \geq 9$.
   (b) Determine when equality holds.

4. If a curve joining two points on the perimeter of a unit square bisects its area, prove that its length is not less than 1.

5. For a given positive integer m, define the sequence $\{a_n\}$ as follows : $a_0 = m$ and $n \geq 1$, $a_n = \frac{a_{n-1}}{2}$ if a_{n-1} is even, and $a_n = a_{n-1} + 3$ is a_{n-1} is odd.
   (a) Prove that for any m, at least one term of the sequence is equal to 1 or 3.
   (b) Determine all values of m for which at least one term of the sequence is equal to 3.
   (c) Determine all values of m for which at least one term of the sequence is equal to 1.

6. Two circles Ω_1 and Ω_2 intersect at A and B. A line through B intersects Ω_1 at C and Ω_2 at E. A second line through B intersects Ω_1, at D and Ω_2 at F.
   (a) If $\angle ABC = \angle ABD$, prove that $CE = DF$.
   (b) If $CE = DF$, prove that $\angle ABC = \angle ABD$.

7. In a mathematics competition, all scores are integers and their total is 8250. The highest three scores are 88, 85 and 80, and the lowest score is 30. No four students have the same score. Determine the total number of students with scores not less than 60.

ANSWERS

PAPER I

2. $x^2 - y^2 = \pm 1$,
3. circle with diameter BC.
5. $(x, y, z, u, v) = (0, 6, 7, 3, -1)$.
6. $\pm\frac{\sqrt{10}}{5}$.
8. $\angle A = \angle B = \angle C = 60°$.
9. $(x-4)^2 + (y+1)^2 = 5$ and $\left(x - \frac{4}{5}\right)^2 + \left(y - \frac{3}{5}\right)^2 = 5$.
10. the square with two vertices on AB.

PAPER II

1. Divide the circle $(x-3)^2 + (y-3)^2 = 8$ into quadrants by the lines $x = y$ and $x + y = 6$. The desired set consists of the "east" and "west" quadrants.
2. false.
3. (b) $\alpha = \arctan\sqrt{2}$ and $\beta = \frac{\pi}{4}$.
5. (b) all multiples of 3; (c) all non-multiples of 3.
7. 61.

Best Problems in Algebra

Problem 1.

Find all functions $f: R \to R$ such that the equality

$$f(f(x)+y) = f(x^2 - y) + 4f(x)y$$

holds for all pairs of real numbers (x, y).

Problem 2.

Solve the system of equations :

$$x + \frac{3x - y}{x^2 + y^2} = 3;\ y - \frac{x + 3y}{x^2 + y^2} = 0.$$

Problem 3.

Mr. Fat and Mr. Taf play a game with a polynomial of degree at least 4 :

$$x^{2n} + \_x^{2n-1} + \_x^{2n-2} + \ldots + \_x + 1.$$

They fill the real numbers to empty spaces in turn.

If the resulting polynomial has no real root, Mr. Fat wins; otherwise, Mr. Taf wins.

If Mr. Fat goes first, who has a winning strategy?

Problem 4.

Find all positive integers k for which the following statement is true : if $F(x)$ is a polynomial with integer coefficients satisfying the condition

$0 \le F(c) \le k$ for $c = 0, 1, \ldots, k+1$, then $F(0) = F(1) = \ldots = F(k+1)$.

Problem 5.

The Fibonacci sequence F_n is given by

$$F_1 = F_2 = 1,\ F_{n+2} = F_{n+1} + F_n\ (n \in N).$$

Prove that

$$F_{2n} = \frac{F_{2n+2}^3 + F_{2n-2}^3}{9} - 2F_{2n}^3;\ \text{for all } n \ge 2.$$

SOLUTIONS

1. Clearly, $f(x) = x^2$ satisfies the functional equation. Now assume that there is a nonzero value a such that $f(a) \ne a^2$.

Let $y = \dfrac{x^2 - f(x)}{2}$ in the functional equation to find that

$$f\left(\frac{f(x)+x^2}{2}\right) = f\left(\frac{f(x)+x^2}{2}\right) + 2f(x)(x^2 - f(x))$$

or $0 = 2f(x)(x^2 - f(x))$. Thus, for each x, either $f(x) = 0$ or $f(x) = x^2$.

In both cases, $f(0) = 0$.

Setting $x = a$, it follows from above that either $f(a) = 0$ or $f(a) = 0$ or $f(a) = a^2$.

The latter is false, so $f(a) = 0$.

Now, let $x = 0$ and then $x = a$ in the functional equation to find that $f(y) = f(-y)$, $f(y) = f(a^2 - y)$

and so $f(y) = f(-y) = f(a^2 + y)$;

that is, the function is periodic with non-zero period a^2.

Let $y = a^2$ in the original functional equation to obtain

$f(f(x)) = f(f(x) + a^2) = f(x^2 - a^2) + 4a^2 f(x) = f(x^2) + 4a^2 f(x)$.

However, putting $y = 0$ in the functional equation gives $f(f(x)) = f(x^2)$ for all x.

Thus, $4a^2 f(x) = 0$ for all x. Since a is non-zero, $f(x) = 0$ for all x. Therefore, either $f(x) = x^2$ or $f(x) = 0$.

2. **Alternative 1**

Multiplying the second equation by i and adding it to the first equation yields

$$x + yi + \frac{(3x - y) - (x + 3y)i}{x^2 + y^2} = 3,$$

or, $$x + yi + \frac{3(x - yi)}{x^2 + y^2} - \frac{i(x - yi)}{x^2 + y^2} = 3.$$

Let $z = x + yi$. Then $\dfrac{1}{z} = \dfrac{x - yi}{x^2 + y^2}$.

Thus the last equation becomes

$$z + \frac{3 - i}{z} = 3 \quad \text{or} \quad z^2 - 3z + (3 - i) = 0.$$

Hence $$z = \frac{3 \pm \sqrt{-3 + 4i}}{2} = \frac{3 \pm (1 + 2i)}{2},$$

that is, $(x, y) = (2, 1)$ or $(x, y) = (1, -1)$.

Alternative 2

Multiplying the first equation by y, the second by x, and adding up yields

$$2xy + \frac{(3x - y)y - (x + 3y)x}{x^2 + y^2} = 3y,$$

or $2xy - 1 = 3y$. It follows that $y \ne 0$ and $x = \dfrac{3y + 1}{2y}$.

Substituting this into the second equation of the given system gives

$$y\left[\left(\frac{3y+1}{2y}\right)^2 + y^2\right] - \left(\frac{3y+1}{2y}\right) - 3y = 0,$$

or $4y^4 - 3y^2 - 1 = 0$.

It follows that $y^2 = 1$ and that the solutions to the

system are (2, 1) and (1, –1).

3. Mr. Taf has a winning strategy.

We say a blank space is odd (even) if it is the coefficient of an odd (even) power of x.

First Mr. Taf will fill in arbitrary real numbers into one of the remaining even spaces, if there are any.

Since there are only $n-1$ even spaces, there will be at least one odd space left after $2n-3$ plays, that is, the given polynomial becomes $p(x) = q(x) + \_x^s + \_x^{2t-1}$, where s and $2t-1$ are distinct positive integers and $q(x)$ is a fixed polynomial.

We claim that there is a real number a such that

$$p(x) = q(x) + ax^s + \_x^{2t-1}$$

will always have a real root regardless of the coefficient of x^{2t-1}.

Then Mr. Taf can simply fill in a in front of x^s and win the game.

Now we prove our claim. Let b be the coefficient of x^{2t-1} in $p(x)$. Note that $\frac{1}{2^{2t-1}}p(2) + p(-1)$

$$= \left(\frac{1}{2^{2t-1}}q(2) + 2^{s-2t+1}a + b\right) + [q(-1) + (-1)^s a - b]$$

$$= \left(\frac{1}{2^{2t-1}}q(2) + q(-1)\right) + a[2^{s-2t+1} + (-1)^s].$$

Since $s \neq 2t-1$, $2^{s-2t+1} + (-1)s \neq 0$.

Thus

$$a = -\frac{\frac{1}{2^{2t-1}}q(2) + q(-1)}{2^{s-2t+1} + (-1)^s}$$

is well defined such that a is independent of b and

$$\frac{1}{2^{2t-1}}p(2) + p(-1) = 0.$$

It follows that either $p(-1) = p(2) = 0$ or $p(-1)$ and $p(2)$ have different signs, which implies that there is a real root of $p(x)$ in between –1 and 2.

In either case, $p(x)$ has a real root regardless of the coefficient of x^{2t-1}, as claimed.

Our proof is thus complete.

4. The statement is true if and only if $k \geq 4$.

We start by proving that it does hold for each $k \geq 4$.

Consider any polynomial $F(x)$ with integer coefficients satisfying the inequality $0 \leq F(c) \leq k$ for each $c \in \{0, 1, ..., k+1\}$.

Note first that $F(k+1) = F(0)$, since $F(k+1) - F(0)$ is a multiple of $k+1$ not exceeding k in absolute value.

Hence $F(x) - F(0) = x(x-k-1)G(x)$,

where $G(x)$ is a polynomial with integer coefficients.

Consequently,

$$k \geq |F(c) - F(0)| = c(k+1-c)|G(c)| \quad ...(1)$$

for each $c \in \{1, 2, ..., k\}$.

The equality $c(k+1-c) > k$ holds for each $c \in \{2, 3, ..., k-1\}$, as it is equivalent to $(c-1)(k-c) > 0$.

Note that the set $\{2, 3, ..., k-1\}$ is not empty if $k \geq 3$, and for any c in this set, (1) implies that $|G(c)| < 1$.

Since $G(c)$ is an integer, $G(c) = 0$. Thus

$$F(x) - F(0) = x(x-2)(x-3)....(x-k+1)(x-k-1)H(x), \quad ...(2)$$

where $H(x)$ is a polynomial with integer coefficients.

To complete the proof of our claim, it remains to show that $H(1) = H(k) = 0$.

Note that for $c = 1$ and $c = k$, (2) implies that

$$k \geq |F(c) - F(0)| = (k-2)! \cdot k \cdot |H(c)|.$$

For $k \geq 4$, $(k-2)! > 1$. Hence $H(c) = 0$.

We established that the statement in the question holds for any $k \geq 4$. But the proof also provides information for the smaller values of k as well.

More exactly, if $F(x)$ satisfies the given condition then 0 and $k+1$ are roots of $F(x)$ and $F(0)$ for any $k \geq 1$ and if $k \geq 3$ then 2 must also be a root of $F(x) - F(0)$.

Taking this into account, it is not hard to find the following counter examples :

$F(x) = x(2-x)$ for $k = 1$,

$F(x) = x(3-x)$ for $k = 2$,

$F(x) = x(4-x)(x-2)^2$ for $k = 3$.

5. Note that

$F_{2n+2} - 3F_{2n} = F_{2n+1} - 2F_{2n} = F_{2n-1} - F_{2n} = -F_{2n-2}$,

whence $3F_{2n} - F_{2n+2} - F_{2n-2} = 0$ ...(1)

for all $n \geq 2$.

Setting $a = 3F_{2n}$, $b = -F_{2n+2}$, and $c = -F_{2n-2}$ in the algebraic identity

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

gives $27F_{2n}^3 - F_{2n+2}^3 - F_{2n-2}^3 - 9F_{2n+2}F_{2n}F_{2n-2} = 0$.

Applying (1) twice gives

$$F_{2n+2}F_{2n-2} - F_{2n}^2 = (3F_{2n} - F_{2n-2})F_{2n-2} - F_{2n}^2$$

$$= F_{2n}(3F_{2n-2} - F_{2n}) - F_{2n-2}^2 = F_{2n}F_{2n-4} - F_{2n-2}^2$$

$$= ... = F_6F_2 - F_4^2 = -1.$$

The desired result follows from

$$9F_{2n+2}F_{2n}F_{2n-2} - 9F_{2n}^3 = 9F_{2n}(F_{2n+2}F_{2n-2} - F_{2n}^2) = -9F_{2n}.$$

MATHS MUSING

SOLUTION SET - 99

1. **(b) :** $a_5 = a_4 - a_3 + a_2 - a_1 = 23 - 20 + 75 - 11 = 67$

$a_6 = a_5 - a_4 + a_3 - a_2 = -a_1$

$a_7 = a_6 - a_5 + a_4 - a_3 = -a_2$

$a_8 = -a_3,\ a_9 = -a_4,\ a_{10} = -a_5$

$a_{11} = a_1,\ a_{12} = a_2,\ a_{13} = a_3,\ a_{14} = a_4,\ a_{15} = a_5$

$\therefore\ a_{31} - a_{53} + a_{75} = a_1 - a_3 + a_5 = 11 - 20 + 67 = 58.$

2. **(d) :** $x = \frac{1}{\alpha}\sum_{r=1}^{6}\binom{6}{r}\alpha^r = \frac{1}{\alpha}((1+\alpha)^6 - 1)$

$$\alpha = \text{cis}\frac{\pi}{3} \Rightarrow x = \frac{1}{2}(1 - i\sqrt{3})\left(\left(1 + \text{cis}\frac{\pi}{3}\right)^6 - 1\right)$$

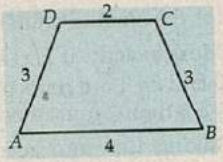
$$= \frac{1}{2}(1 - i\sqrt{3})\left(2^6\cos^6\frac{\pi}{6}\left(\text{cis}\frac{\pi}{6}\right)^6 - 1\right)$$

$$= \frac{1}{2}(1 - i\sqrt{3})(-27 - 1) = -14(1 - i\sqrt{3})$$

$\therefore\ |x| = 28$

3. **(b) :** $A = (-2, 0),\ B = (2, 0)$

$C = (1, 2\sqrt{2}),\ D = (-1, 2\sqrt{2})$



If the desired circle has centre $(0, r)$ and radius r, then

$$4 + r^2 = (r+2)^2,\ 1 + (r - 2\sqrt{2})^2 = (r+1)^2$$

Subtracting, $r = \frac{r+4}{2\sqrt{2}}$

$$\therefore\ \frac{(r+4)^2}{8} = r^2 + 4r \Rightarrow 7r^2 + 24r - 16 = 0$$

$$\Rightarrow\ r = \frac{4}{7}.$$

4. **(b) :** $\frac{dy}{dx} = xy + x^3y^3$

$$\Rightarrow -\frac{2}{y^3}\frac{dy}{dx} + \frac{2x}{y^2} = -2x^3$$

Solving the linear equation in

$$\frac{1}{y^2}e^{x^2} = -\int 2x^3 \cdot e^{x^2}dx = (1 - x^2)e^{x^2} + c$$

$$y\cdot(0) = 1 \Rightarrow c = 0.\quad \therefore\ y = \pm\frac{1}{\sqrt{1-x^2}}.$$

$$\text{Area} = \int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = \pi$$

5. **(a, d) :**

$$\frac{x^2}{2} + \frac{y^2}{1} = 1 \Rightarrow P = \left(-1, \frac{1}{\sqrt{2}}\right) \text{ and } Q = \left(1, \frac{1}{\sqrt{2}}\right).$$

If PQ is the latus rectum of the parabola, $PQ = 2$.

Vertex is $\left(0, \frac{1}{\sqrt{2}} + \frac{1}{2}\right), \left(0, \frac{1}{\sqrt{2}} - \frac{1}{2}\right)$

The parabolas are

$$x^2 = -2\left(y - \frac{1}{\sqrt{2}} - \frac{1}{2}\right)\ \text{ i.e., }\ x^2 + 2y = \sqrt{2} + 1 \text{ and}$$

$$x^2 = 2\left(y - \frac{1}{\sqrt{2}} + \frac{1}{2}\right)\ \text{ i.e., }\ x^2 - 2y = 1 - \sqrt{2}.$$

6. **(d) :** $2(r + R) = 2(s - a)\tan\frac{C}{2} + c = 2s - c = a + b$

$\therefore\ a + b = 31 \Rightarrow (a+b)^2 = 961$

$\Rightarrow\ 25^2 + 4\Delta = 961 \Rightarrow \Delta = 84$

7. **(b) :** $r_1 + r_2 = 4R\cos^2\frac{C}{2} = 2R = 25$

8. **(c) :** $r_3 = \frac{\Delta}{s - c} = \frac{84}{28 - 25} = 28$

9. **(a) → p, (b) → (q), (c) → (q), (d) → (q, r)**

(a) $f(x) = 1 + \sum_{r=1}^{4}\frac{x^r}{r!}.\ f(x) \to \infty$ as $x \to \pm\infty$.

Let $f(x)$ has minimum at $x = \alpha$.

$f'(\alpha) = 0 \Rightarrow \frac{\alpha^4}{4!} > 0 \Rightarrow f(x) > 0$ for all x.

(b) $\int_{\sqrt{2}}^{x}\frac{dt}{t\sqrt{t^2 - 1}} = \frac{\pi}{12} \Rightarrow \sec^{-1}x - \frac{\pi}{4} = \frac{\pi}{12}$

or $\sec^{-1}x = \frac{\pi}{3},\ x = 2.$

(c) $5(2\cos^2\theta - 1) + 1 + \cos\theta + 1 = 0$

$$10\cos^2\theta + \cos\theta - 3 = 0 \Rightarrow \cos\theta = \frac{1}{2}, -\frac{3}{5}.$$

Two values of θ in $(0, \pi)$.

(d) $3^{2n+2} - 8n - 9 = (1 + 8)^{n+1} - 8n - 9 = \binom{n+1}{2}8^2 + \ldots,$

which is divisible by $8^2 = 2^6 \Rightarrow m = 6$.

10. **(1) :** Writing numbers from 0 to 999

000, 001, 002, ..., 997, 998, 999

Writing these in the reverse order

999, 998, 997, ..., 002, 001, 000

Adding these 2000 numbers, we get

$$\frac{9 \times 3000}{2} = 13500$$

Omitting 0 and adding 1000 to the above numbers, we get $N = 13501$.

MOCK TEST PAPER

ISI 2012

Indian Statistical Institute

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PART-A

SECTION - I

Single Correct Answer Type

1. If in a right angle triangle ABC, $4\sin A\cos B - 1 = 0$ and $\tan A$ is finite, then

(a) angles are in A.P. (b) angles are in G.P.
(c) angles are in H.P. (d) none of these

2. If $a^2 + b^2 + c^2 - 2ab = 0$, then the point of concurrency of family of lines $ax + by + c = 0$ lies on the line

(a) $y = x$ (b) $y = x + 1$
(c) $y = -x$ (d) $3x = y$

3. If in an isosceles triangle with base 'a', vertical angle $20°$ and lateral side each of length 'b' is given then value of $a^3 + b^3$ equals

(a) $3ab$ (b) $3ab^2$ (c) $3a^2b$ (d) 3

4. If the positive integers are written in a triangular array as shown

			1			
		2		3		
	4		5		6	
7		8		9		10

Then the row in which the number 2010 will be, is

(a) 58 (b) 61 (c) 63 (d) 65

5. The values of k for which the inequality $k\cos^2 x - k\cos x + 1 \geq 0, \forall x \in (-\infty, \infty)$ holds is

(a) $k < -\frac{1}{2}$ (b) $k > 4$
(c) $-\frac{1}{2} \leq k \leq 4$ (d) $\frac{1}{2} \leq k \leq 5$

6. Let two non-collinear vectors $\vec{a}$ and $\vec{b}$ inclined at an angle $\frac{2\pi}{3}$ be such that $|\vec{a}| = 3$ and $|\vec{b}| = 2$. If a point P moves so that at any time t its position vector $\overline{OP}$ (where O is the origin) is given as $\overline{OP} = \left(t + \frac{1}{t}\right)\vec{a} + \left(t - \frac{1}{t}\right)\vec{b}$, then least distance of P from the origin is

(a) $\sqrt{2\sqrt{133} - 10}$ (b) $\sqrt{2\sqrt{133} + 10}$
(c) $\sqrt{5 + \sqrt{133}}$ (d) none of these

7. Let k be a fixed positive integer. The n^{th} derivative of $\frac{1}{x^k - 1}$ has the form $\frac{p_n(x)}{(x^k - 1)^{n+1}}$ where $p_n(x)$ is a polynomial of degree n with $p_n(1) = 1$. Then the value of $p_n(1)$ is

(a) $(n-1)!(-k)^n$ (b) $n!(-k)^{n-1}$
(c) $(n-1)!(-k)^{n-1}$ (d) $n!(-k)^n$

8. Number of triangles with each side having integral length and the longest side is of 11 unit is equal to k^2. Then the value of k is equal to

(a) 3 (b) 4 (c) 5 (d) 6

9. If ${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^r . {}^nC_r = 28$, then n is equal to

(a) 6 (b) 7 (c) 8 (d) 9

10. The length of the sub-tangent to the hyperbola $x^2 - 4y^2 = 4$ corresponding to the normal having slope unity is $\frac{1}{\sqrt{k}}$, then k is equal to

(a) 1 (b) 2 (c) 3 (d) 4

11. If $x \in \left(0, \frac{\pi}{2}\right)$ and $\cos x = \frac{1}{3}$ then the value of $\sum_{n=0}^{\infty} \frac{\cos nx}{3^n}$ is equal to

(a) 1 (b) -1 (c) 2 (d) -2

12. Let S_1 and S_2 denote the circles

$x^2 + y^2 + 10x - 24y - 87 = 0$ and

$x^2 + y^2 - 10x - 24y + 153 = 0$ respectively.

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(Let m be the smallest positive value of 'a' for which the line $y = ax$ contains the centre of a circle which touches S_2 externally and S_1 internally). Given that $m^2 = \frac{p}{q}$, where p and q are relatively prime integers, $(p + q)$ is
(a) 169 (b) 256 (c) 144 (d) 81

13. The number of values of k for which the equation $x^3 - 3x + k = 0$ has two distinct roots lying in the interval (0, 1) are
(a) three (b) two
(c) infinitely many
(d) no value of k satisfies the requirement

14. If $\frac{dy}{dx} = f(x) + \int_0^1 f(x)dx$, then the equation of the curve $y = f(x)$ passing through (0, 1) is
(a) $f(x) = \frac{2e^x - e + 1}{3 - e}$ (b) $f(x) = \frac{3e^x - 2e + 1}{2(2 - e)}$
(c) $f(x) = \frac{e^x - 2e + 1}{e + 1}$ (d) none of these

15. A staircase has 10 steps. A person can go up the steps one at a time, two at a time, or any combination of 1's and 2's. The number of ways in which the person can go up the stairs is
(a) 89 (b) 144 (c) 132 (d) 211

16. Let f be a function such that $f(xy) = f(x)\cdot f(y)$ for all $x, y \in R^+$ and $f(1 + x) = 1 + x(1 + (g(x)))$, where $\lim_{x\to 0} g(x) = 0$. The value of $\int_1^2 \frac{f(x)}{f'(x)}\cdot\frac{1}{1+x^2}dx$ is $\frac{1}{2}\log_e\left(\frac{a}{b}\right)$ where a & b are co-prime. Find $a + b$.
(a) 7 (b) 8 (c) 9 (d) 10

17. If $\int_0^x f(x)\sin t\, dt$ = constant, $0 < x < 2\pi$ and $f(\pi) = 2$ then find the value of $f(\pi/2)$.
(a) 2 (b) 4 (c) 6 (d) 8

18. For $a \in R$ if $|x + a - 3| + |x - 2a| = |2x - a - 3|$ is true for all $x \in R$, then exhaustive set of a is
(a) $a \in [-4, 4]$ (b) $a \in [-3, 2]$
(c) $a \in \{-2, 2\}$ (d) $a \in \{1\}$

19. If A is a skew symmetric matrix, then $B = (I - A)(I+A)^{-1}$ is (where I is an identity matrix of same order as of A)
(a) idempotent matrix (b) symmetric matrix
(c) orthogonal matrix (d) none of these

20. If function $f(x) = \cos(nx)\sin\left(\frac{5x}{n}\right)$ satisfies $f(x + 3\pi) = f(x)$, then find the number of integral values of n.
(a) 8 (b) 9 (c) 10 (d) 11

21. Find the number of integral values of 'a' for which the inequality $3 - |x - a| > x^2$ is satisfied by at least one negative x.
(a) 1 (b) 3 (c) 6 (d) 8

22. $A(z_1)$, $B(z_2)$, $C(z_3)$ are three points in the argand plane where $|z_1 + z_2| = \left||z_1| - |z_2|\right|$ and $|(1 - i)z_1 + iz_3| = |z_1| + |z_3 - z_1|$, then
(a) A,B,C lie on a circle with centre $\left(\frac{z_2 + z_3}{2}\right)$
(b) A,B,C are collinear
(c) ABC form an equilateral triangle
(d) ABC form an obtuse angle triangle

23. $\left[\cot^{-1} x\right]\left[\cos^{-1} x\right] = 0$ where 'x' is a non negative and [.] is the greatest integer of x, then all solutions of x belongs to
(a) (cos 1, 1] (b) (cos 1, cot 1)
(c) (cot 1, 1] (d) none of these

24. Let OA, OB, OC be coterminous edges of a cuboid. If l, m, n be the shortest distances between the sides OA, OB, OC and their respective skew body diagonals to them, respectively, then find $\frac{\left(\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2}\right)}{\left(\frac{1}{OA^2} + \frac{1}{OB^2} + \frac{1}{OC^2}\right)}$.
(a) 1 (b) 2 (c) 3 (d) 4

25. If the median AM, angle bisector AD and altitude AH drawn from vertex A of a ΔABC divide angle A into four equal parts and D lies in between H and M, then
(a) $\angle A = \frac{\pi}{3}$ (b) $\angle A = 90°$
(c) $\frac{AC}{AB} = \sqrt{2} - 1$ (d) $\frac{AC}{AB} = \frac{1}{\sqrt{2} + 2}$

26. If $f(x) = x + \sin x$, then find $\frac{2}{\pi^2}\cdot\int_\pi^{2\pi} (f^{-1}(x) + \sin x)dx$
(a) 2 (b) 3 (c) 6 (d) 9

27. Find number(s) of point(s) whose perpendicular distances from yz, zx and xy planes are in A.P. and whose distances from x,y,z axes are $\sqrt{13}, \sqrt{10}$ and $\sqrt{5}$

(a) 2 (b) 4 (c) 6 (d) 8

28. If $f(x) = \max\left(\frac{1}{\pi}\cos^{-1}(\cos \pi x), \{x\}\right)$ and $g(x) = \min\left(\frac{1}{\pi}\cos^{-1}(\cos \pi x), \{x\}\right)$ (where {.} represents fractional part of x). Then find the value of $\frac{\int_{1/2}^{2} f(x)dx}{\int_{1}^{2} g(x)dx}$

(a) 1 (b) 3
(c) 5 (d) 7

29. If sin (sin x + cos x) = cos(cos x – sin x) and largest possible value of sin x is $\frac{\pi}{k}$, then the value of k is

(a) 4 (b) 8 (c) 5 (d) 6

30. The number of solution(s) of the equation $z^2 - z - |z|^2 - \frac{64}{|z|^5} = 0$ is/are

(a) 0 (b) 1 (c) 2 (d) 3

SECTION - II

Multiple Correct Answer Type

1. If $f(x)$ is continuous in [0, 2] and $f(0) = f(2)$, then the equation $f(x) = f(x + 1)$ has (where $f(1) \neq f(0)$)
(a) non-real root in [0, 2]
(b) at least one real root in [0, 1]
(c) at least one real root in [0, 2]
(d) at least one real root in [1, 2]

2. If $A\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right), B\left(\frac{-3}{\sqrt{2}}, \sqrt{2}\right), C\left(\frac{-3}{\sqrt{2}}, -\sqrt{2}\right)$ and $D(3\cos\theta, 2\sin\theta)$ are four points. If the area of quadrilateral $ABCD$ is maximum $\left(\text{where } \theta \in \left(\frac{3\pi}{2}, 2\pi\right)\right)$ then
(a) maximum area is 10 sq. units
(b) $\theta = \frac{7\pi}{4}$
(c) $\theta = 2\pi - \sin^{-1}\left(\frac{3}{\sqrt{85}}\right)$
(d) maximum area is 12 sq. units

3. The solution of the differential equation $f(x)\frac{dy}{dx} + f'(x)y = 1$
(a) $x = yf(x) + c$ (b) $xf^{-1}(x) + c = 0$
(c) $y = \frac{x+c}{f(x)}$ (d) $y = xf(x)$

4. The function $f(x) = \cos^{-1}\left(\frac{2\left[|\sin x| + |\cos x|\right]}{\sin^2 x + 2\sin x + \frac{11}{4}}\right)$ is defined if x belongs to (where [.] represents greatest integer function)
(a) $\left[0, \frac{7\pi}{6}\right]$ (b) $\left[0, \frac{\pi}{6}\right]$
(c) $\left[\frac{11\pi}{6}, 2\pi\right]$ (d) $[\pi, 2\pi]$

5. A function $f: R \to R^+$ satisfies $f(x + y) = f(x) \cdot f(y)$, for all $x, y \in R$, $f(0) = 1$, $f'(0) = 2$, then
(a) $\int_0^{\ln 3}\left[f(x)e^{-x}\right]dx = \ln 4.5$ (where [.] denotes greatest integer function).
(b) $\lim_{x\to 0}\left[f(x)\right]$ does not exist (where [.] denotes greatest integer function)
(c) $f^{-1}(x) = \ln\sqrt{x}, \forall x > 0$
(d) $f(x) < e^{x^2-4x}$ has infinite solution in (0, 6)

6. If a,b,c,d are in A.P. and $\int_0^2 f(x)dx = -4$ where $f(x) = \begin{vmatrix} x+a & x+b & x+a-c \\ x+b & x+c & x-1 \\ x+c & x+d & x-b+d \end{vmatrix}$, then the common difference of the A.P. may be
(a) 1 (b) –1 (c) 2 (d) 4

7. Let z_1 and z_2 be two complex numbers such that $z_1^2 - 4z_2 = 16 + 20i$. If α and β are roots of $x^2 + z_1x + z_2 + M = 0$ (where M is complex number) and $|(\alpha - \beta)^2| = 28$, then
(a) maximum value of $|M|$ is $7 + \sqrt{41}$
(b) maximum value of $|M|$ is $5 + \sqrt{41}$
(c) minimum value of $|M|$ is $7 - \sqrt{41}$
(d) minimum value of $|M|$ is $5 - \sqrt{41}$

8. Vertex of parabola(s) having common chord to the circles $(x-1)^2 + (y-2)^2 = 5$ and $(x-3)^2 + (y-4)^2 = 25$; as directrix, center of either as the focus, is/are
(a) $\left(-\frac{1}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{1}{4}, \frac{5}{4}\right)$ (c) $\left(\frac{5}{4}, \frac{9}{4}\right)$ (d) (5, 6)

9. If $f(x + y) = f(x) \cdot f(y)$ for all x,y and $f(0) \neq 0$ and $F(x) = \frac{f(x)}{1 + (f(x))^2}$ then
(a) $\int_{-2010}^{2011} F(x)dx = \int_0^{2011} F(x)dx$

(b) $\int_{-2010}^{2011} F(x)dx - \int_{0}^{2010} F(x)dx = \int_{0}^{2011} F(x)dx$

(c) $\int_{-2010}^{2011} F(x)dx = 0$

(d) $\int_{-2010}^{2011} (2F(-x) - F(x))dx = 2\int_{0}^{2010} F(x)dx$

10. Let $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ be the unit vectors such that $\hat{\alpha}$ and $\hat{\beta}$ are mutually perpendicular and $\hat{\gamma}$ is equally inclined to $\hat{\alpha}$ and $\hat{\beta}$ at an angle θ. If $\hat{\gamma} = x\hat{\alpha} + y\hat{\beta} + z(\hat{\alpha}\times\hat{\beta})$, then

(a) $z^2 = 1 - 2x^2$ (b) $z^2 = 1 - 2y^2$

(c) $z^2 = 1 - x^2 - y^2$ (d) $x^2 = y^2$

PART - B

Short Answer Type

1. Equations of the diagonals of a rectangle are $y + 8x - 17 = 0$ and $y - 8x + 7 = 0$. If the area of the rectangle is 8 sq. units find the equations of the sides of the rectangle.

2. Two sides of a triangle have the joint equation $x^2 - 2xy - 3y^2 + 8y - 4 = 0$. The third side, which is variable, always passes through the point $(-5, -1)$. Find the range of values of the slope of the third side, so that the origin is an interior point of the triangle.

3. Let $P(\sin\theta, \cos\theta)$ $(0 \le \theta \le 2\pi)$ be a point and let OAB be a triangle with vertices $(0, 0), \left(\sqrt{\frac{3}{2}}, 0\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$. Find θ if P lies inside the ΔOAB.

4. Consider two lines $L_1 : x - y = 0$ and $L_2 : x + y = 0$ and a moving point $P(x, y)$. Let $d(P, L_i)$, $i = 1, 2$ represents the distance of point P, from the line L_i. If point P moves in certain region R in such a way that, $2 \le d(P,L_1) + d(P, L_2) \le 4$. Find the area of region R.

5. Consider the circle $x^2 + y^2 = a^2$. Let $A \equiv (a, 0)$ and D be a given interior point of the circle. If BC be any arbitrary chord of the circle through point D, prove that the locus of the centroid of triangle ABC is a circle whose radius is less than $a/3$.

6. Consider the inequation $x^2 + |x + a| - 9 < 0$, find the values of the real parameter 'a' so that the given inequation has atleast one negative solution.

7. If $x^2 + px - 444p = 0$ has integral roots where p is a prime number then find the value(s) of p.

8. Find the value(s) of 'a' for which the inequality $\tan^2 x + (a + 1)\tan x - (a - 3) < 0$, is true for at least one $x \in \left(0, \frac{\pi}{2}\right)$.

9. If t be a real number satisfying the equation $2t^3 - 9t^2 + 30 - a = 0$, then find the values of the parameter 'a' for which the equation $x + \frac{1}{x} = t$ gives six real and distinct values of x.

10. (a) Sum the series

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left[\sum_{k=0}^{n} (k+1)\int_0^1 2^{-(k+1)x}\,dx\right]$$

(b) Find the value of $\sum_{i=0}^{\infty}\sum_{\substack{j=0 \\ (i\ne j\ne k)}}^{\infty}\sum_{k=0}^{\infty} \frac{1}{3^i 3^j 3^k}$.

11. Find the remainder when $1690^{2608} + 2608^{1690}$ is divided by 7.

12. (a) In a knockout tournament, 2^n equally skilled players namely $S_1, S_2, S_3, \ldots S_{2^n}$ are participating. In each round, players are divided in pairs at random and winner from each pair moves in the next round. If S_2 reaches semi-final, then find the probability that S_1 will win the tournament.

(b) Let $A = \{0, 5, 10, 15, \ldots, 195\}$. Let B be any subset of A with atleast 15 elements. What is the probability that B has at least one pair of elements whose sum is divisible by 15 ?

SOLUTIONS

1. **(a) :** $4\sin A\cos B = 1$, so A and B cannot be $\frac{\pi}{2}$ [as if $B = \frac{\pi}{2}$, then $\cos B = 0$ and if $A = \frac{\pi}{2}$, $\tan A$ is not defined]

$$\Rightarrow C = \frac{\pi}{2}, B = \frac{\pi}{2} - A \Rightarrow 4\sin A\cos\left(\frac{\pi}{2} - A\right) = 1$$

$$\Rightarrow \sin^2 A = \frac{1}{4} \Rightarrow \sin A = \frac{1}{2} \Rightarrow A = \frac{\pi}{6} \Rightarrow B = \frac{\pi}{3}$$

$\therefore$ so angles are in A.P.

2. **(c) :** $(a - b)^2 - c^2 = 0 \Rightarrow (a - b - c)(a - b + c) = 0$

If $a - b = c \Rightarrow ax + by + (a - b) = 0$

$\Rightarrow a(x + 1) + b(y - 1) = 0 \Rightarrow x = -1, y = 1$

If $a - b = -c \Rightarrow ax + by + (b - a) = 0$

$\Rightarrow a(x - 1) + b(y + 1) = 0 \Rightarrow x = 1, y = -1$

3. **(b) :** $\sin 10° = \frac{a}{2b} \Rightarrow \sin 30° = 3\sin 10° - 4\sin^3 10°$

$$\Rightarrow \frac{1}{2}=\frac{3a}{2b}-\frac{4a^3}{8b^3} \Rightarrow 1=\frac{3a}{b}-\frac{a^3}{b^3} \Rightarrow a^3+b^3=3ab^2$$

4. **(c)** : Let 2010 be in k^{th} row

$\Rightarrow$ k^{th} term of series 1,2,4,7,... ≤ 2010

and $(k+1)^{th}$ term of series 1,2,4,7,... > 2010

$$S_n = 1+2+4+7+...+T_n$$
$$S_n = 1+2+4+...+T_{n-1}+T_n$$
$$\Rightarrow 0 = 1+(1+2+3+...(n-1) \text{ terms}) - T_n$$
$$\Rightarrow T_n = \frac{n^2-n+2}{2}$$
$$\Rightarrow \frac{k^2-k+2}{2} \le 2010 \text{ and } \frac{k^2+k+2}{2} > 2010$$
$$\Rightarrow k^2-k-4018 \le 0 \text{ and } k^2+k-4018>0$$
$$\Rightarrow \left(k-\frac{1}{2}\right)^2 \le \frac{16073}{4} \text{ and } \left(k+\frac{1}{2}\right)^2 > \frac{16073}{4}$$
$$\Rightarrow k-\frac{1}{2} \le 63.3 \text{ and } k+\frac{1}{2} > 63.3 \Rightarrow k=63.$$

5. **(c)** : $k\cos^2 x - k\cos x + 1 \ge 0 \quad \forall\, x \in (-\infty, \infty)$

$$\Rightarrow k(\cos^2 x - \cos x)+1 \ge 0 \qquad ...(i)$$

But $\cos^2 x - \cos x = \left(\cos x - \frac{1}{2}\right)^2 - \frac{1}{4}$

$$\Rightarrow -\frac{1}{4} \le \cos^2 x - \cos x \le 2$$

$\therefore$ From (i), we get $2k+1 \ge 0 \Rightarrow k \ge -\frac{1}{2}$

$$\Rightarrow -\frac{k}{4}+1 \ge 0 \Rightarrow k \le 4 \Rightarrow -\frac{1}{2} \le k \le 4$$

6. **(b)** : We have

$$|\overline{OP}|^2 = \left(t+\frac{1}{t}\right)^2 |\vec{a}|^2 + \left(t-\frac{1}{t}\right)^2 |\vec{b}|^2 + 2\left(t^2-\frac{1}{t^2}\right)|\vec{a}||\vec{b}|\cos\left(\frac{2\pi}{3}\right)$$
$$|\overline{OP}|^2 = 9\left(t+\frac{1}{t}\right)^2 + 4\left(t-\frac{1}{t}\right)^2 + 2\left(t^2-\frac{1}{t^2}\right)3\cdot 2\cdot\left(\frac{-1}{2}\right)$$
$$= 9\left(t^2+\frac{1}{t^2}+2\right)+4\left(t^2+\frac{1}{t^2}-2\right)-6\left(t^2-\frac{1}{t^2}\right)$$
$$= 7t^2+\frac{19}{t^2}+10$$
$$\Rightarrow |\overline{OP}|^2 \ge 2\sqrt{7t^2.\frac{19}{t^2}}+10 \qquad (\because A.M. \ge G.M.)$$

$\therefore$ Minimum value of $|\overline{OP}| = \sqrt{10+2\sqrt{133}}$

7. **(d)** : For $n \ge 1$

$$(x^k-1)^{-n-1}p_n(x) = \frac{d}{dx}\left[(x^k-1)^{-n}\cdot p_{(n-1)}(x)\right]$$
$$= (-n)(x^k-1)^{-n-1}(kx^{k-1})p_{n-1}(x)+(x^k-1)^{-n}p'_{n-1}(x)$$

or $p_n(x) = (x^k-1)p'_{n-1}(x) - n(kx^{k-1})p_{n-1}(x)$

at $x=1$, $p_n(1) = -nk\,p_{(n-1)}(1)$

$$\Rightarrow p_n(1) = -nk[-(n-1)k\,p_{n-2}(1)]$$

By doing so on we get $p_n(1) = n!(-k)^n p_0(1)$

8. **(d)** : Let the three sides be $a \le b \le c = 11$

Then $6 \le b < 11$ and $c-b < a \le b$. As b decreased by 1 unit the range of a decrease by 2.

When $b = 11$, we have $1 \le a \le 11$.

Hence the total number of triangles is $11+9+7+5+3+1=36$

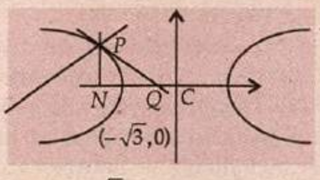
9. **(d)** : ${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + ... + (-1)^r.{}^nC_r$

$$= {}^{n-1}C_0 - ({}^{n-1}C_0 + {}^{n-1}C_1) + ({}^{n-1}C_1 + {}^{n-1}C_2)$$
$$-({}^{n-1}C_2 + {}^{n-1}C_3) + ... + (-1)^r.({}^{n-1}C_{r-1} + {}^{n-1}C_r)$$
$$= (-1)^r.{}^{n-1}C_r$$

$(-1)^r.{}^{n-1}C_r = 28 \Rightarrow r$ must be even

$${}^{n-1}C_r = 28 \Rightarrow {}^{n-1}C_r = 7\times 4 = \frac{7\times 8}{2} = {}^8C_2 \Rightarrow n-1=8 \Rightarrow n=9$$

10. **(c)** : The corresponding normals are $y = x \pm \frac{5\sqrt{3}}{3}$.



Considering $y = x+\frac{5\sqrt{3}}{3}$, corresponding tangent is $y=-x-\sqrt{3}$

$\Rightarrow$ Point of contact is $\left(\frac{-4}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\Rightarrow$ Length of sub tangent

$$= NQ = NC - QC = \frac{4}{\sqrt{3}} - \sqrt{3} = \frac{1}{\sqrt{3}} \Rightarrow k=3$$

11. **(a)** : Let $C = 1+\frac{\cos x}{3}+\frac{\cos 2x}{3^2}+.....$

and $S = \frac{\sin x}{3}+\frac{\sin 2x}{3^2}+.....$

$$\Rightarrow C+iS = 1+\frac{e^{ix}}{3}+\frac{e^{2ix}}{3^2}+...$$
$$\Rightarrow C+iS = \frac{1}{1-\frac{e^{ix}}{3}} = \frac{3}{3-\cos x - i\sin x}$$

Comparing real parts

$$C = \frac{3(3-\cos x)}{(3-\cos x)^2+\sin^2 x} \Rightarrow C=1 \quad \left(\because \cos x = \frac{1}{3}\right)$$

12. (a) : $S_1 : C_1(-5, 12)$, $S_2 : C_2(5, 12)$,
$r_1 = 16, r_2 = 4$
$CC_2 = r + 4$ (where C is the centre of circle touching C_2 externally and C_1 internally)
$CC_1 = 16 - r$ (Not $r - 16$, $\because S_2$ is contained by S_1)
$\Rightarrow CC_1 + CC_2 = 20$
$\therefore$ Locus of C is the ellipse with foci at C_1 and C_2 and length of major axis = 20
Locus of C is

$$\frac{x^2}{100} + \frac{(y-12)^2}{75} = 1 \qquad ...(i)$$

According to question $y = ax$ is tangent to this ellipse from (0, 0)
Equation of tangent to (i) is

$$y - 12 = mx + \sqrt{100m^2 + 75}$$

But this is passing through (0, 0)

$$\Rightarrow -12 = \sqrt{100m^2 + 75} \Rightarrow m^2 = \frac{69}{100} \Rightarrow p + q = 169.$$

13. (d) : Let there be a value of k for which $x^3 - 3x + k = 0$ has two distinct roots between 0 and 1. Let a, b be two distinct roots of $x^3 - 3x + k = 0$ lying between 0 and 1 such that $a < b$.
Let $f(a) = f(b) = 0$. Since between any two roots of a polynomial $f(x)$ there exists at least one roots of its derivative $f'(x)$. Therefore $f'(x) = 3x^3 - 3$ has at least one root between a and b. But $f'(x) = 0$ has two roots equal to ± 1 which do not lie between a and b. Hence $f(x) = 0$ has no real roots lying between 0 and 1 for any value of k.

14. (a) : $f''(x) = f'(x) \Rightarrow \dfrac{f''(x)}{f'(x)} = 1$

On integrating, $f'(x) = Ce^x$
Which gives $f(x) = Ce^x + D$
But $f(0) = 1 \Rightarrow C + D = 1 \quad \therefore f(x) = Ce^x + 1 - C$

So, $f'(x) = Ce^x$. Putting it in $f'(x) = f(x) + \int_0^1 f(x)dx$

$$\Rightarrow Ce^x = Ce^x + 1 - C + \int_0^1 (Ce^x + 1 - C)dx \Rightarrow C = \frac{2}{3-e}$$

So, $f(x) = \dfrac{2e^x - e + 1}{3 - e}$

15. (a) : $x + 2y = 10$, where x is the number of times he takes single steps, and y is the number of times he takes two steps

	Cases	Total number of ways
1	$x = 0, y = 5$	$5!/5! = 1$
2	$x = 2, y = 4$	$6!/2!4! = 15$
3	$x = 4, y = 3$	$7!/4!3! = 35$
4	$x = 6, y = 2$	$8!/2!6! = 28$
5	$x = 8, y = 1$	$9!/8! = 9$
6	$x = 10, y = 0$	$10!/10! = 1$

$\therefore \ p = 89$

16. (a) :

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{f\left(x\left(1 + \frac{h}{x}\right)\right) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)f\left(1 + \frac{h}{x}\right) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)\left(\frac{h}{x}\left(1 + g\left(\frac{h}{x}\right)\right)\right)}{h} = \frac{f(x)}{x}\lim_{h\to 0}\left(1 + g\left(\frac{h}{x}\right)\right) = \frac{f(x)}{x}$$

$$\Rightarrow \frac{f(x)}{f'(x)} = x \Rightarrow \int_1^2 \frac{f(x)}{f'(x)}\frac{dx}{1+x^2} = \int_1^2 \frac{x}{1+x^2}dx = \frac{1}{2}\log\left(\frac{5}{2}\right)$$

$\Rightarrow a = 5, b = 2$

17. (b) : Differentiate both side, we get
$f'(x)(1 - \cos x) + f(x)\sin x = 0$

$$\Rightarrow \int\frac{f'(x)}{f(x)}dx = \int\frac{\sin x}{\cos x - 1}dx$$

$\Rightarrow \ln|f(x)| = -2\ln\sin x/2$

$$\Rightarrow f(x) = \frac{e}{\left(\sin\frac{x}{2}\right)^2} \Rightarrow f(\pi) = 2 \Rightarrow c = 2$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 4$$

18. (d) : $|x| + |y| = |x + y|$
$\Rightarrow xy \ge 0$, therefore $(x - (3 - a))(x - 2a) \ge 0, \forall x \in R$
$\Rightarrow x^2 - x(3 + a) + 2a(3 - a) \ge 0 \forall x \in R$
$\Rightarrow (a+3)^2 - 8a(3 - a) \le 0 \Rightarrow (a-1)^2 \le 0 \Rightarrow a = 1$
which is true, $\forall x \in R$.

19. (c) : $B = (I - A)(I + A)^{-1} \Rightarrow B^T = (I + A^T)^{-1}(I - A^T)$
$= (I - A)^{-1}(I + A)$

$BB^T = (I - A)(I + A)^{-1}(I - A)^{-1}(I + A)$
$= (I - A)(I - A)^{-1}(I + A)^{-1}(I + A) = I$
(As $(I - A)\cdot(I + A) = (I + A)(I - A)$)

20. (a) : $f(x + \lambda) = f(x)$

$$\Rightarrow \cos n(x+\lambda)\sin\left(\frac{5(x+\lambda)}{n}\right) = \cos(nx)\sin\left(\frac{5x}{n}\right)$$

at $x = 0$, $\cos n\lambda \sin\left(\dfrac{5\lambda}{n}\right) = 0$

if $\cos n\lambda = 0,\ n\lambda = r\pi + \frac{\pi}{2},\ r \in I$

$n(3\pi) = r\pi + \frac{\pi}{2}$ $\quad(\because \lambda = 3\pi)$

$(3n - r) = \frac{1}{2}$ [not possible]

$\therefore \cos n\lambda \neq 0 \ \therefore \sin\left(\frac{5\lambda}{n}\right) = 0 \Rightarrow \frac{5\lambda}{n} = p\pi (p \in I) \Rightarrow n = \frac{15}{p}$

For $p = \pm 1, \pm 3, \pm 5, \pm 15$

$n = \pm 15, \pm 5, \pm 3, \pm 1$

21. **(c)** : $|x - a| < 3 - x^2 \Rightarrow -3 + x^2 < x - a < 3 - x^2$

$\Rightarrow x^2 - x + a - 3 < 0$ and $x^2 + x - (a + 3) < 0$

both equation $x^2 - x + a - 3 = 0$ and $x^2 + x - (a+3) = 0$ should have real & unequal roots, then

$a < \frac{13}{4}$ and $a > -\frac{13}{4}$ roots are

$\frac{1 \pm \sqrt{13 - 4a}}{2}$ and $\frac{-1 \pm \sqrt{13 + 4a}}{2}$

one of the roots is negative if $a < 3 \Rightarrow \frac{-13}{4} < a < 3$

Integral values of 'a' $= -3, -2, -1, 0, 1, 2$

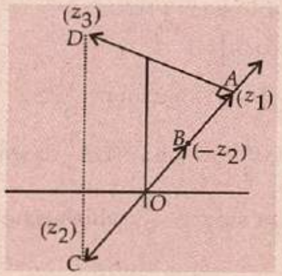
22. **(a)** : According to given conditions

(i) $|z_1 + z_2| = ||z_1| - |z_2||$ we can conclude that $\overline{OA}$ and $\overline{OC}$ are anti parallel

(ii) $|z_1 + i(z_3 - z_1)| = |z_1| + |z_3 - z_1| : \overline{OA}$ and $\overline{AD}$ are perpendicular

$\Rightarrow (z_2 - z_1)$ and $(z_3 - z_1)$ are perpendicular

Hence (a) is true.



23. **(a)** : Given equation is valid only in $[-1, 1]$

$\Rightarrow \cot^{-1} x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

The possible solution is

$\left[\cot^{-1} x\right] = 0 \Rightarrow \cot^{-1} x \in \left[\frac{\pi}{4}, 1\right) \Rightarrow (\cot 1) < x \le 1$

Similarly $\left[\cos^{-1} x\right] = 0 \Rightarrow (\cos 1) < x \le 1$

$\Rightarrow$ Option (a) is correct, as $\cos 1 < \cot 1$.

24. **(b)** : Let $\hat{i}, \hat{j}, \hat{k}$ be the unit vectors along OA, OB, OC. CN, BM are diagonals skew to OA.

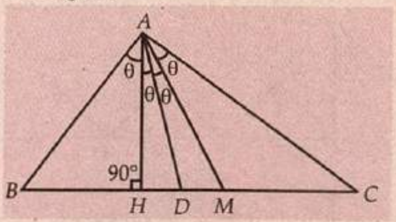
Let 1 be the S.D between OA and CN and let $\overline{OA} = x\hat{i}$; $\overline{OC} = z\hat{k}$; $\overline{OB} = y\hat{j} \Rightarrow \overline{ON} = x\hat{i} + y\hat{j}$ equation of OA and CN are $\vec{r} = x\hat{i}$ and $\vec{r} = z\hat{k} + p(x\hat{i} + y\hat{j} - z\hat{k})$ respectively. The shortest distance between those two lines,

$l = \frac{yz}{\sqrt{y^2 + z^2}} \Rightarrow \frac{1}{l^2} = \frac{1}{y^2} + \frac{1}{z^2}$

Similarly, $\frac{1}{m^2} = \frac{1}{z^2} + \frac{1}{x^2}$; $\frac{1}{n^2} = \frac{1}{x^2} + \frac{1}{y^2}$

$\Rightarrow \frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2} = 2\left(\frac{1}{OA^2} + \frac{1}{OB^2} + \frac{1}{OC^2}\right)$

25. **(b)** : According to given question consider ΔABC; as shown in figure



From sine rule in ΔAMB

$$\frac{AM}{BM} = \frac{\sin B}{\sin 3\theta} = \frac{\sin(\pi/2 - \theta)}{\sin 3\theta} = \frac{\cos\theta}{\sin 3\theta} \quad \text{...(i)}$$

Similarly from sine rule in ΔAMC

$$\frac{AM}{CM} = \frac{\sin C}{\sin\theta} = \frac{\sin(\pi/2 - 3\theta)}{\sin\theta} = \frac{\cos 3\theta}{\sin\theta} \quad \text{...(ii)}$$

$\therefore$ $BM = CM$ {as AM is the median}

From (i) and (ii), we get

$\frac{\cos\theta}{\sin 3\theta} = \frac{\cos 3\theta}{\sin\theta} \Rightarrow \sin 2\theta = \sin 6\theta \Rightarrow \quad 2\cos 4\theta \cdot \sin 2\theta = 0$

Either $\cos 4\theta = 0 \Rightarrow 4\theta = \frac{\pi}{2} \Rightarrow \angle A = \frac{\pi}{2}$

or $\sin 2\theta = 0$ (not possible for a triangle)

26. **(b)** : Let $x = f(t) \Rightarrow dx = f'(t)dt$

$$\Rightarrow \int_{\pi}^{2\pi} f^{-1}(x)dx = \int_{\pi}^{2\pi} t f'(t)dt = \left(t\left[f(t)\right]\right)_{\pi}^{2\pi} - \int_{\pi}^{2\pi} f(t)dt$$

$$= (4\pi^2 - \pi^2) - \int_{\pi}^{2\pi} f(t)dt$$

$$I = \int_{\pi}^{2\pi} (f^{-1}(x) + \sin x)dx = \int_{\pi}^{2\pi} f^{-1}(x)dx + \int_{\pi}^{2\pi} \sin x\,dx$$

$$= 3\pi^2 - \int_{\pi}^{2\pi} f(t)dt + \int_{\pi}^{2\pi} \sin x\,dx$$

$$= 3\pi^2 - \int_{\pi}^{2\pi} (f(x) - \sin x)dx$$

$$= 3\pi^2 - \int_{\pi}^{2\pi} x\,dx = 3\pi^2 - \frac{1}{2}(4\pi^2 - \pi^2) = \frac{3}{2}\pi^2$$

$$\Rightarrow \frac{2}{\pi^2} I = 3$$

27. (d) : Co-ordinates of required point can be taken as $(\pm(a-d), \pm a, \pm(a+d))$

Also according to question

$(a-d)^2 + a^2 = 5$, $(a-d)^2 + (a+d)^2 = 10$

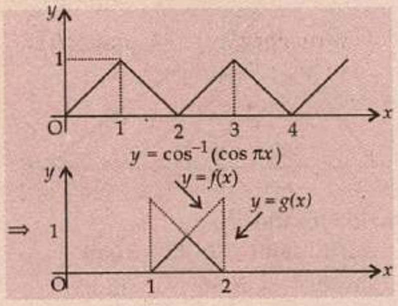
and $a^2 + (a+d)^2 = 13$

Solving these equations we will get

$a - d = 1, a = 2, a + d = 3$

So 8 points are possible.

28. (b) :



$$\Rightarrow \int_1^2 f(x)dx = \frac{3}{4} \text{ and } \int_1^2 g(x)dx = \frac{1}{4} \Rightarrow \text{Ratio} = 3$$

29. (a) : $\sin(\sin x + \cos x) = \cos(\cos x - \sin x)$

$\cos(\cos x - \sin x) = \cos\left(\frac{\pi}{2} - (\sin x + \cos x)\right)$

$\therefore \cos x - \sin x = 2n\pi \pm \left(\frac{\pi}{2} - \sin x - \cos x\right)$

Taking +ve sign

$\cos x - \sin x = 2n\pi + \frac{\pi}{2} - \sin x - \cos x$

$\cos x = n\pi + \frac{\pi}{4}$, for $n = 0$, $\cos x = \frac{\pi}{4}$, which is the only possible value

$$\Rightarrow \sin x = \frac{(\sqrt{16-\pi^2})}{4} \quad ...(i)$$

Taking –ve sign

$$\sin x = \frac{\pi}{4} \quad ...(ii)$$

From (i) and (ii), we get $\frac{\pi}{4}$ as the largest value

Hence $k = 4$

30. (b) : Here $z^2 - z = |z|^2 + \frac{64}{|z|^5}$ ...(1)

$\Rightarrow z^2 - z = \bar{z}^2 - \bar{z}$ ($\because z^2 - z$ is purely real number)

$\Rightarrow (z - \bar{z})(z + \bar{z} - 1) = 0$

$\Rightarrow z = \bar{z}$ as $z + \bar{z} = 1$ is not possible $\left(\because x \neq \frac{1}{2}\right)$

$\Rightarrow z = x$

$\therefore$ Equation (i), given as $x^2 - x - |x|^2 + \frac{64}{|x|^5} = 0$

$\Rightarrow x = 2$

$\therefore$ Only one solution.

SECTION - II

1. (b, c) : Let $G(x) = f(x) - f(x+1)$

$G(0) = f(0) - f(1)$ and $G(1) = f(1) - f(2)$

Since $f(2) = f(0) \Rightarrow G(0) + G(1) = 0$

$\Rightarrow G(0)$ and $G(1)$ are of opposite sign

$\Rightarrow f(x) = f(x+1)$ at least once in $[0, 1]$

2. (b, d) : Area of quadrilateral $ABCD$ is maximum when area of ACD is maximum

$\Rightarrow$ Distance of D from AC is maximum

i.e., $\cos\theta - \sin\theta$ is maximum

$= \sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$ is maximum

$\Rightarrow \theta = \frac{7\pi}{4}$ and area $= \frac{6}{\sqrt{2}} \cdot 2\sqrt{2} = 12$ sq.units

(Since $ABCD$ is a rectangle)

3. (a, c) : The given differential equation can be written as $f(x)\,dy + f'(x)y\,dx = dx$

i.e., $d(f(x).\,y) = d(x)$

Integrating, we get $y.f(x) = x + c$ or $f(x) = \frac{x+c}{y}$

4. (a, b, c) : $1 \le |\sin x| + |\cos x| \le \sqrt{2}$

$\Rightarrow [|\sin x| + |\cos x|] = 1$

$f(x)$ is defined if $\sin^2 x + 2\sin x + \frac{11}{4} \ge 2$

$\Rightarrow (\sin x + 1)^2 \ge \frac{1}{4} \Rightarrow \sin x + 1 \ge \frac{1}{2}$ or $\sin x + 1 \le -\frac{1}{2}$

$\Rightarrow \sin x \ge -\frac{1}{2}$ or $\sin x \le -\frac{3}{2}$, which is not true.

5. (a, b, c) : $f(x) = e^{2x}$

(a) $\int_0^{\ln 3} [e^x]dx = \ln 2 + 2(\ln 3 - \ln 2) = \ln 9 - \ln 2 = \ln 4.5$

(b) $\lim_{x\to 0^+} [e^{2x}] = 1$; $\lim_{x\to 0^-} [e^{2x}] = 0 \Rightarrow \lim_{x\to 0} [e^{2x}]$ does not exist

(c) $f^{-1}(x) = \ln\sqrt{x}, \ \forall x > 0$

(d) $e^{2x} < e^{x^2-4x}; 2x < x^2 - 4x$

$\Rightarrow x^2 - 6x > 0 \Rightarrow x < 0$ or $x > 6$.

6. (a, b) : $b - a = c - b = d - c = \lambda$ (say)

$$f(x)=\begin{vmatrix} x+a & x+b & x+a-c \\ x+b & x+c & x-1 \\ x+c & x+d & x-b+d \end{vmatrix}$$

$$=\begin{vmatrix} x+a & x+b & x+a-c \\ \lambda & \lambda & -1+2\lambda \\ \lambda & \lambda & 1+2\lambda \end{vmatrix} = \begin{vmatrix} x+a & \lambda & x+a-c \\ \lambda & 0 & -1+2\lambda \\ \lambda & 0 & 1+2\lambda \end{vmatrix}$$

$(R_2 \to R_2 - R_1; R_3 \to R_3 - R_2)$ $(C_2 \to C_2 - C_1)$

$\Rightarrow f(x) = -\lambda(\lambda + 2\lambda^2 + \lambda - 2\lambda^2) = -2\lambda^2$

$\Rightarrow \int_0^2 -2\lambda^2 dx = -4 \Rightarrow 2\lambda^2 \left[x\right]_0^2 = 4$

$\Rightarrow 2\lambda^2 \times 2 = 4 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

7. **(a, c) :**

$(\alpha-\beta)^2 = (16+20i-4M) \Rightarrow |16+20i-4M| = 28$

$\Rightarrow |M-4-5i| = 7$

8. **(b, c) :** The common chord to the circles will be $x + y = 0$

$\Rightarrow$ Equation of Axis of the parabolas will be $x - y + 1 = 0$. As vertex is the mid point of focus and point of intersection of axis and directrix of the parabola. Hence required points will be $\left(\frac{1}{4}, \frac{5}{4}\right)$ and $\left(\frac{5}{4}, \frac{9}{4}\right)$.

9. **(b, d) :** We can conclude that $F(x)$ is an even function

$\because \int_{-2010}^{2011} F(x)dx = \int_{-2010}^{2010} F(x)dx + \int_{2010}^{2011} F(x)dx$

$= 2\int_0^{2010} F(x)dx + \int_{2010}^{2011} F(x)dx = \int_0^{2010} F(x)dx + \int_0^{2011} F(x)dx$

Hence (b, d) are correct.

10. **(a, b, c, d) :** $\hat{\alpha}.\hat{\gamma} = x(\hat{\alpha}\cdot\hat{\alpha}) + y(\hat{\alpha}\cdot\hat{\beta}) + z\hat{\alpha}\cdot(\hat{\alpha}\times\hat{\beta})$

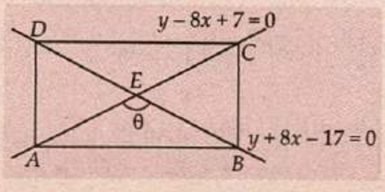
$\Rightarrow x = \cos\theta$, similarly $y = \cos\theta$

Now, $\hat{\gamma}\cdot(\hat{\alpha}\times\hat{\beta}) = z(\hat{\alpha}\times\hat{\beta})\cdot(\hat{\alpha}\times\hat{\beta}) \Rightarrow z = [\hat{\alpha}\,\hat{\beta}\,\hat{\gamma}]$

$$\Rightarrow [\hat{\alpha}\,\hat{\beta}\,\hat{\gamma}]^2 = \begin{vmatrix} 1 & 0 & \cos\theta \\ 0 & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix} = 1 - 2\cos^2\theta$$

PART - B

1. Points of intersection of the diagonals is $E\left(\frac{3}{2}, 5\right)$.



Now, $\Delta AEB = \Delta BEC = \Delta CED = \Delta DEA = 2$ sq.units

Let $\angle AEB = \theta$ and $EA = EB = l$

$\tan\theta = \frac{16}{63} \Rightarrow \sin\theta = \frac{16}{65}$ and $\cos\theta = \frac{63}{65}$

$\Rightarrow$ Area of $\Delta AEB = 2 = \frac{1}{2}\cdot l^2 \cdot \frac{16}{65}$

$\Rightarrow l^2 = \frac{65}{4} \Rightarrow l = \frac{\sqrt{65}}{2}$

Now equations of bisectors of diagonals are

$y-8x+7 = \pm(y+8x-17)$ or $x = \frac{3}{2}, y = 5$

$AB^2 = 2l^2(1-\cos\theta)$

$\Rightarrow AB^2 = 2\cdot\frac{65}{4}\left(1-\frac{63}{65}\right) \Rightarrow AB = 1 \Rightarrow BC = 8$

Since the sides of the rectangle will be parallel to the bisectors of the diagonals, their equations are $x = \frac{3}{2} \pm \frac{1}{2}$ and $y = 5 \pm 4$ or $x = 1, 2$ and $y = 1, 9$.

2. Combined equation of lines is

$x^2 - 2xy - 3y^2 + 8y - 4 = 0$

$\Rightarrow (x-y)^2 = 4y^2 - 8y + 4 \Rightarrow x - y = \pm 2(y-1)$

Thus two sides of the triangle are;

$L_1 : 3y - x - 2 = 0$ and $L_2 : y + x - 2 = 0$

and these intersect at $A \equiv (1, 1)$

Let the third side be $(y+1) = m(x+5)$

$\Rightarrow L_3 : y = mx + 5m - 1$

Let L_3 meet the lines L_1 and L_2 at B and C.

Then $B \equiv \left(\frac{15m-5}{1-3m}, -1\right)$ and $C \equiv \left(\frac{3-5m}{1+m}, \frac{7m-1}{1+m}\right)$

Now the origin has to be the interior point of triangle ABC.

Hence O and A should lie on the same side of side BC

$\Rightarrow (1-5m)(1-m+1-5m) > 0 \Rightarrow 2(1-5m)(1-3m) > 0$

$\Rightarrow m > \frac{1}{3}$ or $< \frac{1}{5}$ ...(1)

Similarly, points O and C should lie on the same side of line AB.

$\Rightarrow -2\left(\frac{3(7m-1)}{1+m} - \frac{3-5m}{1+m} - 2\right) > 0$

$\Rightarrow -1 < m < \frac{1}{3}$ ...(2)

Finally points O and B should lie on the same side of line AC.

$-2\left(\frac{15m-5}{1-3m} + (-1) - 2\right) > 0$

$\Rightarrow \left(m - \frac{1}{3}\right)^2 > 0 \Rightarrow m \in R$ ...(3)

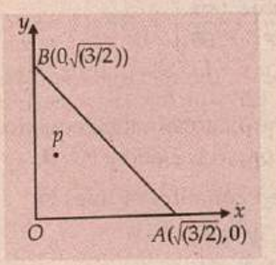
From (1), (2) and (3) we get $m \in \left(-1, \frac{1}{5}\right)$.

3. Equations of lines along OA, OB and AB are $y = 0$, $x = 0$ and $x + y = \sqrt{\frac{3}{2}}$ respectively.

Now P and B will lie on the same side of $y = 0$ if $\cos\theta > 0$.

Similarly P and A will lie on the same side of $x = 0$ if $\sin\theta > 0$ and P and O will lie on the same side of $x + y = \sqrt{\frac{3}{2}}$ if $\sin\theta+\cos\theta<\sqrt{\frac{3}{2}}$.

Hence P will lie inside the ΔABC if $\sin\theta > 0$, $\cos\theta > 0$ and $\sin\theta+\cos\theta<\sqrt{\frac{3}{2}}$.



Now, $\sin\theta+\cos\theta<\sqrt{\frac{3}{2}} \Rightarrow \sin\left(\theta+\frac{\pi}{4}\right)<\frac{\sqrt{3}}{2}$

i.e., $0<\theta+\frac{\pi}{4}<\frac{\pi}{3}$ or $\frac{2\pi}{3}<\theta+\frac{\pi}{4}<\pi$

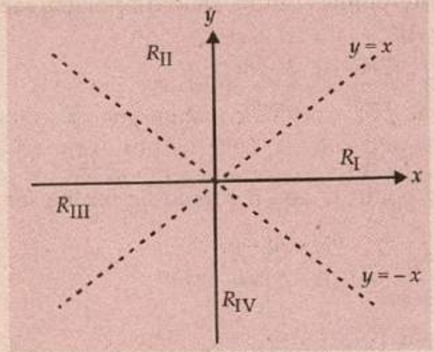
Since $\sin\theta>0$ and $\cos\theta>0$, so $0<\theta<\frac{\pi}{12}$ or $\frac{5\pi}{12}<\theta<\frac{\pi}{2}$

4. $d(P, L_1)=\frac{|x-y|}{\sqrt{2}}$ and $d(P, L_2)=\frac{|x+y|}{\sqrt{2}}$

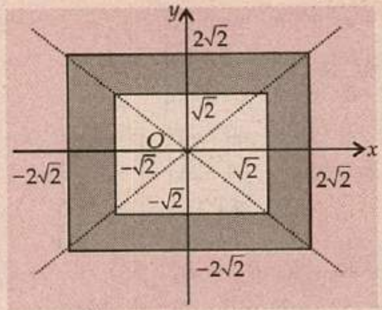
Now we have $2\le d(P, L_1)+d(P, L_2)\le 4$

$\Rightarrow 2\sqrt{2}\le|x-y|+|x+y|\le 4\sqrt{2}$ ...(1)

Now let us consider the four regions, namely R_1, R_2, R_3 and R_4 in the lines L_1 and L_2 divided the coordinate plane.



In R_I, we have $y < x$, $y > -x$. In R_2 we have $y > x$, $y > -x$. Similarly in R_3, we have $y > x$, $y < -x$.



Finally in R_4 we have $y < x$, $y < -x$.

Thus for R_1 equation (1) becomes

$$2\sqrt{2}\le x-y+x+y\le 4\sqrt{2} \Rightarrow \sqrt{2}\le x\le 2\sqrt{2}$$

Similarly for R_2 equation (1) becomes

$$2\sqrt{2}\le y-x+x+y\le 4\sqrt{2} \Rightarrow \sqrt{2}\le y\le 2\sqrt{2}$$

In R_3 equation (1) will becomes

$$2\sqrt{2}\le y-x-x-y\le 4\sqrt{2} \Rightarrow -\sqrt{2}\le y\le -2\sqrt{2}$$

Finally in R_4 equation (1) will become

$$2\sqrt{2}\le x-y-x-y\le 4\sqrt{2} \Rightarrow -\sqrt{2}\le y\le -2\sqrt{2}$$

Thus region 'R' will be the region between concentric squares formed by the line

$x=\pm 2\sqrt{2}, y=\pm 2\sqrt{2}$ and $x=\pm 2\sqrt{2}, y==\pm\sqrt{2}$

Thus the required area

$$=(4\sqrt{2})^2-(2\sqrt{2})^2=24 \text{ sq. units}$$

5. Let $D \equiv (\alpha, \beta)$, where $\alpha^2+\beta^2-a^2<0$

$B\equiv(a\cos\theta_1, a\sin\theta_1), C\equiv(a\cos\theta_2, a\sin\theta_2)$

Equation of line BC,

$$x\cos\left(\frac{\theta_1+\theta_2}{2}\right)+y\sin\left(\frac{\theta_1+\theta_2}{2}\right)=a\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

It passing through (α, β)

$$\Rightarrow \alpha\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\beta\sin\left(\frac{\theta_1+\theta_2}{2}\right)=a\cos\left(\frac{\theta_1-\theta_2}{2}\right) \quad ...(1)$$

Let centroid of ΔABC be (h, k)

$\Rightarrow 3h=a\cos\theta_1+a\cos\theta_2+a$, $3k=a\sin\theta_1+a\sin\theta_2$

$\Rightarrow \frac{3h}{a}=\cos\theta_1+\cos\theta_2+1$, $\frac{3k}{a}=\sin\theta_1+\sin\theta_2$

$$\Rightarrow 2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cdot\cos\left(\frac{\theta_1-\theta_2}{2}\right)=\frac{3h}{a}-1 \quad ...(2)$$

$$\text{and } 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cdot\cos\left(\frac{\theta_1-\theta_2}{2}\right)=\frac{3k}{a} \quad ...(3)$$

Multiplying (1) with $2\cos\left(\frac{\theta_1-\theta_2}{2}\right)$, we get

$$\alpha 2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cdot\cos\left(\frac{\theta_1-\theta_2}{2}\right)+2\beta\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

$$=2a\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)\alpha\left(\frac{3h-a}{a}\right)+\beta\left(\frac{3k}{a}\right)=2a\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)$$

Now from (2) and (3) we get,

$$2a\cos^2\left(\frac{\theta_1-\theta_2}{2}\right)=\frac{1}{2}\left(\frac{9k^2}{a^2}+\left(\frac{3h-a}{a}\right)^2\right)$$

$$\Rightarrow \alpha(3h-a)+3\beta k=\frac{a^2}{2}\left(\frac{9k^2}{a^2}+\left(\frac{3h-a}{a}\right)^2\right)$$

$$\Rightarrow 6\alpha h-2a\alpha+6\beta k=9k^2+9h^2+a^2-6ah$$

$\Rightarrow$ Locus of centroid is

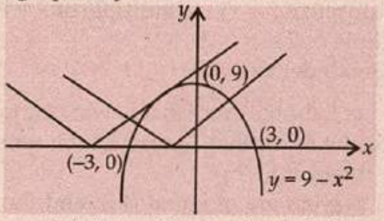
$$9x^2+9y^2-6x(a+\alpha)-6y\beta+a^2+2a\alpha=0$$

which is clearly a circle of radius

$$\sqrt{\frac{(a+\alpha)^2}{9}+\frac{\beta^2}{9}-\frac{a^2+2a\alpha}{9}}=\frac{1}{3}\sqrt{\alpha^2+\beta^2}<\frac{a}{3}\cdot$$

6. $9-x^2>|x+a|$

As per the question, we have to make sure that for atleast one negative x, graph of $y=9-x^2$ must lie above the graph of $y=|x+a|$



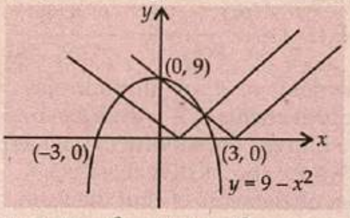
Case - I : $x+a\geq 0$, let us take up the limiting case when $y=|x+a|$ touches the parabola $y=9-x^2 \Rightarrow 9-x^2=x+a$ should have equal roots

$\Rightarrow x^2+x+(a-9)=0$ has equal roots

$\Rightarrow 1-4(a-9)=0 \Rightarrow a=\frac{37}{4} \Rightarrow 0\leq a<\frac{37}{4}$

Case - II: $x+a<0$

In the limiting case the left branch of $y=|x+a|$ *i.e.*, $y=-x-a$ will pass through the vertex of the parabola.



$\Rightarrow 9=0-a \Rightarrow a=-9 \quad \Rightarrow a>-9$

That means required set of values of a is $\left(-9,\frac{37}{4}\right)$

Alternative Solution : $x^2+|x+a|-9<0$

Case - I : Let $x+a\geq 0 \Rightarrow$ For all $x<0$, $a>0$

Also $x^2+x+a-9<0 \Rightarrow a<9-x^2-x$

$\Rightarrow a<\frac{37}{4}-\left(x+\frac{1}{2}\right)^2 \Rightarrow a<\frac{37}{4}$ *i.e.*, $0<a<\frac{37}{4}$ ...(1)

Case - II : Let $x+a<0 \Rightarrow a\leq 0$

Also $x^2-x-a-9<0 \Rightarrow a>x^2-x-9$

$\Rightarrow a>-9$, so $-9<a\leq 0$ ...(2)

From (1) and (2), $-9<a<\frac{37}{4}$.

7. The equation has integral hence

$$x=\frac{-p\pm\sqrt{p^2+4\times 444p}}{2}$$

Since $p=2$ does not give the integral roots

$\Rightarrow$ D must be perfect square of an odd integer

i.e., $D^2=p^2+1776p=p(p+1776)$

Since D is perfect square

$\Rightarrow$ $p+1776$ must be a multiple of p

$\Rightarrow$ 1776 must be a multiple of p

Now $1776=2^4\,3\cdot 37$ whence $p=2$ or 3 or 37

(i) $p=2$ then $p(p+1776)=2(2+1776)=3556$
$=4\times 7\times 127$ which is not a perfect square.

(ii) $p=3$ then $p(p+1776)=3(3+1776)=5337$ which is not a perfect square as its last digit is 7.

(iii) $p=37$ then $p(p+1776)=37(37+1776)=37^2\,.\,7^2$ which is odd

Hence $p=37$.

8. The required condition will be satisfied if

(i) The quadratic expression (quadratic in $\tan x$)

$f(x)=\tan^2 x+(a+1)\tan x-(a-3)$ has positive discriminant, and

(ii) Atleast are root of $f(x)=0$, is positive, as $\tan x>0,\ \forall\, x\in\left(0,\frac{\pi}{2}\right)$

For (i) Discriminant >0 or $(a+1)^2+4(a-3)>0$

$\Rightarrow a>2\sqrt{5}-3$ or $a<-(2\sqrt{5}+3)$ ...(1)

For (ii), we first find the condition, that both the roots of $t^2+(a+1)t-(a-3)=0$ ($t=\tan x$) are non-positive, for which sum of roots <0 and product of roots ≥ 0

$\Rightarrow -(a+1)<0$ and $-(a-3)\geq 0 \Rightarrow -1<a\leq 3$ and

Condition (ii) will be fullfilled if $a\leq -1$ or $a>3$ ...(2)

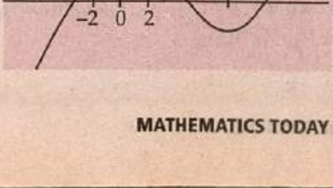
Required values of a is given by intersection of (1) and (2). Hence $a\in(-\infty,-3-2\sqrt{5})\cup(3,\infty)$.

9. We have, $2t^3-9t^2+30-a=0$

Any real root t_0 of this equation gives two real and distinct values of x if $|t_0|>2$. Thus, we need to find the condition for the equation in t to have three real and distinct roots none of which lies in $[-2, 2]$.

Let $f(t)=2t^3-9t^2+30-a$

$\therefore\ f'(t)=6t^2-18t=0 \Rightarrow t=0, 3$



So the equation $f(t) = 0$ has three real and distinct roots if $f(0)\cdot f(3) < 0$

$\Rightarrow (30-a)(54-81+30+a) < 0 \Rightarrow (30-a)(3-a) < 0$

$\Rightarrow (a-3)(a-30) < 0 \Rightarrow a \in (3, 30)$ ...(1)

Also, none of the roots lies in $[-2, 2]$ if $f(-2) > 0$ and $f(2) > 0$

$-16-36+30-a > 0 \Rightarrow a+22 < 0$ and $a-10 < 0$

$-22-a > 0$ and $10-a > 0 \Rightarrow a+22 < 0$ and $a-10 < 0$

$\Rightarrow a < -22$ and $a < 10$

$\Rightarrow a < -22$ ...(2)

From (1) and (2) no real value of a exists.

10.

(a) $$\int_0^1 2^{-(k+1)x}\,dx = -\frac{1}{(k+1)}\left[\frac{2^{-(k+1)x}}{\ln 2}\right]_0^1$$

$$= -\frac{1}{(k+1)}\left[\frac{2^{-(k+1)}}{\ln 2} - \frac{1}{\ln 2}\right]$$

Now required sum $$= \frac{1}{\ln 2}\sum_{n=0}^{n}\frac{1}{n!}\sum_{k=0}^{n}\left(1-\frac{1}{2^{k+1}}\right)$$

$$= \frac{1}{\ln 2}\left[\sum_{n=0}^{\infty}\frac{n+1}{n!} - \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\left(\frac{1}{2}\right)^{k+1}\right]$$

$$= \frac{1}{\ln 2}\left[\sum_{n=0}^{\infty}\frac{1}{(n-1)!} + \sum_{n=0}^{\infty}\frac{1}{n!} - \sum_{n=0}^{\infty}\frac{1}{n!}\left(1-\frac{1}{2^{n+1}}\right)\right]$$

$$= \frac{1}{\ln 2}\left[2e - \sum_{n=0}^{\infty}\frac{1}{n!} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n!}\right]$$

$$= \frac{1}{\ln 2}\left[2e - e + \frac{1}{2}e^{\frac{1}{2}}\right] = \frac{1}{2\ln 2}(2e+\sqrt{e})$$

(b) $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k},\ (i\neq j\neq k)$$

Let us first of all find the sum without any restriction i, j, k.

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k} = \left(\sum_{i=0}^{\infty}\frac{1}{3^i}\right)^3 = \frac{27}{8}$$

For the requirement sum we have to remove the cases when $i = j = k$ or when any two of them are equal and not equal to other variable (say $i = j \neq k$)

Case – 1 : When $i = j = k$

In this case $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k} = \sum_{i=0}^{\infty}\frac{1}{3^{3i}} = \frac{27}{26}$$

Case – II : $i = j \neq k$

In this case $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{3^i3^j3^k} = \left(\sum_{i=0}^{\infty}\frac{1}{3^{2i}}\right)\left(\sum_{\substack{k=0\\k\neq i}}^{\infty}\frac{1}{3^k}\right)$$

$$= \sum_{i=0}^{\infty}\frac{1}{3^{2i}}\left(\frac{3}{2}-\frac{1}{3^i}\right) = \frac{3}{2}\cdot\frac{9}{8} - \frac{27}{26} = \frac{135}{8\cdot 26}$$

Hence required sum $$= \frac{27}{8} - \frac{27}{26} - \left(\frac{135}{8\cdot 26}\right)\cdot 3 = \frac{81}{208}.$$

11. $1690 = 7 \times 241 + 3,\ 2608 = 7 \times 372 + 4$

Let $S = 1690^{2608} + 2608^{1690}$

$= (7 \times 241 + 3)^{2608} + (7 \times 372 + 4)^{1690}$

= a number multiple of $7 + 3^{2608} + 4^{1690}$

Let $S' = 3^{2608} + 4^{1690}$

Clearly remainder in S and S' will be the same when divided by 7

$S' = 3 \times 3^{3\times 867} + 4 \times 4^{3\times 563} = 3 \times 27^{867} + 4 \times 64^{563}$

$= 3(28-1)^{867} + 4(63+1)^{563}$

= 3[multiple of 7 – 1] + 4[multiple of 7 + 1]

= multiple of 7 + 1

Hence remainder is 1.

12. (a) Let E_1 be the event that S_1 wins the tournament and E_2 be the event that S_2 reaches the semifinal. We have to obtain $P(E_1/E_2)$.

Since all players are of equal skill and there will be four persons in the semifinal

$$\Rightarrow P(E_2) = \frac{{}^{2^n-1}C_3}{{}^{2^n}C_4} = \frac{4}{2^n}$$

$P(E_1 \cap E_2)$ = Probability that S_1 and S_2 both are in the semifinal and then S_1 wins in semifinal and also in

final $$= \frac{{}^{2^n-2}C_2}{{}^{2^n}C_4}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{3}{2^n(2^n-1)}$$

Hence, $$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{3\cdot 2^n}{2^n(2^n-1)4} = \frac{3}{4(2^n-1)}$$

(b) The elements of A are all multiples of 5. Sum of every pair of elements of A is divisible by 5. Therefore, we have to find the probability that B has two distinct elements whose sum is divisible by 3.

Let A_0 = set of elements of A of the form

$3k = \{0, 15, 30, ..., 195\}$

A_1 = Set of elements of A of the form

$3k + 1 = \{10, 25, ..., 190\}$

A_2 = Set of elements of A of the form

$3k + 2 = \{5, 20, 35, ..., 185\}$

$n(A_0) = 14,\ n(A_1) = n(A_2) = 13$

If B has at least two elements from A_0, then we are done. If B contains at most one element of A_0, then it must have at least one element from each of A_1 and A_2 for which the sum of these two elements will be divisible by 3.

So, required probability = 1.

PROBLEMS FROM NATIONAL OLYMPIADS

1. Determine all non-negative integral pairs (x, y) for which $(xy-7)^2 = x^2+y^2$

2. Prove that $1 < \frac{1}{1001} + \frac{1}{1002} + \frac{1}{1003} + ... + \frac{1}{3001} < \frac{4}{3}$.

3. Determine all functions $f: R \setminus \{0,1\} \to R$ (here R denotes the set of real numbers) satisfying the functional relation $f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}$, for $x \neq 0$ and $x \neq 1$.

4. Let $p(x) = x^2 + ax + b$ be a quadratic polynomial in which a and b are integers. Given any integer n, show that there is an integer M such that $p(n)p(n+1) = p(M)$.

5. Suppose a and b are two positive real numbers such that the roots of the cubic equation $x^3 - ax + b = 0$ are all real. If α is a root of this cubic with minimal absolute value prove that $\frac{b}{a} < \alpha \leq \frac{3b}{2a}$.

6. Given a triangle ABC in a plane Σ find the set of all points P lying in the plane Σ such that the circumcircles of triangles ABP, BCP and CAP are congruent.

7. Suppose P is an interior point of a triangle ABC and AP, BP, CP meet the opposite sides BC, CA, AB in D, E, F respectively. Show that $\frac{AF}{FB} + \frac{AE}{EC} = \frac{AP}{PD}$.

8. A triangle ABC has incentre I. Its incircle touches the side BC at T. The line through T parallel to IA meets the incircle at S and the tangent to the incircle at S meets sides AB, AC in points C', B' respectively. Prove that triangle $AB'C'$ is similar to triangle ABC.

9. Suppose $A_1A_2A_3$ .... A_n is an n-sided regular polygon such that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$. Determine n, the number of sides of the polygon.

10. Let P be an interior point of a triangle ABC and let BP and CP meet AC and AB in E and F respectively. If $[BPF] = 4$, $[BPC] = 8$ and $[CPE] = 13$, find $[AFPE]$. (Here [ ] denotes the area of a triangle or a quadrilateral as the case may be.)

11. There are two urns each containing an arbitrary number of balls. Both are non empty to begin with. We are allowed two types of operations :
(i) Remove an equal number of balls simultaneously from both urns;
(ii) Double the number of balls in any one of them.
Show that after performing these operations finitely many times, both the urns can be made empty.

12. How many increasing 3-term geometric progressions can be obtained from the sequence 1, 2, $2^2, 2^3, ..., 2^n$?
(*e.g.*, $\{2^2, 2^5, 2^8\}$ is a 3-term geometric progression for $n \geq 8$.)

13. There are seventeen distinct positive integers such that none of them has a prime factor exceeding 10. Show that the product of some two of them is a square.

14. Show that the number of 3-element subsets $\{a, b, c\}$ of the set $\{1, 2, 3, ..., 63\}$ with $a+b+c < 95$ is less than the number of those with $a+b+c > 95$.

15. For which positive integral values of n can the set $\{1, 2, 3, ..., 4n\}$ be split into n disjoint 4-element subsets $\{a, b, c, d\}$ such that in each of these sets $a = \frac{(b+c+d)}{3}$.

16. A staircase has n steps. A man climbs either one step or two steps at a time. Prove that the number of ways in which he can climb up the staircase, starting from the bottom, is $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right], n \geq 1$.

17. Let $A = \{1, 2, 3, ..., n\}$. If a_i is the minimum element of the set A_i (where A_i denotes the subset of A containing exactly three elements) and X denotes the set of A_i's, then evaluate $\sum_{A_i \in X} a_i$.

18. Two players P_1 and P_2 are playing the final of a chess championship, which consists of a series of matches. Probability of P_1 winning a match is $\frac{2}{3}$ and for P_2 is $\frac{1}{3}$. The winner will be the one who is

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He trains IIT and Olympiad aspirants.

ahead by 2 games as compared to the other player and wins atleast 6 games. Now, if the player P_2 wins first four matches, find the probability of P_1 winning the championship.

19. Prove that $\cot^2\frac{\pi}{2n+1}, \cot^2\frac{2\pi}{2n+1}, \ldots, \cot^2\frac{n\pi}{2n+1}$

are the roots of the equation

$x^n - \frac{^{2n+1}C_3}{^{2n+1}C_1}x^{n-1} + \frac{^{2n+1}C_5}{^{2n+1}C_1}x^{n-2} - \ldots = 0$. And hence show that $\lim_{x\to\infty}\left(\sum_{r=1}^{n}\frac{1}{r^2}\right) = \frac{\pi^2}{6}$.

20. Let the area of a given triangle ABC be Δ. Points A_1, B_1 and C_1 are the mid points of the sides BC, CA and AB respectively. Point A_2 is the mid point of CA_1. Lines C_1A_1 and AA_2 meet the median BB_1 at points E and D respectively. If Δ_1 be the area of the quadrilateral A_1A_2DE, using vectors or otherwise prove that $\frac{\Delta_1}{\Delta} = \frac{11}{56}$.

21. Find the number of elements in the range of $f(x) = [x] + [2x] + \left[\frac{2}{3}x\right] + [3x] + [4x] + [5x]$ for $0 \le x \le n$ where $n \in N$ and $[\cdot]$ denotes the greatest integer function.

SOLUTIONS

1. We have the obvious solution (7, 0) and (0, 7). So suppose $x \ne 0$ and $y \ne 0$. We have $(xy - 7)^2 = x^2 + y^2$
or, $(xy)^2 - 14xy + 49 = x^2 + y^2$
or, $(xy)^2 - 12xy + 36 + 13 = x^2 + y^2 + 2xy$
or, $(xy - 6)^2 + 13 = (x + y)^2$
or, $13 = [(x + y) + (xy - 6)][(x + y) - (xy - 6)]$
Since 13 is a prime number the only possible factors are ± 1 and ± 13, *i.e.*,
(i) $(x + y) + (xy - 6) = 13$ and $(x + y) - (xy - 6) = 1$
or
(ii) $(x + y) - (xy - 6) = -13$ and $(x + y) + (xy - 6) = -1$
When solved, these alternatives give the solutions (3, 4) and (4, 3). Thus, (7, 0), (0, 7), (3, 4) and (4, 3) are all the solutions (in non-negative integers) of $(xy - 7)^2 = x^2 + y^2$.

2. Consider 2001 numbers $\frac{1}{k}$, $1001 \le k \le 3001$.
Using A.M. - H.M. inequality, we get

$$\left(\sum_{k=1001}^{3001} k\right)\left(\sum_{k=1001}^{3001}\frac{1}{k}\right) > (2001)^2.$$

But $\sum_{k=1001}^{3001} k = (2001)^2$.
Hence we get the inequality $\sum_{k=1001}^{3001}\frac{1}{k} > 1$.

On the other hand grouping 500 terms at a time, we also have $S = \sum_{k=1001}^{3001}\frac{1}{k} < \frac{500}{1000} + \frac{500}{1500} + \frac{500}{2000} + \frac{500}{2500} + \frac{1}{3001}$
$< \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{3000} = \frac{3851}{3000} < \frac{4}{3}$.

Note : We can sharpen the above inequality. Consider the sum $S = \sum_{k=n+1}^{3n+1}\frac{1}{k}$

There are $2n + 1$ terms in the sum and the middle term is $\frac{1}{2n+1}$. We can write the sum in the form

$$S = \frac{1}{2n+1} + \sum_{k=1}^{n}\left(\frac{1}{2n+1+k} + \frac{1}{2n+1-k}\right)$$
$$= \frac{1}{2n+1} + \frac{2}{(2n+1)} + \sum_{k=1}^{n}\frac{1}{1-\left(\frac{k}{2n+1}\right)^2}.$$

For $0 < a < \frac{1}{2}$, we have $1 + a < \frac{1}{1-a} < 1 + 2a$.
Thus we get the bounds

$$\frac{1}{2n+1} + \frac{2}{2n+1}\sum_{k=1}^{n}\left[1 + \left(\frac{k}{2n+1}\right)^2\right] < S \text{ and}$$
$$S < \frac{1}{2n+1} + \frac{2}{2n+1}\sum_{k=1}^{n}\left[1 + 2\left(\frac{k}{2n+1}\right)^2\right].$$

This on simplification gives

$$1 + \frac{2}{(2n+1)^3}\sum_{k=1}^{n}k^2 < S < 1 + \frac{4}{(2n+1)^3}\sum_{k=1}^{n}k^2.$$

Now using the identity $\sum_{k=1}^{n}k^2 = \frac{n(n+1)(2n+1)}{6}$ the inequality simplifies to

$$1 + \frac{n(n+1)}{3(2n+1)^2} < S < 1 + \frac{2}{3}\frac{n(n+1)}{3(2n+1)^2}.$$

But for $n \ge 1$, we also have $\frac{2}{9} \le \frac{n(n+1)}{(2n+1)^2} \le \frac{1}{4}$.
This leads to $\frac{29}{27} < S < \frac{7}{6}$.

3. Putting $y = \frac{1}{(1-x)}$, the given functional equation can be written as $f(x) + f(y) = 2\left(\frac{1}{x} - y\right)$

If we set $z = \frac{1}{(1-y)}$, then $x = \frac{1}{(1-z)}$. Hence we also have the relations $f(y) + f(z) = 2\left(\frac{1}{y} - z\right)$
and $f(z) + f(x) = 2\left(\frac{1}{z} - x\right)$

Adding the first and third relations, we get

$$2(f(x) + f(y) + f(z)) = 2\left(\frac{1}{x} - x\right) - 2y + \frac{2}{z}.$$

Using the second relation, this reduces to

$$2f(x) = 2\left(\frac{1}{x} - x\right) - 2\left(y + \frac{1}{y}\right) + 2\left(z + \frac{1}{z}\right).$$

Now using $y+\frac{1}{y}=\frac{1}{1-x}+1-x,\ z+\frac{1}{z}=\frac{x-1}{x}+\frac{x}{x-1}$

we get $f(x)=\frac{x+1}{x-1}$.

Thus $f(x)=\frac{(x+1)}{(x-1)}$ is the only function satisfying the given functional equation.

4. **1st Solution :** We can write $p(n)p(n+1)$

$= (n^2+an+b)((n+1)^2+a(n+1)+b)$
$= n^2(n+1)^2+a\{n(n+1)^2+n^2(n+1)\}+b\{n^2+(n+1)^2\}+a^2n(n+1)+b^2+ab(2n+1)$
$= n^2(n+1)^2+a^2n^2+b^2+2an^2(n+1)+2bn(n+1)+2nab+a^2n+an(n+1)+ab+b$
$= (n(n+1)+an+b)^2+a(n(n+1)+an+b)+b$
$= p(n(n+1)+an+b)$

2nd Solution : If α and β are the roots of the equation $p(x)=0$ we can write : $p(x)=(x-\alpha)(x-\beta)$.
Then $p(n)p(n+1)=(n-\alpha)(n-\beta)(n+1-\alpha)(n+1-\beta)$
$= (n-\alpha)(n-\beta+1)\,(n-\beta)\,(n-\alpha+1)$
$= \{n(n-\beta)+n-\alpha(n-\beta)-\alpha\}\times\{n(n-\alpha)+n-\beta(n-\alpha)-\beta\}$
$= (M-\alpha)(M-\beta)=p(M)$
where $M=n^2-n(\alpha+\beta)+\alpha\beta+n=n^2+na+b+n$.

Remark : Another way is to consider the quadratic equation $M^2+aM+b=(n^2+an+b)((n+1)^2+a(n+1)+b)$ and to show that this equation has integer roots of equivalently, that the discriminant $a^2-4[b-(n^2+an+b)((n+1)^2+a(n+1)+b))]$ is a square.

5. Let α, β, γ be the roots of the given cubic $x^3-ax+b=0$, where $a>0$ and $b>0$. We have then

$$\left.\begin{aligned}\alpha+\beta+\gamma&=0\\ \alpha\beta+\beta\gamma+\gamma\alpha&=-a\\ \alpha\beta\gamma&=-b\end{aligned}\right\}. \qquad (*)$$

From the last of these equations, we see that either all the roots are negative or two are positive and one negative. However the second equation in (*) shows that all three cannot be negative. So two of α, β, γ are positive and the remaining root is negative. The first equation in (*) implies that the negative root is numerically larger than the other two positive roots. Hence we may assume that $\gamma<0<\alpha\le\beta$ where $|\alpha|\le|\beta|\le|\gamma|$.

We have $b-a\alpha=-\alpha\beta\gamma+\alpha(\alpha\beta+\beta\gamma+\gamma\alpha)$
$= \alpha^2(\beta+\gamma)=-\alpha^3<0$.

Since a is positive, we get $\frac{b}{a}<\alpha$ proving the first inequality.

Again, we have $3b-2a\alpha=-3\alpha\beta\gamma+2\alpha(\alpha\beta+\beta\gamma+\gamma\alpha)$
$= -\alpha\beta\gamma+2\alpha^2\beta+2\alpha^2\gamma=\alpha[2\alpha(\beta+\gamma)-\beta\gamma]$
$= \alpha[-2(\beta+\gamma)^2-\beta\gamma]$ (since $\alpha=-(\beta+\gamma)$)
$= -\alpha(2\beta^2+5\beta\gamma+2\gamma^2)=-\alpha(2\beta+\gamma)(\beta+2\gamma)$
$= -\alpha(\beta-\alpha)(\gamma-\alpha)$

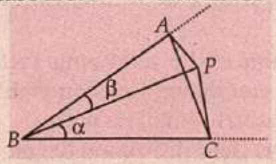
Observe that $-\alpha<0$, $\beta\ge\alpha$, $\gamma-\alpha<0$. Hence $3b-2a\alpha$ is non-negative. This proves the second inequality, $\alpha\le\frac{3b}{2a}$.

6. **1st Solution :** We shall show that the locus of all such points is the union of the circumcircle and the orthocentre of the triangle ABC.

Let P be any point in the cone determined by two sides, say, BA and BC. Using the sine rule in the triangles PAC and PBC, we get $\angle CAP=\alpha$ or $180°-\alpha$.

Similarly, using the triangles CAP and BAP, we also get $\angle ACP=\beta$ or $180°-\beta$.

Consider the case $\angle CAP=\alpha$ and $\angle ACP=180°-\beta$.



Here we get, $\angle APC=180°-(\alpha+180°-\beta)=\beta-\alpha$

Again the triangles BPC and BPA gives $\angle BAP=\angle BCP$ or $\angle BAP=180°-\angle BCP$.

If $\angle BAP=\angle BCP=\gamma$, then the sum of the angles of the quadrilateral is equal to $2\beta+2\gamma$. This implies that $\beta+\gamma=180°$. Since β and γ are angles of a triangle, this is impossible. If $\angle BAP=180°-\angle BCP=180°-\gamma$, then we get $-2\beta+360°=180°$. Hence $\beta=90°$. This forces that $\angle PCA=90°$ and AP is a diameter of the circle through A, B, C and P, *i.e.*, P is on the circumcircle of ABC. Similarly, we can dispose off the case $\angle CAP=180°-\alpha$, $\angle ACP=\beta$. Finally consider the case, $\angle CAP=180°-\alpha$ and $\angle ACP=180°-\beta$. Considering the triangle ACP, we see that $\angle APC=180°-\angle ABC$.

Similarly, the case $\angle CAP=\alpha$, $\angle ACP=\beta$ gives that $\angle APC$ and $\angle ABC$ are supplementary angles. Thus, A, B, C and P are concyclic.

On the other hand, suppose P is in the cone determined by the lines, say, CB and AB extended. Since $\angle PBC+\angle PAC=\angle PBA+\angle PCA=180°$, it follows that $\angle ABC$ and $\angle APC$ are supplementary angles. Thus, triangles ABC and APC, and hence triangles ABC and BPC, have the same circumradii. Now sine rule gives $\angle CPB=\beta$ or $180°-\beta$, $\angle APB=\gamma$ or $180°-\gamma$.

Also, if $\angle BAP=\alpha$, then $\angle BCP=\alpha$ or $180°-\alpha$. Consider the case, then $\angle APC=\beta+180°-\gamma$, $\angle PAC+\angle PCA=\beta+\gamma+2\alpha$ and hence $\beta+\gamma+2\alpha=\gamma-\beta$ or $\alpha+\beta=0$ which is impossible. If $\angle BCP=180°-\alpha$, then we have $\angle APC=\beta+180°-\gamma$, $\angle PAC+\angle PCA=\beta+\gamma+180°$, which is impossible. Similarly we can dispose off the cases $\angle CPB=180°-\beta$, $\angle APB=\gamma$, $\angle BCP=\alpha$ or $180°-\alpha$.

Finally if $\angle CPB=\beta$, $\angle APB=\gamma$, $\angle BCP=180°-\alpha$, then again we get $\angle APC=\beta+\gamma$, $\angle PAC+\angle PCA=180°+\beta+\gamma$.

This forces $2(\beta+\gamma)=0$ which is impossible. We conclude that the only possibility is $\angle APB=\gamma$, $\angle CPB=\beta$ and $\angle BCP=\alpha$.

In this case, we get $\angle APC=\beta+\gamma$, $\angle PAC+\angle PCA=2\alpha+\beta+\gamma$.

This gives us $\alpha = 90° - (\beta + \gamma)$.

Thus $\beta + \alpha = 90° - \gamma$ and $\alpha + \gamma = 90° - \beta$. These imply that AP is perpendicular to CB and CP is perpendicular to AB. Hence P is the orthocentre.

Similarly we can consider other regions determined by BA and CA or BC and AC.

Finally if P is a point inside the triangle, we can show that P is the orthocentre of the triangle ABC in the similar way.

Thus if P is any point satisfying the hypothesis, then either P is the orthocentre of the triangle ABC or P must be on the circumcircle of the triangle ABC.

2nd Solution :

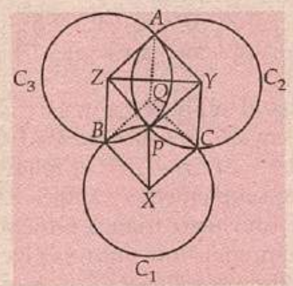
We need to know the following facts about three equal circles intersecting in a common point. If three congruent (that is, equal) circles C_1, C_2, C_3 have a common point P and A, B, C are the other three points of intersections, then

(a) the circumcircle of triangle ABC has the same radius as the three circles; and

(b) the point P is the orthocentre of triangle ABC.

A brief proof of (a) and (b) follows :

Let X, Y, Z be the centres of the circles C_1, C_2, C_3 respectively. Complete the quadrilaterals $PXBZ$ and $PXCY$, join AP and ZY. Observe that $PXBZ$ and $PXCY$ are rhombuses and so ZB is parallel and equal to YC. Hence so are BC and ZY. Since AP is perpendicular to ZY, AP is perpendicular to BC. Similarly BP and CP are perpendicular to CA and AB respectively. Hence P is the orthocentre of triangle ABC. This proves (b).

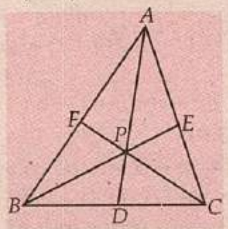


To prove (a), complete the parallelogram $AYCQ$, which is in fact a rhombus. So $AQ = CQ$. It is easily seen that $AZBQ$ is also a rhombus. So $AQ = BQ$. Thus Q is circumcentre of triangle ABC and its radius $(= AQ = CY)$ is the same as that of each of the three circles. Note that we can have a configuration of three equal circles such that P falls outside triangle ABC, but statements (a) and (b) are still true.

Coming to the problem, let (XYZ) denote the circle through any three non collinear points X, Y, Z. It is given that three equal circles pass through P. Hence by (a) above, the four circles (PAB), (PBC), (PCA) and (ABC) are congruent to one another. Observe that either the three circles (PAB), (PBC), (PCA) coincide [and hence coincide with (ABC)] or the three circles are all distinct passing through the point P. Thus either P is on the circumcircle of ABC of P is the orthocentre of ABC.

7. **1st Solution :** We use the fact that the areas of two triangles having the same height are in the ratio of their bases. We also use some simple properties of equal fractions.

Specifically, if $\frac{a}{b} = \frac{c}{d}$, then each fraction is also equal to $\frac{(a+c)}{(b+d)}$ as well as $\frac{(a-c)}{(b-d)}$.



Now $\frac{[ACF]}{[BCF]} = \frac{AF}{FB} = \frac{[APF]}{[BPF]}$

So $\frac{AF}{FB} = \frac{[ACF]-[APF]}{[BCF]-[BPF]} = \frac{[ACP]}{[BCP]}$ .....(1)

Similarly from $\frac{[ABE]}{[CBE]} = \frac{AE}{EC} = \frac{[APE]}{[CPE]}$

one obtains $\frac{AE}{EC} = \frac{[ABE]-[APE]}{[CBE]-[CPE]} = \frac{[ABP]}{[CBP]}$. .....(2)

From (1) and (2), by addition, we get

$\frac{AF}{FB} + \frac{AE}{EC} = \frac{[ACP]+[ABP]}{[BCP]}$. .....(3)

Again, $\frac{AP}{PD} = \frac{[ABP]}{[DBP]} = \frac{[CAP]}{[DCP]} = \frac{[ABP]+[ACP]}{[BCP]}$. ....(4)

From (3) and (4), we have the desired result.

2nd Solution : Applying Ceva's theorem to the Cevians AD, BE, CF which are concurrent at P, we have $\frac{AF}{FB} \cdot \frac{BC}{DC} \cdot \frac{CE}{EA} = 1$.

Therefore, $\frac{AE}{EC} = \frac{AF}{FB} \cdot \frac{BD}{DC}$.

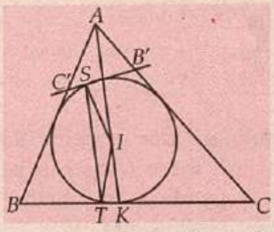
Hence $\frac{AF}{FB} + \frac{AE}{AC} = \frac{AF}{FB}\left(1 + \frac{BD}{DC}\right) = \frac{AF}{FB} \cdot \frac{BC}{DC}$ .....(1)

Now applying 'Menelaus' Theorem to triangle ABD, whose sides are cut by the line FPC, we have $\frac{AF}{FB} \cdot \frac{BC}{DC} \cdot \frac{DP}{PA} = +1$

Consequently, $\frac{AF}{FB} \cdot \frac{BC}{DC} = \frac{AP}{PD}$ .....(2)

Comparing (1) and (2), we have the desired relation.

8. Let AI meet BC in K. Join IS. We do some angle-chasing now. Since AK is parallel to ST, we have $\angle STB = \angle AKB = \angle KCA + \angle KAC = C + \frac{A}{2}$.



So, $\angle STI = 90° - \angle STB = 90° - \left(C + \frac{A}{2}\right)$

But $\angle TSI = \angle STI$ since SIT is an isosceles triangle.

Therefore $\angle C'ST = 90° - \angle TSI = C + \frac{A}{2}$

In the quadrilateral $BTSC'$.

$\angle SC'B = 360° - (\angle C'BT + \angle BTS + TSC')$

$= 360° - \left(B + C + \frac{A}{2} + C + \frac{A}{2}\right)$

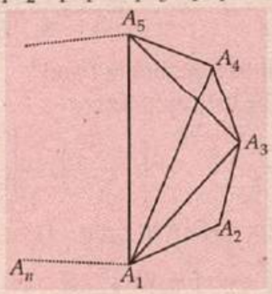
$= 360° - (A + B + C + C) = 180° - C.$

Hence $\angle AC'B' = 180° - \angle SC'B = 180° - (180° - C) = C$.

Similarly, $\angle AB'C' = B$. Thus it follows that triangles ABC and $AB'C'$ are similar.

9. **1st Solution :** From the given relation, we have

$A_1A_2 \cdot A_1A_3 + A_1A_2 \cdot A_1A_4 = A_1A_3 \cdot A_1A_4$ ..... (1)



Also in the cyclic quadrilateral $A_1A_3A_4A_5$, we have, by Ptolemy's theorem,

$A_4A_5 \cdot A_1A_3 + A_3A_4 \cdot A_1A_5 = A_3A_5 \cdot A_1A_4$ .....(2)

Since $A_1A_2 \ldots A_n$ is a regular polygon, we have $A_1A_2 = A_4A_5$, $A_1A_2 = A_3A_4$, $A_1A_3 = A_3A_5$.

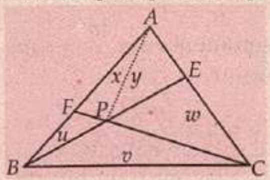
Comparing (1) and (2), we have $A_1A_4 = A_1A_5$.

Since the two diagonals A_1A_4 and A_1A_5 are equal, it follows that there must be the same number of vertices between A_1 and A_4 as between A_1 and A_5. That is the polygon must be 7-sided, that is $n = 7$.

2nd Solution :

If O is the centre of the circle in which $A_1A_2 \ldots A_n$ is inscribed and θ is the angle which each side of the polygon subtends at O then using the relation $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$ obtain an equation in θ. Solve the equation to get $\theta = \frac{2\pi}{7}$. This means $n = 7$.

10. More generally, let $[BPF] = u$, $[BPC] = v$ and $[CPE] = w$. Join AP. Let $[AFP] = x$ and $[AEP] = y$.



Using the triangle AFC and BFC, we get

$$\frac{x}{y+w} = \frac{FP}{PC} = \frac{u}{v}$$

This gives the equation $vx - uy = uw$.

Again using the triangles AEB and CEB we get another equation $wx - vy = -uw$.

Solving these equations, we obtain

$$x = \frac{uw(u+v)}{v^2 - uw},\ y = \frac{uw(w+v)}{v^2 - uw}$$

Hence we obtain $x + y = \frac{uw(u+2v+w)}{v^2 - uw}$.

Putting the values $u = 4$, $v = 8$, $w = 1$, we get $[AFPE] = 143$.

11. If both the urns have the same number of balls, then we can empty both the urns in one operation. Else, we remove the same number of balls from each of the urns so that one of the urns contains exactly one ball. (If m and n denote the number of balls in the urns, and say $m > n$, then take out $n - 1$ balls from each.) We now double the number of balls of the urn which contains only one ball and remove one ball from each of the urn. This process decreases the number of balls in the other urn by 1. Continuing this way we reach a stage when both the urns contain one ball each whence we can empty the urns removing one ball from each of the two urns.

12. Let us start counting 3-term G.P.'s with common ratios $2, 2^2, 2^3, \ldots$

The 3-term G.P.'s with common ratio 2 are

$1, 2, 2^2; 2, 2^2, 2^3; \ldots ; 2^{n-2}, 2^{n-1}, 2^n$.

They are $(n - 1)$ in number. The 3-term GP's with common ratio 2^2 are $1, 2^2, 2^4; 2, 2^3, 2^5; \ldots ; 2^{n-4}, 2^{n-2}, 2^n$.

They are $(n - 3)$ in number. Similarly we see that the 3-term GP's with common ratio 2^3 are $(n - 5)$ in number and so on. Thus the number of 3-term GP's which can be formed from the sequence $1, 2, 2^2, 2^3, \ldots, 2^n$ is equal to $S = (n - 1) + (n - 3) + (n - 5) + \ldots$

Here the last term is 2 or 1 according as n is odd or even.

If n is odd, then $S = (n - 1) + (n - 3) + (n - 5) + \ldots + 2$

$$= 2\left(1+2+3+...+\frac{n-1}{2}\right) = \frac{n^2-1}{4}.$$

If n is even, then $S = (n-1) + (n-3) + ... + 1 = \frac{n^2}{4}$.

Here the required number is $\frac{(n^2-1)}{4}$ or $\frac{n^2}{4}$ according as n is odd or even.

13. Since none of the 17 integers has a prime factor exceeding 10, all of them have the form $2^a3^b5^c7^d$, where a, b, c and d are non-negative integers. The product of two such numbers, say $2^a3^b5^c7^d$ and $2^{a'}3^{b'}5^{c'}7^{d'}$, is $2^{a+a'}3^{b+b'}5^{c+c'}7^{d+d'}$. Thus if $a + a'$, $b + b'$, $c + c'$ and $d + d'$, are all even then the product would be a square. For this to happen the 4-tuples (a, b, c, d) and (a', b', c', d') should have the parity (that is to say a and a' should both be odd or both even, b and b' should be both odd or both even etc.). Since each of the numbers a, b, c and d can either be odd or even, the total number of patterns of the 4-tuples (a, b, c, d) is $2^2 = 16$. As we have seventeen 4-tuples (a, b, c, d), each corresponding to the 17 given numbers, it follows, by the pigeon-hole principle, that at least two of these seventeen 4-tuples should have the same parity. The product of the numbers corresponding to these 4-tuples will then be a square.

14. Suppose that (a, b, c) is a subset of $\{1, 2, 3, ..., 63\}$ with $a + b + c < 95$. Then $(64 - a, 64 - b, 64 - c)$ is a subset of $\{1, 2, 3, ..., 63\}$ with $(64-a) + (64-b) + (64-c) = 192 - (a + b + c) > 192 - 95 = 97$. Conversely, if (a, b, c) is a subset of $\{1, 2, 3, ..., 63\}$ with $a + b + c < 97$, then $(64 - a, 64 - b, 64 - c)$ is such that $(64 - a) + (64 - b) + (64 - c) = 192 - (a + b + c) < 95$. Thus there is a one-one correspondence between 3-element subsets (a, b, c) with $a + b + c < 95$ and those such that $a + b + c > 97$. Hence the number of subsets with $a + b + c < 95$ is equal to that with $a + b + c > 97$. Thus the set of 3-element subsets (a, b, c) with $a + b + c > 95$ will contain those with $a + b + c > 97$ and a few more.

15. Suppose $\{a, b, c, d\}$ is a group in which $a = (b + c + d)/3$. Then $a + b + c + d = 4a$. Hence, if such an n exists, then 4 divides $1 + 2 + ... + 4n$. However this sum is $2n(4n + 1)$. Thus a necessary condition for existence of such a set is that n be even.

We show that this condition is also sufficient; *i.e.*, if $n = 2k$ for some k, then it is possible to partition $\{1, 2, 3, ..., 8k\}$ into groups of 4 elements $\{a, b, c, d\}$ such that $a = \frac{(b+c+d)}{3}$. To this end, divide $\{1, 2, 3, ..., 8k\}$ in to groups of 8 integers such that each group contains 8 consecutive integers. If $\{a + 1, a + 2, a + 3, ..., a + 8\}$ is one such set, we can divide this set into two sets of 4 integers each as follows : $\{a + 4, a + 1, a + 3, a + 8\}$, $\{a + 5, a + 2, a + 6, a + 7\}$.

The desired partition is obtained since

$$a + 4 = \frac{a+1+a+3+a+8}{3}$$

and $a + 5 = \frac{a+2+a+6+a+7}{3}$.

16. For climbing up the n stairs, let the number of ways be $f(n)$. All these ways can be divided into two categories, depending on whether the last leap includes one stair or two stairs.

Number of ways of climbing first $(n - 1)$ stairs (when the last leap is that one stair) $= f(n - 1)$.

Number of ways of climbing first $(n - 2)$ stairs (when the last leap is that two stair) $= f(n - 2)$.

Since the total number of ways $= f(n)$.

$f(n) = f(n - 1) + f(n - 2)$.

Here we can see that $f(1) = 1$, $f(2) = 2$

For $n = 1$,

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = \frac{1}{\sqrt{5}}\left[\frac{6+\sqrt{5}}{4} - \frac{6-\sqrt{5}}{4}\right] = 1$$

$$n = 2, \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{1+3\sqrt{5}+3\times5+5\sqrt{5}}{8} - \frac{1+3\sqrt{5}+3\times5-5\sqrt{5}}{8}\right] = \frac{2\sqrt{5}}{\sqrt{5}} = 2$$

Hence the result is true for $n = 1$ and 2

Let the results be true for $n \le m$ *i.e.*

$$f(m) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m+1}\right]$$

For $n = m + 1$, $f(m + 1) = f(m) + f(m - 1)$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m+1}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m} - \left(\frac{1-\sqrt{5}}{2}\right)^{m}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{m}\left(\frac{1-\sqrt{5}}{2}+1\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{m}\left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{m+2}\right]$$

Hence, by the method of mathematical induction, the result is true for all $n \in N$.

17. Since we can take three elements out of n in ${}^nC_3$ ways, X will contains ${}^nC_3$ subsets. There are exactly ${}^{n-1}C_2$ elements of X having 1 as least element; exactly ${}^{n-2}C_2$ elements of X having 2 as least element and so on. In general there are exactly ${}^{n-r}C_2$ elements of X having $r(1 \le r \le n-2)$ as least element.

Thus, $\sum_{a_i \in X} \min(a_i) = 1({}^{n-1}C_2) + 2({}^{n-2}C_2) + 3({}^{n-3}C_2) + \dots + (n-2)({}^2C_2)$

$$\sum_{r=1}^{n-2} r\left({}^{n-r}C_2\right) = \sum_{r=1}^{n-2} \frac{(n-r)(n-r-1)}{2}$$

$$= \frac{1}{2}\sum_{r=1}^{n-2} r\left[r^2-(2n-1)r+n(n-1)\right]$$

$$= \frac{1}{2}\left[\sum_{r=1}^{n-2} r^3-(2n-1)\sum_{r=1}^{n-2} r^2+n(n-1)\sum_{r=1}^{n-2} r\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}(n-1)^2(n-2)^2 - \frac{1}{6}(2n-1)(n-2)(n-1)(2n-3) + \frac{1}{2}n(n-1)(n-2)(n-1)\right]$$

$$= \frac{1}{24}(n-2)(n-1)[3(n-2)(n-1)-2(2n-1)(2n-3)+6n(n-1)]$$

$$= \frac{1}{24}(n-2)(n-1)\left[3n^2-9n+6-2\left(4n^2-8n+3\right)+6n^2-6n\right]$$

$$= \frac{1}{24}(n-2)(n-1)\left(n^2+n\right) = \frac{1}{24}(n+1)n(n-1)(n-2) = {}^{n+1}C_4$$

18. P_1 can win in the following mutually exclusive ways

(a) P_1 wins the next six matches

(b) P_1 wins five out of next six matches, so that after next six matches scores of P_1 and P_2 are tied up. This tie continues up to next '$2n$' matches ($n \ge 0$) and finally P_1 wins 2 consecutive matches

Now probability of case (a) $= \left(\frac{2}{3}\right)^6$ and probability of tie after 6 matches (in case (b)) $= {}^6C_5\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right) = 6.\frac{2^5}{3^6} = \frac{2^6}{3^5}$

now probability that scores are still tied up after another next two matches $= \frac{2}{3}.\frac{1}{3}+\frac{1}{3}.\frac{2}{3} = \frac{4}{9}$

[1st match won by P_1 and 2nd by P_2 or 1st by P_2 2nd by P_1]

Similarly probability that scores are still tied up after another $2n$ matches $= \left(\frac{4}{9}\right)^n$

$\Rightarrow$ total probability of P_1 winning the championship

$$= \left(\frac{2}{3}\right)^6 + \frac{2^6}{3^5}.\left(\sum_{n=0}^{\infty}\left(\frac{4}{9}\right)^2.\left(\frac{2}{3}\right)^2\right)$$

$$= \left(\frac{2}{3}\right)^6 + \frac{2^6}{3^5}.\left(\frac{2}{3}\right)^2\left(\frac{1}{1-\frac{4}{9}}\right) = \frac{17}{5}\left(\frac{2}{3}\right)^6 = \frac{1088}{3645}$$

19. From DeMoivre's Theorem, we know that

$$\sin(2n+1)\alpha = {}^{2n+1}C_1(\cos\alpha)^{2n}\sin\alpha - {}^{2n+1}C_3(\cos\alpha)^{2n-2}\sin^3\alpha + \dots + (-1)^n\sin^{2n-1}\alpha$$

$$\Rightarrow \sin(2n+1)\alpha = \sin^{2n+1}\alpha\left[{}^{2n+1}C_1\cot^{2n}\alpha - {}^{2n+1}C_3\cot^{2n-2}\alpha + {}^{2n+1}C_5\cot^{2n-4}\alpha - \dots\right]$$

It follows that for $\alpha = \frac{\pi}{2n+1}, \frac{2\pi}{2n+1}, \dots, \frac{n\pi}{2n+1}$

$$\Rightarrow {}^{2n+1}C_1\cot^{2n}\alpha - {}^{2n+1}C_3\cot^{2n-2}\alpha + {}^{2n+1}C_5\cot^{2n-4}\alpha - \dots = 0$$

$\Rightarrow \cot^2\frac{\pi}{2n+1}, \cot^2\frac{2\pi}{2n+1}, \dots, \cot^2\frac{n\pi}{2n+1}$ are the roots of the equation ${}^{2n+1}C_1x^n - {}^{2n+1}C_3x^{n-2} + \dots = 0$ if $0 < \theta < \frac{\pi}{2}$ $\sin\theta < \theta < \tan\theta$.

$$\Rightarrow \cot^2\theta < \frac{1}{\theta^2} < \operatorname{cosec}^2\theta$$

Now $\cot^2\frac{\pi}{2n+1} + \cot^2\frac{2\pi}{2n+1} + \dots + \cot^2\frac{n\pi}{2n+1} < \left(\frac{2n+1}{\pi}\right)^2 + \left(\frac{2n+1}{2\pi}\right)^2 + \left(\frac{2n+1}{3\pi}\right)^2 + \dots + \left(\frac{2n+1}{n\pi}\right)^2$

$< \operatorname{cosec}^2\frac{\pi}{2n+1} + \operatorname{cosec}^2\frac{2\pi}{2n+1} + \dots + \operatorname{cosec}^2\frac{n\pi}{2n+1}$

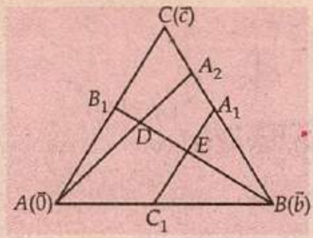
$$\Rightarrow \frac{n(2n-1)}{3} < \frac{(2n-1)^2}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}\right) < \frac{2n(n+1)}{3}$$

(since $\cot^2\frac{\pi}{2n+1}, \cot^2\frac{2\pi}{2n+1}, \dots, \cot^2\frac{n\pi}{2n+1}$ are the roots of given equation)

$$\Rightarrow \left(1-\frac{1}{2n+1}\right)\left(1-\frac{2}{2n+1}\right)\frac{\pi^2}{6} < \left(1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}\right) < \left(1-\frac{1}{2n+1}\right)\left(1+\frac{1}{2n+1}\right)\frac{\pi^2}{6}$$

Taking limit $n \to \infty$ and applying Sandwitch theorem, we get the desired result.

20. Let position vector of A, B and C be $\vec{0}, \vec{b}$ and $\vec{c}$ respectively.



We have, $\overline{AC_1} = \dfrac{\vec{b}}{2}, \overline{AB_1} = \dfrac{\vec{c}}{2}, \overline{AA_1} = \dfrac{\vec{b}+\vec{c}}{2}, \overline{AA_2} = \dfrac{3\vec{c}+\vec{b}}{4}$

Equation of lines BB_1, AA_2, and C_1A_1 are respectively $\vec{r} = \vec{b} + \lambda_1\left(\dfrac{\vec{c}}{2} - \vec{b}\right)$, $\vec{r} = \lambda_2 \dfrac{3\vec{c}+\vec{b}}{4}$ and $\vec{r} = \dfrac{\vec{b}}{2} + \lambda_3\left(\dfrac{\vec{c}}{2}\right)$.

For point D, we have $\vec{b} + \lambda_1\left(\dfrac{\vec{c}}{2} - \vec{b}\right) = \lambda_2\left(\dfrac{3\vec{c}+\vec{b}}{4}\right)$

$$\Rightarrow \vec{b}\left(1 - \lambda_1 - \frac{\lambda_2}{4}\right) + \frac{\vec{c}}{4}(2\lambda_1 - 3\lambda_2) = \vec{0}$$

$$\Rightarrow \lambda_1 = \frac{6}{7}, \lambda_2 = \frac{4}{7} \Rightarrow \overline{AD} = \frac{3\vec{c}+\vec{b}}{7}$$

For point E we have $\vec{b} + \lambda_1\left(\dfrac{\vec{c}}{2} - \vec{b}\right) = \dfrac{\vec{b}}{2} + \dfrac{\lambda_3}{2}\vec{c}$

$$\Rightarrow \vec{b}\left(\frac{1}{2} - \lambda_1\right) + \frac{\vec{c}}{2}(\lambda_1 - \lambda_3) = \vec{0}$$

$$\Rightarrow \lambda_1 = \lambda_3 = \frac{1}{2} \Rightarrow \overline{AE} = \frac{2\vec{b}+\vec{c}}{4}$$

Now $\overline{EA_2} = \dfrac{3\vec{c}+\vec{b}-2\vec{b}-c}{4} = \dfrac{2\vec{c}-\vec{b}}{4}$,

$$\overline{DA_1} = \frac{\vec{b}+\vec{c}}{2} - \frac{3\vec{c}+\vec{b}}{7} = \frac{5\vec{b}+\vec{c}}{14}$$

Area of quadrilateral $EA_1A_2D = \dfrac{1}{2}\left|\overline{EA_2} \times \overline{DA_1}\right|$

$$\frac{1}{112}\left|(2\vec{c}-\vec{b}) \times (5\vec{b}+\vec{c})\right| = \frac{1}{112}\left|10\vec{c}\times\vec{b} - \vec{b}\times\vec{c}\right|$$

$$= \frac{11}{112}\left|\vec{c}\times\vec{b}\right| = \frac{11}{56}\cdot\frac{1}{2}\left|\vec{c}\times\vec{b}\right| = \frac{11}{56} \text{ area of } \Delta ABC.$$

Thus required ratio is $\dfrac{11}{56}$.

21. Given $f(x) = [x] + [2x] + \left[\dfrac{2}{3}x\right] + [3x] + [4x] + [5x]$

Since $[kx]$ changes its value at every integral multiple of $\dfrac{1}{k}$

$\Rightarrow$ $[x]$ will change at every integral multiple of 1

$\Rightarrow$ $[2x]$ will change at every integral multiple of $\dfrac{1}{2}$

$\Rightarrow$ $[3x]$ will change at every integral multiple of $\dfrac{1}{3}$

$\Rightarrow$ $[4x]$ will change at every integral multiple of $\dfrac{1}{4}$

$\Rightarrow$ $[5x]$ will change at every integral multiple of $\dfrac{1}{5}$

$\Rightarrow$ $\left[\dfrac{2}{3x}\right]$ will change at every integral multiple of $\dfrac{3}{2}$

They will change simultaneously at every multiple of LCM of $\left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dfrac{1}{5}, \dfrac{3}{2}\right\} = 3$

Number of total points at which $f(x)$ will change its value in the interval $[0, 3)$ will depend on the total number of different terms in the following series:

$[x] = 0, 1, 2,$

$$\left.\begin{array}{l} [2x] = 0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{5}{2} \\ [3x] = 0, \frac{1}{3}, \frac{2}{3}, \ldots, \frac{8}{3} \\ [4x] = 0, \frac{1}{4}, \frac{2}{4}, \ldots, \frac{11}{4} \\ [5x] = 0, \frac{1}{5}, \frac{2}{5}, \ldots, \frac{14}{5} \\ \left[\frac{2}{3}x\right] = 0, \frac{3}{2}, \frac{6}{2} \end{array}\right\} \quad \ldots\ldots (1)$$

$\therefore$ Total number of different terms in (1) = (9 + 12 + 15) – (2 + 2 + 2)

Hence number of terms in the range of $f(x)$ for $0 \le x < 3$ is 30.

If $x \in [0, n]$, then:

(i) If $n = 3k$, then number of terms in the range $= 30k + 1$

(ii) If $n = 3k + 1$, then number of terms in the range $= 30k + (3 + 4 + 5 + 1) - 2 = 30k + 11$

(iii) If $n = 3k + 2$, then number of terms in the range $= 30k + (6 + 8 + 10 + 1) - (2 + 2) = 30k + 21$.

90. (4): $\int_0^1 x^5(1-x)^6 dx = \lambda \Rightarrow \frac{1}{\lambda} = \frac{1}{7\cdot 8\cdot 9\cdot 11}$

$\Rightarrow \lambda = 7\cdot 2^3\cdot 3^2\cdot 11$

$\Rightarrow$ Number of prime factors = 4

91. (8): Let $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}\, dx = \frac{\pi}{8}(\alpha-\beta)^2$

$$\Rightarrow \int_{k/n}^{\frac{k+1}{n}} \sqrt{\left(x-\frac{k}{n}\right)\left(\frac{k+1}{n}-x\right)}\, dx = \frac{\pi}{8}\left(\frac{k}{n}-\frac{k+1}{n}\right)^2$$

Now, $\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{\pi}{8}\cdot\frac{1}{n^2} = \frac{\pi}{8}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} 1dx$

$$= \frac{\pi}{8}\lim_{n\to\infty}\frac{n}{n} = \frac{\pi}{8}$$

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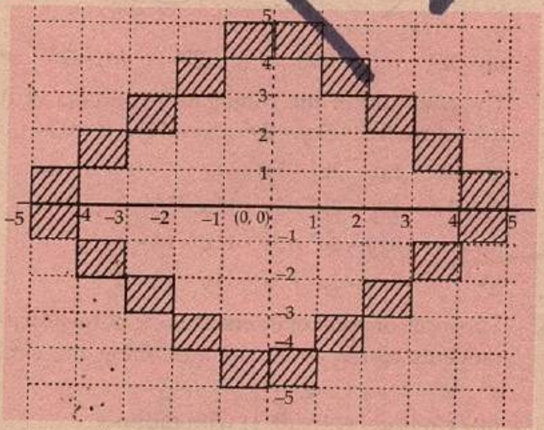
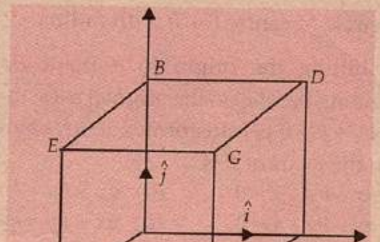
92. (1): Put $x = \tan A$, $y = \tan B$

93. (5)



94. (5): *OA, OB, OC* are along *x, y* and *z* axes respectively.



Let $\overrightarrow{OA} = \hat{i}, \overrightarrow{OB} = \hat{j}, \overrightarrow{OC} = \hat{k}$

OD, OE, OF are the diagonals of the cube.

$$\overrightarrow{OD} = u\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right); \overrightarrow{OE} = 2u\left(\frac{\hat{j}+\hat{k}}{\sqrt{2}}\right); \overrightarrow{OF} = 3u\left(\frac{\hat{i}+\hat{k}}{\sqrt{2}}\right)$$

95. (5): Let $h(x) = f(x) - g(x)$

$h'(x) = f'(x) - g'(x)$

$h''(x) = f''(x) - g''(x)$

$\Rightarrow h''(x) = 0 \Rightarrow h'(x) = a$ (constant)

$\Rightarrow f'(x) - g'(x) = a \Rightarrow f'(1) - g'(1) = a$

$\Rightarrow a = 2; h'(x) = 2 \Rightarrow h(x) = 2x + \lambda$

$\Rightarrow h(2) = f(2) - g(2) = 9 - 3 = 6$

$\Rightarrow 4 + \lambda = 6 \Rightarrow \lambda = 2 \Rightarrow h(x) = 2x + 2$

$\Rightarrow h\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right) + 2 = 5$

96. (8): We know that $\int_a^b |\sin x|\, dx$ represents area under the curve from $x = a$ to $x = b$

We know that $\int_a^{a+\pi} |\sin x|\, dx = 2$

Since $\int_a^b |\sin x|\, dx = 8 \Rightarrow b - a = 4\pi$ ... (i)

Similarly, $\int_0^{a+b} |\cos x|\, dx = 9 \Rightarrow a + b = \frac{9\pi}{2}$ ... (ii)

Solving (i) and (ii), $a = \frac{\pi}{4}$, $b = \frac{17\pi}{4}$

$$\int_a^b x\sin x dx = \int_{\pi/4}^{17\pi/4} x\sin x dx = -2\sqrt{2}\pi$$

$[-k] = [2\sqrt{2}\pi] = 8$

MOCK TEST PAPER 2013

ISI

Indian Statistical Institute

Exam on 12th May

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SECTION-I

Multiple Choice Type Questions

Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct.

1. The chord of contact of tangents from any point of circle $x^2+y^2=a^2$ with respect to the circle $x^2+y^2=b^2$ touches the circle $x^2+y^2=c^2$ where $(a, b, c>0)$ then

(a) $b<\frac{a+c}{2}$

(b) $\frac{1}{1+\log a}, \frac{1}{1+\log b}, \frac{1}{1+\log c}$ are in A. P.

(c) a, b, c are in A.P. (d) $b>\sqrt{ac}$

2. The locus of mid-point of the chord of the circle with diameter as minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$ which subtends right angle at centre of ellipse is

(a) $x^2+y^2=2b^2$ (b) $x^2+y^2=\frac{a^2+b^2}{2}$

(c) $2(x^2+y^2)=b^2$ (d) $x^2+y^2=\frac{b^2}{4}$

3. Let α, β, γ are the roots of the equation $x^3+3x^2-6x-8=0$. If $\left(\frac{1}{\alpha}, \alpha\right), \left(\frac{1}{\beta}, \beta\right)$ and $\left(\frac{1}{\gamma}, \gamma\right)$ are the vertices of the triangle, then

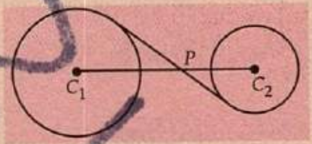
(a) centroid of the triangle is $\left(-\frac{3}{4}, -1\right)$

(b) orthocentre of the triangle is $\left(-\frac{1}{8}, -8\right)$

(c) circumcentre of the triangle is $\left(\frac{29}{8}, -\frac{23}{16}\right)$

(d) centroid of the triangle is $\left(\frac{1}{4}, -1\right)$

4. In the figure, two circles with centres C_1 and C_2 are 35 units apart, *i.e.* $C_1C_2=35$. The radii of the circles with centres C_1 and C_2 are 12 and 9 respectively. If P is the intersection of C_1C_2 and a common transverse tangent to the circles, then the area of the square whose side is equal to C_1P is



(a) 324 (b) 400 (c) 20 (d) 225

5. The equation of the conjugate hyperbola of the hyperbola $3x^2-5xy-2y^2+5x+11y-8=0$ is

(a) $3x^2-5xy-2y^2+5x+11y+8=0$

(b) $3x^2-5xy-2y^2+5x+11y+16=0$

(c) $3x^2-5xy-2y^2+5x+11y-16=0$

(d) $3x^2-5xy-2y^2+5x+11y-12=0$

6. The locus of point of intersection of perpendicular tangents to the circle circumscribing the circles $x^2+y^2-2|y|=0$ is

(a) circle of centre (0, 1) with radius = $\sqrt{2}$

(b) circle of centre (1, 0) with radius = 2

(c) circle of centre (0, 0) with radius = $2\sqrt{2}$

(d) circle of centre (0, 0) with radius = 2

7. On shifting the origin to a point P, the axes remaining parallel to the original axes, the equation $ax+by+c=0$ is transformed to $ax+by+c+k=0$. Then the locus of P is

(a) $ax-by+c=0$ (b) $ax+by+k=0$

(c) $ax-by-k=0$ (d) $ax+by-k=0$

8. The points (α, β), (γ, δ), (α, δ) and (γ, β) are always

(a) collinear (b) concyclic

(c) square (d) vertices

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9. If the normals at P and Q on the parabola $y^2 = 4ax$ meet on it at R. Then the directrix of the locus of mid-point of PQ is
(a) $2x + 3a = 0$ (b) $2x + 5a = 0$
(c) $2x + a = 0$ (d) $2x - 5a = 0$

10. Which of the following is wrong?
(a) The portion of a tangent to a parabola intercepted between the directrix and the curve subtend a right angle at the focus.
(b) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.
(c) The area of the triangle formed by three points on a parabola is thrice the area of the triangle formed by the tangents at these points.
(d) The foot of perpendicular from the focus on any tangent to a parabola lies on the tangent at the vertex.

11. Angle subtended by common tangents of two ellipses $9(x-2)^2 + 4y^2 = 36$ and $9(x+1)^2 + y^2 = 9$ at the origin is
(a) $\frac{\pi}{2}$ (b) $\tan^{-1}\left(\frac{7}{9}\right)$
(c) $\tan^{-1}\left(\frac{9}{7}\right)$ (d) $\frac{\pi}{4}$

12. C_1: $x^2 + y^2 = r^2$ and C_2: $\frac{x^2}{16} + \frac{y^2}{9} = 1$ intersect at four distinct points A, B, C and D. Their common tangents form a square $A'B'C'D'$. Then the ratio of area of the circle C_1 to the area of circumcircle of $\Delta A'B'C'$ is
(a) $\frac{9}{16}$ (b) $\frac{3}{4}$ (c) 1 (d) $\frac{1}{2}$

13. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is such that it has the least area but contains the circle $(x-1)^2 + y^2 = 1$ Then the eccentricity of the ellipse is
(a) $\sqrt{\frac{2}{3}}$ (b) $\sqrt{\frac{1}{3}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

14. The ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $a^2x^2 - y^2 = 4$ intersect at right angles then the equation of the circle through the point of intersection of two conic is
(a) $x^2 + y^2 = 5$
(b) $\sqrt{5}(x^2 + y^2) = 3x + 4y$
(c) $\sqrt{5}(x^2 + y^2) - 3x + 4y = 0$
(d) $x^2 + y^2 = 25$

15. If S_1 and S_2 are the foci of the hyperbola whose transverse axis length is 4 and conjugate axis length is 6, S_3 and S_4 are the foci of the conjugate hyperbola, then the area of the quadrilateral $S_1S_2S_3S_4$ is
(a) 24 (b) 26 (c) 22 (d) 28

16. Let I be the incentre of the triangle ABC, where $\frac{\overrightarrow{BC}}{|\overrightarrow{BC}|} + \frac{\overrightarrow{BA}}{|\overrightarrow{BA}|} = \frac{\overrightarrow{BI}}{k}$ then the diameter of the circumcircle of the triangle is
(a) $k(\cos A/2 + \cos C/2)$ (b) $k(\sin A/2 + \sin C/2)$
(c) $k(\cot A/2 + \cot C/2)$ (d) $k(\tan A/2 + \tan C/2)$

17. The reciprocal of the distance between two points, one on each of the lines $\frac{x-2}{3} = \frac{y-4}{2} = \frac{z-5}{5}$ and $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$
(a) cannot be less than 9
(b) having minimum value $5\sqrt{3}$
(c) cannot be greater than 78
(d) cannot be $2\sqrt{19}$

18. Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is
(a) $x + 2y - 2z = 0$ (b) $3x + 2y - 2z = 0$
(c) $x - 2y + z = 0$ (d) $5x + 2y - 4z = 0$

19. The distance between the plane $x - 2y + z - 6 = 0$ and the plane containing the sets of points $(1 + 2\lambda, 2 + 3\lambda, 3 + 4\lambda)$ and $(2 + 3\mu, 3 + 4\mu, 4 + 5\mu)$, where λ, μ are parameters, is
(a) $\sqrt{3/2}$ (b) $\sqrt{6}$ (c) $\sqrt{12}$ (d) $2\sqrt{6}$

20. Three numbers a, b, c are chosen randomly from the set of natural numbers. The probability that $a^2 + b^2 + c^2$ is divisible by 7 is
(a) 1/3 (b) 1/4 (c) 1/5 (d) 1/7

21. If the equation $z^2 + z + \alpha = 0$ has a purely imaginary root and α lies on the circle $|z| = 1$ then the imaginary part of that root, is (are)
(a) $\pm\sqrt{2}$ (b) 0
(c) $\pm\sqrt{2-\sqrt{2}}$ (d) $\pm\sqrt{\frac{\sqrt{5}-1}{2}}$

22. If $|z| = 1$ and $z \neq \pm 1$ then one of the possible values of $\arg(z) - \arg(z+1) - \arg(z-1)$, is
(a) $-\pi/6$ (b) $\pi/3$ (c) $-\pi/2$ (d) $\pi/4$

23. Let X be a set containing n elements. The number of all the ordered triplets (A, B, C) such that C is a subset of B and B is a proper subset of A where $A \subseteq X$, is
(a) 4^n (b) 3^n (c) $4^n - 3^n$ (d) $3^n - 2^n$

24. If two events A and B are such that $P(\bar{A}) = 0.3$, $P(B) = 0.4$, $P(A \cap \bar{B}) = 0.5$, then the value of $P(B / (A \cup \bar{B}))$ is
(a) 1/2 (b) 1/4 (c) 3/4 (d) 4/5

25. If the third power of a natural number ends with a prime digit then the probability of its fourth power ending not with a prime digit, is
(a) 3/10 (b) 9/10 (c) 4/9 (d) 3/4

26. A fair coin is tossed 10 times and the outcomes are listed. Let H_i be the event that the i^{th} outcome is a head and A_m be the event that the list contains exactly m heads, then
(a) H_3 and A_4 are independent
(b) A_1 and A_9 are independent
(c) H_2 and A_5 are independent
(d) H_4 and H_8 are not independent

SECTION-II

Short Answer Type Questions

27. Given a triangle ABC. The perpendiculars erected to AB and BC at their mid-points intersect the line AC at points M and N such that $|MN| = |AC|$. The perpendiculars erected to AB and AC at their mid-points intersect BC at points K and L such that $|KL| = \frac{1}{2}|BC|$. Find the smallest angle of the triangle ABC.

28. The radii of the circles inscribed in and circumscribed about a triangle are equal to r and R, respectively. Find the area of the triangle if the circle passing through the centres of the inscribed and circumscribed circles and the intersection point of the altitudes of the triangle is known to pass atleast through one of the vertices of the triangle.

29. If $a^3 + b^3 + c^3 = (b + c)(a + c)(a + b)$ and $(b^2 + c^2 - a^2)x = (c^2 + a^2 - b^2)y = (a^2 + b^2 - c^2)z$, then $x^3 + y^3 + z^3 = (x + y)(x + z)(y + z)$.

30. Show that

$$\left(\sqrt[5]{\frac{32}{5}} - \sqrt[5]{\frac{27}{5}}\right)^{\frac{1}{3}} = \sqrt[5]{\frac{1}{25}} + \sqrt[5]{\frac{3}{26}} - \sqrt[5]{\frac{9}{25}};$$

31. Solve the system

$$\frac{x}{a+\lambda} + \frac{y}{b+\lambda} + \frac{z}{c+\lambda} = 1$$

$$\frac{x}{a+\mu} + \frac{y}{b+\mu} + \frac{z}{c+\mu} = 1$$

$$\frac{x}{a+\upsilon} + \frac{y}{b+\upsilon} + \frac{z}{c+\upsilon} = 1$$

32. Solve and analyze the equation
$\sin 3x + \sin 2x = m \sin x$.

33. Let f be a bijective (one-one and onto) function from the set $A = \{1, 2, 3, ..., n\}$ to itself. Show that there is a positive integer $M > 1$ such that $f^M(i) = f(i)$ for each $i \in A$.
[f^M denotes the composite function *fofo... of* repeated M times.]

34. Three congruent circles have a common point O and lie inside a triangle such that each circle touches a pair of sides of the triangle. Prove that the incentre and the circumcentre of the triangle and the point O are collinear.

35. A triangle ABC has incentre I. Its incircle touches the side BC at T. The line through T parallel to IA meets the incircle at S and the tangent to the incircle at S meets sides AB, AC in points C', B' respectively. Prove that triangle $AB'C'$ is similar to triangle ABC.

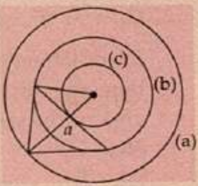
36. If a, b, x and y are integers greater than 1 such that a and b have no common factors except 1 and $x^a = y^b$, show that $x = n^b$ and $y = n^a$ for some integer n greater than 1.

SOLUTIONS

1. **(a):** $\frac{1}{2}a\sqrt{b^2 - c^2} = \frac{1}{2}b\sqrt{a^2 - b^2}$

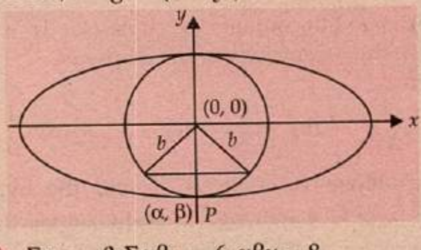
$\Rightarrow b = \sqrt{ac}$

$\because$ A.M. $\geq$ G.M.



2. **(c):** $\alpha^2 + \beta^2 = \frac{b^2}{2}$

Taking locus, we get $2(x^2 + y^2) = b^2$



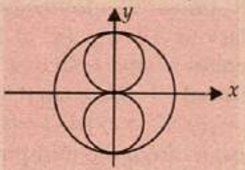
3. **(b):** $\Sigma\alpha = -3, \Sigma\alpha\beta = -6, \alpha\beta\gamma = 8$

Since $\left(\frac{1}{\alpha}, \alpha\right), \left(\frac{1}{\beta}, \beta\right)$ and $\left(\frac{1}{\gamma}, \gamma\right)$ lie on the hyperbola $xy = 1$. Therefore orthocentre will be $\left(-\frac{1}{\alpha\beta\gamma}, -\alpha\beta\gamma\right)$.

4. **(b):** $C_1P = \frac{35 \times 12}{21} = 20$. Area $= (20)^2 = 400$

5. (c) : Equation of asymptotes of hyperbola is given by $3x^2 - 5xy - 2y^2 + 5x + 11y + k = 0$
This gives two lines if $k = -12$.
Hence the equation of conjugate hyperbola is given by 2(equation of asymptotes) – equation of hyperbola = 0

6. (c) : Equation of circle circumscribing the circles $x^2 + y^2 - 2|y| = 0$ is $x^2 + y^2 = 4$
Equation of director circle is $x^2 + y^2 = 8$.



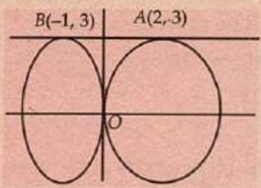
7. (d): $x = X + \alpha,\ y = Y + \beta$
$aX + bY + a\alpha + b\beta + c = 0$
$a\alpha + b\beta + c = c + k$
$a\alpha + b\beta - k = 0$
So, locus of (α, β) is $ax + by - k = 0$

8. (b): The given points are the vertices of a rectangle.

9. (b): Normal at $P(t_1)$ and $Q(t_2)$ meets the curve again at R. So, $t_1t_2 = 2$.
Hence the locus of mid-point of PQ is $y^2 = 2ax + 4a^2$.
Hence the directrix is $2x + 5a = 0$.

10. (c) : Standard properties of parabola.

11. (c) : $m_{OA} = \dfrac{3}{2};\ m_{OB} = -3$



Hence, $\tan\theta = \dfrac{9}{7}$, where $\theta = \angle AOB$

12. (d): If $A'B'C'D'$ is a square, then tangents $y = \pm x \pm 5$ for which diagonal length $A'C' = 10$.
Area of circumcircle of $\Delta A'B'C' = 25\pi$

Area of circle C_1 is $\dfrac{25\pi}{2}$

Hence the ratio is 1 : 2.

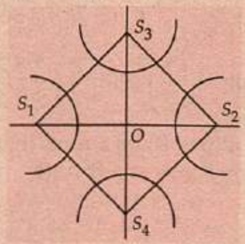
13. (a): Area is least if both the curves touch each other. Hence, solving both the equations together will give us
$$a = \frac{1}{e\sqrt{1-e^2}}$$
Area of ellipse $= \pi ab = \dfrac{\pi}{e^2\sqrt{1-e^2}} = \dfrac{\pi}{\sqrt{e^4 - e^6}}$

Area will be minimum when $e^4 - e^6$ is maximum hence
$$e^2 = \frac{2}{3} \Rightarrow e = \sqrt{\frac{2}{3}}$$

14. (a): Since ellipse and hyperbola intersect orthogonally, they are confocal. Hence $a = 2$.
Let $P(x_1, y_1)$ be the point of intersection of both the curves therefore, $\left.\begin{matrix} 4x_1^2 + 9y_1^2 = 36 \\ 4x_1^2 - y_1^2 = 4 \end{matrix}\right\} \Rightarrow x_1^2 + y_1^2 = 5$

Hence the equation of circle is $x^2 + y^2 = 5$.

15. (b): Area = 4 × Area ΔS_2OS_4
$$= 4 \times \frac{1}{2}ae \times be_1 = 2abee_1$$
$$= 12ee_1 = 12 \times \frac{\sqrt{13}}{2} \times \frac{\sqrt{13}}{3} = 26 \text{ sq. units}$$



16. (c) : Taking modulus both sides
$$2\cos\frac{B}{2} = \frac{1}{k}|\vec{BI}| = \frac{1}{k}\frac{r}{\sin(B/2)} = \frac{4R\sin(A/2)\sin(C/2)}{k}$$
$$\Rightarrow 2R = \frac{k\sin\left(\frac{A+C}{2}\right)}{\sin(A/2)\sin(C/2)} = k\left(\cot\frac{A}{2} + \cot\frac{C}{2}\right)$$

17. (c) : The shortest distance (SD)
$$= \frac{\begin{vmatrix} 2-1 & 4-2 & 5-3 \\ 2 & 3 & 4 \\ 3 & 2 & 5 \end{vmatrix}}{\left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 2 & 5 \end{vmatrix}\right|} = \frac{1}{\sqrt{78}}$$

18. (c) : Vector along the required plane is
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{vmatrix} = 8\hat{i} - \hat{j} - 10\hat{k}$$
So, normal vector $(\vec{n})$ to the plane is
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & -1 & -10 \\ 2 & 3 & 4 \end{vmatrix} = 26\hat{i} - 52\hat{j} + 26\hat{k}$$
So, equation of the plane is $\vec{r} \cdot \vec{n} = 0 \Rightarrow x - 2y + z = 0$.

19. (b): Normal vector : $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -\hat{i}+2\hat{j}-\hat{k}$

Equation of plane is $-1(x-1)+2(y-2)-1(z-3)=0$
$\Rightarrow\ x-2y+z=0$

So, required distance $= \dfrac{|6|}{\sqrt{1+4+1}} = \sqrt{6}$

20. (d): Numbers are of the form: $7k$, $7k+1$, $7k+2$, $7k+3$, $7k+4$, $7k+5$, $7k+6$

Their squares: $7k$, $7k+1$, $7k+4$, $7k+2$, $7k+2$, $7k+4$, $7k+1$.

So, for $a^2+b^2+c^2$ to be a multiple of 7, either all the three squares should be of the form $7k$ or they belong to the categories $7k+1$, $7k+2$, $7k+4$ separately.

So, required prob. $= \left(\dfrac{1}{7}\right)^3 + 3!\left(\dfrac{2}{7}\right)\left(\dfrac{2}{7}\right)\left(\dfrac{2}{7}\right) = \dfrac{1}{7}$

21. (d): Let $z = i\beta(\beta \in R)$ be a root, then
$-\beta^2+i\beta+\alpha=0\ \Rightarrow\ \alpha=\beta^2-i\beta$

Now as $|\alpha|=1\ \Rightarrow \beta^4+\beta^2=1\ \Rightarrow\ \beta^2 = \dfrac{-1+\sqrt{5}}{2}$

22. (c) : $\arg(z)-\arg(z+1)-\arg(z-1)$

$$= \arg\left(\frac{z}{z^2-1}\right) = \arg\left(\frac{z}{z^2-z\bar{z}}\right)$$

$$= \arg\left(\frac{1}{z-\bar{z}}\right) = \arg\text{ (purely imaginary no.)}$$

23. (c) : $C \subseteq B \subset A \subseteq X$

The number of ways $= \displaystyle\sum_{i=0}^{n} {}^nC_i\left[\sum_{j=0}^{i-1} {}^iC_j\left[\sum_{k=0}^{j} {}^jC_k\right]\right]$

$$= \sum_{i=0}^{n} {}^nC_i \sum_{j=0}^{i-1} {}^iC_j 2^j = \sum_{i=0}^{n} {}^nC_i(3^i-2^i) = 4^n-3^n$$

24. (b): $P(B/(A(\cup\bar{B})) = \dfrac{P(B\cap(A\cup\bar{B}))}{P(A\cup\bar{B})}$

$$= \frac{P(A)-P(A\cap\bar{B})}{P(A)+P(\bar{B})-P(A\cap\bar{B})} = \frac{0.7-0.5}{0.7+0.6-0.5} = \frac{0.2}{0.8} = \frac{1}{4}$$

25. (d): Total cases are when numbers ending with 3, 5, 7, or 8

Favourable cases are when numbers are ending with 3, 7, or 8

So, the required probability = 3/4

26. (c) : $P(H_i) = \dfrac{1}{2}$, $P(A_m) = \dfrac{{}^{10}C_m}{2^{10}}$

$P(H_i \cap A_m) = \dfrac{{}^9C_{m-1}}{2^{10}}$

For H_i and A_m to be independent

$$\frac{{}^9C_{m-1}}{2^{10}} = \frac{1}{2}\times\frac{{}^{10}C_m}{2^{10}} \Rightarrow 1 = \frac{1}{2}\times\frac{10}{m}\ \Rightarrow\ m=5$$

27. Let us first prove the following statement. If the perpendiculars to AB and BC at their midpoints intersect AC at points M and N so that $|MN| = \lambda|AC|$, then either $\tan A\tan C = 1-2\lambda$ or $\tan A\tan C = 1+2\lambda$. Let us denote: $|AB| = c$, $|BC| = a$, $|AC| = b$. If the segments of the perpendiculars from the mid-points of the sides to the points M and N do not intersect, then

$$|MN| = b - \frac{c}{2\cos A} - \frac{a}{2\cos C} = \lambda b \Rightarrow 2(1-\lambda)$$

$$\sin B\cos A\cos C = \frac{1}{2}(\sin 2C + \sin 2A)$$

$$\Rightarrow\ 2(1-\lambda)\sin(A+C)\cos A\cos C$$
$$= \sin(A+C)\cos(A-C)$$
$$\Rightarrow\ 2(1-\lambda)\cos A\times\cos C = \cos A\cos C + \sin A\sin C$$
$$\Rightarrow\ \tan A\times\cos C = 1-2\lambda.$$

And if these segments intersect, then

$\tan A\tan C = 1+2\lambda$.

In our case $\lambda = 1$, that is, either $\tan A\times\tan C = -1$ or $\tan A\tan C = 3$. For the angles B and C we get ($\lambda = 1/2$) either $\tan B\tan C = 0$ (this is impossible) or $\tan B\tan C = 2$. The system

$$\begin{cases} \tan A\tan C = -1, \\ \tan B\tan C = 2, \\ A+B+C = \pi \end{cases}$$

has no solution. Hence, $\tan A\tan C = 3$. Solving the corresponding system, we find $\tan A = 3$, $\tan B = 2$, $\tan C = 1$.

Answer: $\pi/4$.

28. Let ABC denote the given triangle, O, K, H the centres of the circumscribed and inscribed circles, and the intersection point of the altitudes of the triangle ABC, respectively. Let us take advantage of the following fact: in an arbitrary triangle the bisector of any of its angles makes equal angles both with the radius of the circumscribed circle and with the altitude emanating from the same vertex. Since the circle passing through O, K and H contains atleast one vertex of the triangle ABC (say, the vertex A), it follows that $|OK| = |KH|$. The point K is situtated inside atleast one of the triangles OBH and OCH. Let it be the triangle OBH. The angle B cannot be obtuse. In the triangles OBK and HBK, we have: $|OK| = |HK|$, KB is a common side, $\angle OBK = \angle HBK$. Hence, $\Delta OBK = \Delta HBK$, since otherwise $\angle BOK + \angle BHK = 180°$ which is impossible (K is inside the triangle

OBH). Consequently, $|BH| = |BO| = R$. The distance from O to AC equals $0.5\ |BH| = 0.5R$, that is, $\angle B = 60°$ ($\angle B$ is acute), $|AC| = R\sqrt{3}$. If now A_1, B_1 and C_1 are the points of tangency of the sides BC, CA and AB to the inscribed circle, respectively, then

$$|BA_1| = |BC_1| = r\sqrt{3},$$

$$|CA_1| + |AC_1| = |CB_1| + |B_1A| = |AC| = R\sqrt{3}$$

The perimeter of the triangle is equal to $2\sqrt{3}(R + r)$. It is now easy to find its area.

Answer : $\sqrt{3}(R + r)r$

29. From the equalities

$$(b^2 + c^2 - a^2)x = (c^2 + a^2 - b^2)y = (a^2 + b^2 - c^2)z$$

follows

$$\frac{x}{\left(\dfrac{1}{b^2 + c^2 - a^2}\right)} = \frac{y}{\left(\dfrac{1}{c^2 + a^2 - b^2}\right)} = \frac{z}{\left(\dfrac{1}{a^2 + b^2 - c^2}\right)}$$

Put for brevity

$b^2 + c^2 - a^2 = A$, $c^2 + a^2 - b^2 = B$, $a^2 + b^2 - c^2 = C$.

It is evident that our problem is equivalent to the following one: if the equation

$x^3 + y^3 + z^3 = (x + y)(x + z)(y + z)$ has the solution

$$x = a, y = b, z = c,$$

then it also has the following solution

$$x = \frac{1}{A}, y = \frac{1}{B}, z = \frac{1}{C}$$

We know the following identity

$(x + y + z)^3 - x^3 - y^3 - z^3 = 3(x + y)(x + z)(y + z)$.

Using this identity, we can easily prove that the equalities

$$x^3 + y^3 + z^3 = (x + y)(x + z)(y + z) \qquad \dots \text{(i)}$$

$$(x + y + z)^3 = 4(x^3 + y^3 + z^3) = 4(x + y)(x + z)(y + z) \qquad \dots \text{(ii)}$$

$$(x + y - z)(x + z - y)(y + z - x) = -4xyz \qquad \dots \text{(iii)}$$

are equivalent, and the existence of any of them involves the existence of the remaining ones. Thus, it is sufficient to prove that

$$\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right)^2 = 4\left(\frac{1}{A} + \frac{1}{B}\right)\left(\frac{1}{A} + \frac{1}{C}\right)\left(\frac{1}{B} + \frac{1}{C}\right),$$

i.e., $(AB + AC + BC)^3 = 4(A + B)(A + C)(B + C) \cdot ABC$

But $A + B = 2c^2$, $A + C = 2b^2$, $B + C = 2a^2$.

Therefore we have to prove

$$(AB + AC + BC)^3 = 32a^2b^2c^2 \cdot ABC.$$

Let us first compute $AB + AC + BC$, and then ABC.

We have $AB + AC + BC = A(B + C) + BC$

$= (b^2 + c^2 - a^2) \cdot 2a^2 + [a^2 + (b^2 - c^2)] \times [a^2 - (b^2 - c^2)]$

$= 2a^2b^2 + 2a^2c^2 - 2a^2 + a^4 - b^4 - c^4 + 2b^2c^2$

$= -a^4 - b^4 - c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2$

$= 4a^2b^2 - (a^2 + b^2 - c^2)^2$

$= (a - b + c)(-a + b + c)(a + b - c)(a + b + c)$

By virtue of equality (iii)

$$(a + c - b)(b + c - a)(a + b - c) = -4abc.$$

Therefore

$$AB + AC + BC = -4abc\ (a + b + c).$$

Compute ABC. Put $a^2 + b^2 + c^2 = s$, then

$ABC = (s - 2a^2)(s - 2b^2)(s - 2c^2)$

$= s^3 - 2(a^2 + b^2 + c^2)s^2 + 4(a^2b^2 + a^2c^2 + b^2c^2)s$

$- 8a^2b^2c^2 = 4(a^2b^2 + a^2c^2 + b^2c^2)s - s^3$

$- 8a^2b^2c^2 = s\{4a^2b^2 + 4a^2c^2 + 4b^2c^2 - (a^2 + b^2 + c^2)^2\}$

$- 8a^2b^2c^2 = -s\{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2\}$

$- 8a^2b^2c^2 = s(a + c - b)(b + c - a) \times (a + b - c)(a^2 + b^2 + c^2)$

$- 8a^2b^2c^2 = -4abc\{a^3 + b^3 + c^3 + a^2(b + c) + b^2(a + c) + c^2(a + b) + 2abc\}$

But, $(a + b)(a + c)(b + c) = a^2(b + c) + b^2(a + c) = c^2(a + b) + 2abc$.

Therefore, by virtue of equality (i), the bracketed expression is equal to $2(a^3 + b^3 + c^3)$.

But, by virtue of equality (ii),

$$2(a^3 + b^3 + c^3) = \frac{1}{2}(a + b + c)^3$$

Therefore $ABC = -2abc(a + b + c)^2$.

But, as has been deduced,

$$AB + AC + BC = -4abc(a + b + c).$$

Therefore, $(AB + AC + BC)^3 = 32a^2b^2c^2 \cdot ABC$

30. It is required to prove that

$$(1 \pm \sqrt[5]{3} - \sqrt[5]{9})^3 = 5(2 - \sqrt[5]{27})$$

Put $\sqrt[5]{3} = \alpha$ *i.e.* $\alpha^5 = 3$

We have $(1 + \alpha - \alpha^2)^2 = 1 + 3\alpha - 5\alpha^3 + 3\alpha^5 - \alpha^6$

But $\alpha^6 = 3\alpha$, $\alpha^5 = 3$

Therefore, $(1 + \alpha - \alpha^2)^3 = 10 - 5\alpha^3 = 5(2 - \sqrt[5]{27})$

31. Consider the following equality:

$$\frac{x}{a + \theta} + \frac{y}{b + \theta} + \frac{z}{c + \theta} - 1 = -\frac{(\theta - \lambda)(\theta - \mu)(\theta - \upsilon)}{(\theta + a)(\theta + b)(\theta + c)}.$$

Let us transform the equality, by reducing its terms to a common denominator and then rejecting the latter. We get a second-degree polynomial in θ with coefficients depending on $x, y, z, \lambda, \mu, \upsilon, a, b, c$, which is equal to zero. If now we substitute successively λ, μ and υ for θ into the original expression, then, by virtue of the given equations, this expression (and, consequently, the second-degree polynomial) vanishes. However, if a second-degree polynomial becomes zero at three different values of the variable, then it is identically equal to zero and, consequently, the equality

$$\frac{x}{a + \theta} + \frac{y}{b + \theta} + \frac{z}{c + \theta} - 1 = -\frac{(\theta - \lambda)(\theta - \mu)(\theta - \upsilon)}{(\theta - a)(\theta + b)(\theta + c)}$$

(by virtue of existence of the three given equations) is an identity with respect to θ, *i.e.*, if holds for any values of θ.

Multiplying both members of this equality by $a + \theta$, put $\theta = -a$. Then we find $x = \dfrac{(a + \lambda)(a + \mu)(a + \upsilon)}{(a - b)(a - c)}$

Likewise we get

$$y = \frac{(b+\lambda)(b+\mu)(b+\upsilon)}{(b-c)(b-a)}, z = \frac{(c+\lambda)(c+\mu)(c+\upsilon)}{(c-a)(c-b)}$$

Of course, we assume here that the given quantities λ, μ, υ as also a, b and c, are not equal to one another.

32. We have $\sin 2x \cos x + \cos 2x \sin x + \sin 2x - m \sin x = 0$. Hence, $\sin x\ [2\cos^2 x + \cos 2x + 2\cos x - m] = 0$, $\sin x\,[4\cos^2 x + 2\cos x - (m+1)] = 0$.
And so, one solution is $x = 0$.
The other is obtained by the formula

$$\cos x = \frac{-1 \pm \sqrt{4m+5}}{4}$$

Hence, first of all, it follows that there must be $4m + 5 \geq 0$.
Further, for one of the roots to exist it is required that

$$|-1+\sqrt{4m+5}| \leq 4,$$

i.e. $-4 \leq -1 + \sqrt{4m+5} \leq +4$ or $-3 \leq \sqrt{4m+5} \leq 5$,

i.e., $m \leq 5$. For the other root to exist it is necessary that

$$|-1-\sqrt{4m+5}| \leq 4, -4 \leq -1-\sqrt{4m+5} \leq 4, m \leq 1$$

Thus if $m < -\frac{5}{4}$, then $\cos x$ has no real values;

At $m = -\frac{5}{4}$ it has one real value $\left(\cos x = -\frac{1}{4}\right)$;

for $-\frac{5}{4} < m \leq 1$; $\cos x$ has two real values

$$\left(\cos x = \frac{-1 \pm \sqrt{4m+5}}{4}\right)$$

and for $1 < m \leq 5$ $\cos x$ again has one real value

$$\left(\cos x = \frac{-1+\sqrt{4m+5}}{4}\right)$$

and at $m > 5$ it has no real values.

33. **1st solution**

Note the following simple properties of bijective functions:

i) If $f : A \to A$ is a bijective function then there is a unique bijective function $g : A \to A$ such that $fog = gof = I_A$, then identity function on A. The function g is called the inverse of f and is denoted by f^{-1}.

Thus, $fof^{-1} = I_A = f^{-1}of$

ii) $foI_A = f = I_A of$.

iii) If f and g are bijections form A to A, then so are gof and fog.

iv) If f, g, h are bijective functions from A to A and $fog = foh$, then $g = h$.
Apply f^{-1} at left to both sides to obtain $g = h$.

Coming to the problem, since A has n elements, we see that there are only finitely many (in fact, $n!$) bijective functions from A to A as each bijective function f gives a permutation of $\{1, 2, 3, ..., n\}$ by taking $\{f(1), f(2), ..., f(n)\}$. Since f is a bijective function from A to A, so is each of the functions in the sequence:

$$fof = f^2, fofof = f^3, ..., f^n, ...$$

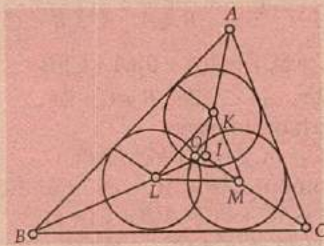
All these cannot be distinct. Since there are only finitely many bijective functions from A to A. Hence for some two distinct positive integers m and n, $m > n$ say, we must have $f^m = f^n$.
If $n = 1$, we take $M = m$, to obtain the result. If $n > 1$, multiply both sides by $(f^{-1})^{n-1}$, to get $f^{m-n+1} = f$. We take $M = m - n + 1$ to get the relation $f^M = f$, $(M > 1)$.
Note this means $f^M(i) = f(i)$ for all $i \in A$.

2nd solution :

Take any element r in the set A and consider the sequence of elements $r, f(r), (fof)(r), (fofof)(r), ...$ obtained by applying f successively. Since A has only n elements there must be repetitions in the above sequence. But when the first repetition occurs, this must be r itself; for, if the above sequence looks (for instance) like $r, a, b, c, d, e, c, ...$
where the first repetition is an element c other than r, this would imply $f(b) = c$ and $f(e) = c$,
contradicting the fact that f is a bijection. Thus for some positive integer $l_r \geq 1$, we have $f^{l_r}(r) = r$.
This is true for each r in the set $A = 1, 2, ..., n$. By taking M to be the l cm of $l_1, l_2, ..., l_r$ we get
$f^M(r) = r$ for each $r \in A$.

[**Note:** If f itself is the identity function the above proof fails because each $l_r = 1$. But in this case we may take M to be any integer greater than or equal to 2].

34. Let K, L, M be the centres of the three circles of equal radii, meeting in a common point O, and pairwise touching some side of the triangle ABC in which the lie. Since the three circles have equal radii, we see that O is equidistant from the points K, L, M and so is the circumcentre of the triangle KLM. Also for same reason, the sides of the triangle KLM are parallel to the corresponding sides of the triangle ABC. (For instance, KL is parallel to AB, as K and L are equidistant from AB.)



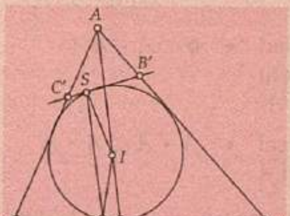
Further AK, BL, CM are the bisectors of angles A, B, C of triangle ABC. So not only they meet in I, the incentre of the triangle ABC but also KI, LI, MI are the bisectors of the angles of triangle KLM, implying that I is also the incentre of triangle KLM.

It follows, from the relation $\frac{IK}{IA}=\frac{IL}{IB}=\frac{IM}{IC}$, that triangle KLM is homothetic to triangle ABC with respect to I, the centre of homothety. (This simply means that triangle ABC is a dilation (or an enlargement) of triangle KLM as seen from I).

A property of homothety is that the centre of homothety and any two corresponding points of the homothetic figures are collinear. Here, in particular, I and the circumcentres of triangles KLM and ABC have to be collinear. This is precisely what was to be proved.

35. Let AI meet BC in K. Join IS. We do some angle-chasing now. Since AK is parallel to ST, we have

$$\angle STB=\angle AKB=\angle KCA+\angle KAC=C+\frac{A}{2}$$



So, $\angle STI=90^\circ-\angle STB=90^\circ-\left(C+\frac{A}{2}\right)$.

But $\angle TSI = \angle STI$ since SIT is an isosceles triangle.

Therefore $\angle C'ST=90^\circ-\angle TSI=C+\frac{A}{2}$.

In the quadrilateral $BTSC'$.

$$\angle SC'B = 360^\circ-(\angle C'BT+\angle BTS+\angle TSC')$$
$$= 360^\circ-\left(B+C+\frac{A}{2}+C+\frac{A}{2}\right)$$
$$= 360^\circ-(A+B+C+C)=180^\circ-C$$

Hence $\angle AC'B'=180^\circ-\angle SC'B=180^\circ-(180^\circ-C)=C$. Similarly, $\angle AB'C' = B$. Thus it follows that triangles ABC and $AB'C'$ are similar.

36. First note that the set of primes dividing x is the same as the set of primes dividing y. Take any prime p dividing x (and hence y also) and suppose it occur to the power α in x and β in y (that is, p^α is the maximum power of p dividing x and p^β is the maximum power of p dividing y). Then $x^a=y^b \Rightarrow p^{\alpha a}=p^{\beta b} \Rightarrow \alpha a=\beta b$

$\Rightarrow \alpha \mid \beta b$ and $b \mid \alpha a$

$\Rightarrow a \mid \beta$ and $b \mid \alpha$ since $(a, b) = 1$

Write $\beta=\alpha\beta_p$ and $\alpha=b\alpha_p$. Then

$$p^{\alpha a}=p^{\beta b} \Rightarrow p^{ab\alpha_p}=p^{ab\beta_p}=\alpha_p=\beta_p.$$

For each prime p dividing x (and hence y) get the integer α_p. Verify that the integer $n=\prod_{p|n}p^{\alpha_p}$ (this notation means n is the product of the numbers p^{α_p} for each prime p dividing n) satisfies the required properties.

SERIES-1

CBSE Board 2014

CHAPTERWISE PRACTICE PAPER — Relations & Functions

RELATIONS

Definition : Let A and B be two non-empty sets. Then a relation R from A to B is a subset of $A \times B$.

TYPES OF RELATIONS

S. No.	Types	Definitions	Examples/Description
1.	Void or Empty Relation	ϕ, a relation $A \times A$ on set A, is called void or empty relation if no element of A is related to any element of A.	• $\phi \subset A \times A$
2.	Universal Relation	A relation $A \times A$ is called universal relation on set A if each element of A is related to every element of A.	• $A \times A \subseteq A \times A$
3.	Identity Relation	Relation $I_A = \{(a, a); a \in A\}$ is called identity relation on set A.	• Let $A = \{1, 2, 3\}$, then $I_A = \{(1, 1), (2, 2), (3, 3)\}$
4.	Inverse Relation	Let $R = \{(a, b); a, b \in A\}$ is a relation on A. Then R^{-1} on A is $R^{-1} = \{(b, a) : a, b \in A\}$	• Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 2), (3, 1), (3, 2)\}$, then $R^{-1} = \{(2, 1), (2, 2), (1, 3), (2, 3)\}$
5.	Reflexive Relation	A relation R on a set A is called reflexive relation if $(a, a) \in R\ \forall\ a \in A$	• Let $A = \{1, 2, 3\}$, If $R = \{(1, 1), (2, 2), (3, 3)\}$ Then R is reflexive.
6.	Symmetric Relation	R is symmetric relation on a set A if $(x, y) \in R \Rightarrow (y, x) \in R\ \forall\ x, y \in A$	• Let $A = \{1, 2\}$ and $R = \{(1, 2), (2, 1)\}$ If $(1, 2) \in R \Rightarrow (2, 1) \in R$ $\Rightarrow$ R is symmetric relation.
7.	Transitive Relation	R is transitive relation on a set A if $(x, y) \in R$, $(y, z) \in R \Rightarrow (x, z) \in R$ $\forall\ x, y, z \in A$	• Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 3), (1, 3)\}$ If $(1, 2) \in R$, $(2, 3) \in R \Rightarrow (1, 3) \in R$ $\Rightarrow$ R is transitive relation.
8.	Antisymmetric Relation	Let A be any set. A relation R on set A is said to be antisymmetric iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ $\forall a, b \in A$	• Identity relation on set A.

EQUIVALENCE RELATION

- If R is reflexive, symmetric and transitive relation, then it is called equivalence relation.

RESULTS ON RELATION

(i) The intersection of two equivalence relations on a set is also an equivalence relation on the set.

(ii) The inverse of an equivalence relation is also an equivalence relation.

DOMAIN AND RANGE OF A RELATION

- If R be a relation from X to Y, then the set of first entries in distinct ordered pairs in R is called domain of R and set of all second entries in

15

Challenging Problems For Entrance Exams

SECTION-I

Single Option Correct

1. $\sum_{r=1}^{63}\{(-1)^{r+1}r^2\} =$

(a) 2013 (b) 2014

(c) 2016 (d) none of these

2. $\sum_{r=1}^{n}\int_{0}^{(2r-1)\pi} f(\sin^4 x)dx =$

(a) $\int_0^{\pi} f(\sin^4 x)dx$ (b) $n\int_0^{\pi} f(\sin^4 x)dx$

(c) $2n^2\int_0^{\pi/2} f(\sin^4 x)dx$ (d) none of these

3. If roots of $x^2 - ax + b = 0$ differ by 2 then locus of (a, b) has which of the following points as vertex?

(a) $(0, -1)$ (b) $(1, 0)$

(c) $(-1, 0)$ (d) $(0, 1)$

4. If a, b, c are in H.P. in ΔABC, then $\dfrac{(\sum ab - 2bc)(\sum ab - 2ca)}{(abc)^2}$ is

(a) $\dfrac{\sin A}{\Delta} + \dfrac{1}{b^2}$ (b) $\dfrac{\sin A}{\Delta} - \dfrac{1}{b^2}$

(c) $\dfrac{\sin A}{2\Delta} - \dfrac{1}{b^2}$ (d) none of these

5. A is the vertex and LL' be length of latus rectum of $y^2 = 4ax$. AB is a chord and $BC \perp AB$ (C is on axis of parabola). Projection of BC on x-axis be α, then α is

(a) $< LL'$ (b) $> LL'$

(c) $= LL'$ (d) none of these

6. If $|z| \le 45 - 2\sqrt{3}$, then $|z^2\cos^2\alpha + \sqrt{48}\cos\alpha \cdot z|$ is

(a) ≤ 2013 (b) ≥ 2013

(c) ≤ 2014 (d) none of these

7. For what values of a, the major axis of $\dfrac{x^2}{\log_{1/3}a^2} + \dfrac{y^2}{\log_a 9 - 5} = 1$ is x-axis?

(a) $(-\infty, 1) \cup (\sqrt{3}, 9)$ (b) $(-\infty, 0) \cup \left(\dfrac{1}{2}, 2\right)$

(c) $(0, 1) \cup (\sqrt{3}, 9)$ (d) $(-\infty, 1) \cup \left(\dfrac{1}{2}, 2\right)$

8. The number of 2 digit numbers in which unit's place is greater than ten's place is

(a) $\sum_{r=1}^{8} r$ (b) $\sum_{r=1}^{5} r^2$

(c) $\sum_{r=1}^{9} r$ (d) none of these

9. In ΔABC, $\dfrac{2bc\cot A(\sin B\cos C + \sin C\cos B)}{b^2 + c^2 - (b\cos C + c\cos B)^2}$ is independent of

(a) a only (b) c only

(c) b and c only (d) all a, b and c

SECTION-II

More Than One Option Correct

10. If $f(x) = \dfrac{\{2x + (1 + x^2)3x^4\}}{1 + 2x^2 + x^4} \cdot e^{x^3}$, $g(x) = \dfrac{x^2 e^{x^3}}{x^2 + 1}$ and $\int f(x)dx = g(x) + \phi(x)$, then $\phi(x)$ may be

(a) zero (b) constant function

(c) even function (d) odd function

11. If $81 + 144a^4 + 16b^4 + 9c^4 = 144abc$ in ΔABC, then

(a) $a > b > c$

(b) $A < B < C$

(c) area $= \dfrac{3\sqrt{3}}{8}$

(d) triangle is right angled

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12. If solution set of $3[\log_2 x - 2] - 2\{\log_2 x\} = \log_2 x$ be $[a, b)$, then primes between a and b are (where $[\cdot]$ and $\{\cdot\}$ are greatest integer function and Fractional part function respectively)

(a) 73 (b) 53 (c) 43 (d) 23

13. If $g(x) = \int_{\frac{5\pi}{4}}^{x} (4\cos t + 3\sin t)dt$ has maximum and minimum as M and m respectively, then in $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$

(a) $m = 0,\ M = \frac{3}{2} - 2\sqrt{3} + \frac{1}{\sqrt{2}}$

(b) $m = 0,\ M = \frac{3}{2} - 2\sqrt{3} + \sqrt{2}$

(c) $m = \frac{3}{2} - 2\sqrt{3} + \frac{1}{\sqrt{2}},\ M = 0$

(d) $m < 0,\ M = 0$

14. Which of the following are solutions of $(\log_2 x)^2 - 6[\log_2 x] = 7$ (where $[\cdot]$ denotes greatest integer function)

(a) 2 (b) 128 (c) $2^{\sqrt{37}}$ (d) $2^{\sqrt{43}}$

15. $2\sin^2(\theta + \alpha) + 2\sin^2(\theta + \beta) + \cos 2(\alpha - \beta) - 4\sin(\theta + \alpha)\sin(\theta + \beta)\cos(\alpha - \beta)$ is independent of

(a) θ (b) α

(c) β (d) none of these

SOLUTIONS

1. **(c)**: $\sum_{r=1}^{63}\{(-1)^{r+1}r^2\} = 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + ... + 63^2$

$= 1^2 - [(2^2 - 3^2) + (4^2 - 5^2) + (6^2 - 7^2) + ...$ to 31 brackets$]$

$= 1 + [5 + 9 + 13 + ...$ to 31 terms$]$

$= 1 + 5 + 9 + ...$ to 32 terms $= \frac{32}{2}\{2 \times 1 + (32 - 1)4\}$

$= 32 \times 63 = 2016$

2. **(c)**: $\sum_{r=1}^{n} \int_0^{(2r-1)\pi} f(\sin^4 x)dx = \int_0^{\pi} f(\sin^4 x)dx + \int_0^{3\pi} f(\sin^4 x)dx + \int_0^{5\pi} f(\sin^4 x)dx + ...$ to n terms

$= \int_0^{\pi} f(\sin^4 x)dx + 3\int_0^{\pi} f(\sin^4 x)dx + 5\int_0^{\pi} f(\sin^4 x)dx + ...$ .... to n terms

$[\because f(\sin^4 x)$ is a periodic function with period $\pi]$

$= [1 + 3 + 5 + ...$ to n terms$]\int_0^{\pi} f(\sin^4 x)dx$

$= n^2 \int_0^{\pi} f(\sin^4 x)dx = n^2 \int_0^{2\cdot\pi/2} f(\sin^4 x)dx$

$= 2n^2 \int_0^{\pi/2} f(\sin^4 x)dx$

[Using the property ; $g(2a - x) = g(x)$

$\Rightarrow \int_0^{2a} g(x)dx = 2\int_0^{a} g(x)dx$]

3. **(a)**: Let roots be $\alpha, \alpha + 2 \Rightarrow \alpha + \alpha + 2 = a$

$\Rightarrow \alpha = \frac{a-2}{2}$

And, $\alpha(\alpha + 2) = b \Rightarrow \frac{a-2}{2}\left(\frac{a-2}{2} + 2\right) = b$

$\Rightarrow a^2 - 4 = 4b \Rightarrow a^2 = 4(b + 1)$

$\Rightarrow$ locus of (a, b) is $x^2 = 4(y + 1)$ which is a parabola having vertex at $(0, -1)$.

4. **(b)**: $\frac{(\sum ab - 2bc)(\sum ab - 2ca)}{(abc)^2}$

$= \frac{(ab - bc + ca)}{abc} \times \frac{(ab + bc - ca)}{abc}$

$= \left(\frac{1}{c} - \frac{1}{a} + \frac{1}{b}\right)\left(\frac{1}{c} + \frac{1}{a} - \frac{1}{b}\right)$

$= \left\{\frac{1}{c} - \left(\frac{2}{b} - \frac{1}{c}\right) + \frac{1}{b}\right\}\left\{\frac{1}{c} + \left(\frac{2}{b} - \frac{1}{c}\right) - \frac{1}{b}\right\}$

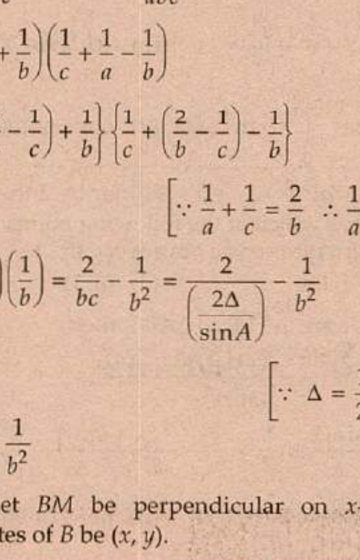
$\left[\because \frac{1}{a} + \frac{1}{c} = \frac{2}{b} \therefore \frac{1}{a} = \frac{2}{b} - \frac{1}{c}\right]$

$= \left(\frac{2}{c} - \frac{1}{b}\right)\left(\frac{1}{b}\right) = \frac{2}{bc} - \frac{1}{b^2} = \frac{2}{\left(\frac{2\Delta}{\sin A}\right)} - \frac{1}{b^2}$

$\left[\because \Delta = \frac{1}{2}bc\sin A\right]$

$= \frac{\sin A}{\Delta} - \frac{1}{b^2}$

5. **(c)**: Let BM be perpendicular on x-axis and co-ordinates of B be (x, y).



$\therefore \quad BM = y,\ AM = x$

Let $\angle BAM = \theta. \quad \therefore \quad \angle BCM = \frac{\pi}{2} - \theta$

In $\Delta ABM,\ \tan\theta = \frac{y}{x}$ ... (i)

In $\Delta BCM,\ \tan\left(\frac{\pi}{2} - \theta\right) = \frac{y}{MC} = \frac{y}{\alpha} \Rightarrow \cot\theta = \frac{y}{\alpha}$

$\Rightarrow \quad \alpha = y\tan\theta = y \cdot \frac{y}{x} \quad$ [from(i)]

$= \frac{4ax}{x} \quad [\because y^2 = 4ax] = 4a = LL'$

6. (a): $\because \ |z| \le 45 - 2\sqrt{3}$ ... (i)

Now,

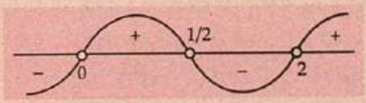
$|z^2\cos^2\alpha + \sqrt{48}\cos\alpha \cdot z| \le |z^2\cos^2\alpha| + |\sqrt{48}\cos\alpha \cdot z|$

$\le |z^2| + |\sqrt{48}z| \le |z|^2 + \sqrt{48}|z| \le |z|(|z| + \sqrt{48})$

$\le (45 - 2\sqrt{3})(45 - 2\sqrt{3} + 4\sqrt{3})$

$\le 2025 - 12 \ \le 2013$

7. (c): For $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the major axis is x-axis if $a^2 > b^2$



$\therefore$ For given ellipse, we have $\log_{1/3} a^2 > \log_a 9 - 5$

$\Rightarrow \ -2\log_3 a > 2\log_a 3 - 5$

$\Rightarrow \ -2k > \frac{2}{k} - 5 \quad$ (where $k = \log_3 a$)

$\Rightarrow \ \frac{2}{k} + 2k - 5 < 0 \ \Rightarrow \ \frac{2k^2 - 5k + 2}{k} < 0$

$\Rightarrow \ \frac{(k-2)(2k-1)}{k} < 0 \Rightarrow \frac{(k-2)\left(k - \frac{1}{2}\right)}{k} < 0$

$\Rightarrow \ k < 0$ or $\frac{1}{2} < k < 2$

$\Rightarrow \ \log_3 a < 0$ or $\frac{1}{2} < \log_3 a < 2$

$\Rightarrow \ a < 3^0$ or $\sqrt{3} < a < 9 \Rightarrow a < 1$ or $\sqrt{3} < a < 9$

But, $a > 0$ for $\log_3 a$ or $\log_a 3$ to be defined

Hence, $a \in (0, 1) \cup (\sqrt{3}, 9)$

8. (a): Let x, y be the unit's and ten's place respectively then the number will be in yx form (where $x > y$).

x	y	**Total**
2	1	1
3	1, 2	2
4	1, 2, 3	3
5	1, 2, 3, 4	4
6	1, 2, 3, 4, 5	5
7	1, 2, 3, 4, 5, 6	6
8	1, 2, 3, 4, 5, 6, 7	7
9	1, 2, 3, 4, 5, 6, 7, 8	8

$\text{Total} = \sum_{r=1}^{8} r = 36$

9. (d): Given expression

$= \frac{2bc\cot A(\sin B\cos C + \sin C\cos B)}{b^2 + c^2 - a^2}$

$= \frac{(b^2 + c^2 - a^2)\sin(B + C)}{(b^2 + c^2 - a^2)\sin A} \quad \left[\because \cos A = \frac{b^2 + c^2 - a^2}{2bc}\right]$

$= \frac{\sin(\pi - A)}{\sin A} = 1$, which is independent of all.

10. (a, b, c) :

$$\because g'(x) = \frac{(x^2+1)(x^2 \cdot e^{x^3} \cdot 3x^2 + 2x \cdot e^{x^3}) - x^2 \cdot e^{x^3} \cdot 2x}{(x^2+1)^2}$$

$$= \frac{e^{x^3}\{(x^2+1)(3x^4+2x) - 2x^3\}}{(x^2+1)^2}$$

$$= \frac{(2x+3x^4+3x^6)e^{x^3}}{1+2x^2+x^4} = \frac{[2x+3x^4(1+x^2)]}{1+2x^2+x^4} \cdot e^{x^3} = f(x)$$

$\therefore \int f(x)dx = g(x) + \phi(x) \Rightarrow \int g'(x)dx = g(x) + \phi(x)$

$\Rightarrow g(x) + c = g(x) + \phi(x)$

[where c is constant of integration]

$\Rightarrow \phi(x) = c$

$\therefore$ $\phi(x)$ is a constant function, even function and may be zero.

11. (b, c, d): $\dfrac{81 + 144a^4 + 16b^4 + 9c^4}{4} = 36abc$

$$= \sqrt[4]{81 \cdot 144a^4 \cdot 16b^4 \cdot 9c^4}$$

$\Rightarrow 81 = 144a^4 = 16b^4 = 9c^4$, as A.M. = G.M. here

$\Rightarrow 2\sqrt{3}a = 2b = \sqrt{3}c = 3$ ... (i)

$\Rightarrow \dfrac{a}{1} = \dfrac{b}{\sqrt{3}} = \dfrac{c}{2} \Rightarrow \dfrac{a}{(1/2)} = \dfrac{b}{(\sqrt{3}/2)} = \dfrac{c}{1}$

$\Rightarrow \dfrac{a}{\sin 30^\circ} = \dfrac{b}{\sin 60^\circ} = \dfrac{c}{\sin 90^\circ}$

$\Rightarrow A = 30^\circ, B = 60^\circ, C = 90^\circ$

$\therefore$ Triangle is right angled and $\angle A < \angle B < \angle C$

From (i), $a = \dfrac{\sqrt{3}}{2}, b = \dfrac{3}{2}, c = \sqrt{3}$

$\therefore$ area $\Delta = \dfrac{1}{2} \times \text{base} \times \text{height} = \dfrac{1}{2} \cdot \dfrac{\sqrt{3}}{2} \cdot \dfrac{3}{2}$

$= \dfrac{3\sqrt{2}}{8}$ sq. units

12. (c, d) : $\because x = [x] + \{x\}$ ... (i)

$\{x + n\} = \{x\} + n$, if $n \in I$, set of integers ... (ii)

and $0 \le \{x\} < 1$ ... (iii)

$\therefore$ We can write

$3((\log_2 x - 2) - \{\log_2 x - 2\}) - 2\{\log_2 x\} = \log_2 x$ [using (i)]

$2\log_2 x - 6 - 3\{\log_2 x - 2\} - 2\{\log_2 x - 2\} = 0$ [using (ii)]

$\Rightarrow \{\log_2 x - 2\} = \dfrac{2\log_2 x - 6}{5} \Rightarrow 0 \le \dfrac{2\log_2 x - 6}{5} < 1$

[using (iii)]

$\Rightarrow 0 \le 2\log_2 x - 6 < 5 \Rightarrow 3 \le \log_2 x < \dfrac{11}{2}$

$\Rightarrow 8 \le x < 32\sqrt{2} \Rightarrow 8 \le x \le 45.25$ (approx.)

13. (c, d) : $g'(x) = 4\cos x + 3\sin x < 0$ in $\left[\dfrac{5\pi}{4}, \dfrac{4\pi}{3}\right]$

$\therefore$ $g(x)$ is a continually diminishing function in $\left[\dfrac{5\pi}{4}, \dfrac{4\pi}{3}\right]$

$\therefore$ Maximum value of $g(x)$ is at $x = \dfrac{5\pi}{4}$ and minimum value is at $x = \dfrac{4\pi}{3}$

$\therefore m = g\left(\dfrac{4\pi}{3}\right) = [4\sin t - 3\cos t]_{5\pi/4}^{4\pi/3}$

$= \left(4\sin\dfrac{4\pi}{3} - 3\cos\dfrac{4\pi}{3}\right) - \left(4\sin\dfrac{5\pi}{4} - 3\cos\dfrac{5\pi}{4}\right)$

$= \left(-2\sqrt{3} + \dfrac{3}{2}\right) - \left(-2\sqrt{2} + \dfrac{3}{\sqrt{2}}\right) = \dfrac{3}{2} - 2\sqrt{3} + \dfrac{1}{\sqrt{2}} < 0$

$M = g\left(\dfrac{5\pi}{4}\right) = \displaystyle\int_{5\pi/4}^{5\pi/4} (4\cos t + 3\sin t)dt = 0$

14. (b, d): Let $[\log_2 x] = k$ (integer)

$\therefore$ $k \le \log_2 x < k + 1 \Rightarrow k^2 \le (\log_2 x)^2 < (k+1)^2$ ... (i)

Given equation becomes $(\log_2 x)^2 = 6k + 7$, which is meaningful if $k \ge -1$ ($\because k \in I$) ... (ii)

From (i), $k^2 \le 6k + 7 < (k+1)^2$

$\Rightarrow k^2 - 6k - 7 \le 0$ and $k^2 - 4k - 6 > 0$

$\Rightarrow (k-7)(k+1) \le 0$ and $(k-2)^2 > 10$

$\Rightarrow -1 \le k \le 7$ ... (iii)

and $|k - 2| > \sqrt{10}$

i.e., $k > 2 + \sqrt{10}$ or $k < 2 - \sqrt{10}$

i.e., $k \ge 6$ or $k \le -2$ [$\because k \in I$] ... (iv)

$\therefore$ From (ii), (iii) and (iv), we have $k = 6, 7$

$\therefore (\log_2 x)^2 = 6k + 7$ gives $\log_2 x = \sqrt{6k+7} = \sqrt{43}, 7$

$\Rightarrow x = 2^{\sqrt{43}}, 2^7 = 2^{\sqrt{43}}, 128$

15. (a, b, c) : Let $\theta + \alpha = x, \theta + \beta = y$ $\therefore \alpha - \beta = x - y$

Now, $4\sin(\theta + \alpha)\sin(\theta + \beta)\cos(\alpha - \beta)$

$= 2(2\sin x \sin y)\cos(x - y)$

$= 2\{\cos(x - y) - \cos(x + y)\}\cos(x - y)$

$= 2\cos^2(x - y) - 2(\cos^2 x - \sin^2 y)$

$= 1 + \cos 2(x - y) - 2(1 - \sin^2 x) + 2\sin^2 y$

$= -1 + 2\sin^2(\theta + \alpha) + 2\sin^2(\theta + \beta) + \cos 2(\alpha - \beta)$

$\therefore$ $2\sin^2(\theta + \alpha) + 2\sin^2(\theta + \beta) + \cos 2(\alpha - \beta) - 4\sin(\theta + \alpha)\sin(\theta + \beta)\cos(\alpha - \beta) = 1$

which is independent of θ, α and β.

21.

	Column I		Column II
(A)	If $f(x) = \sin^2 x + \sin^2\left(x+\frac{\pi}{3}\right) + \cos x \cos\left(x+\frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right)=1$, then $gof\left(\frac{\pi}{8}\right)=$	(p)	3
(B)	Let $f_n(\theta) = \tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)(1+\sec 4\theta)\ldots(1+\sec 2^n\theta)$, then $f_2\left(\frac{\pi}{16}\right)\cdot f_3\left(\frac{\pi}{32}\right)=$	(q)	$-\sqrt{\frac{3}{2}}$
(C)	If $\cot(\theta-\alpha)$, $3\cot\theta$, $\cot(\theta+\alpha)$ are in A.P. and θ is not an integral multiple of $\pi/2$, then $\sin\theta \operatorname{cosec}\alpha$ is equal to	(r)	1
(D)	Number of ordered pairs (a, x) satisfying the equation $\sec^2(a+2)x + a^2 - 1 = 0$, $-\pi < x < \pi$ is	(s)	$\sqrt{\frac{3}{2}}$

22.

	Column I		Column II
(A)	If $\vec{a}, \vec{b}, \vec{c}$ are non coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is	(p)	$\frac{\pi}{3}$
(B)	Four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ such that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=0$. Let P_1 and P_2 be planes determined by the pairs of vectors $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ respectively, then the angles between P_1 and P_2 is	(q)	$\frac{3\pi}{4}$
(C)	If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are perpendicular to each other then the angles between $\vec{a}$ and $\vec{b}$ is	(r)	$\frac{\pi}{6}$
(D)	If $\lvert\vec{a}\rvert=3, \lvert\vec{b}\rvert=5, \lvert\vec{c}\rvert=7$ and $\vec{a}+\vec{b}+\vec{c}=0$. Then angle between $\vec{a}$ and $\vec{b}$ is	(s)	0

ANSWERS KEYS

Paper I

1. (d) 2. (a) 3. (b) 4. (a) 5. (a) 6. (d) 7. (a) 8. (a) 9. (a) 10. (c) 11. (d) 12. (c) 13. (a) 14. (d) 15. (c) 16. (b) 17. (b,c) 18. (d) 19. (a)

20. (A → r); (B → p); (C → q), (D → s)

21. (A → q); (B → r); (C → s), (D → p)

22. (A → s); (B → p); (C → q), (D → r)

Paper II

1. (a) 2. (a) 3. (a) 4. (d) 5. (d) 6. (c) 7. (b) 8. (d) 9. (c) 10. (c) 11. (a) 12. (a) 13. (d) 14. (d) 15. (d) 16. (a) 17. (a) 18. (b,c) 19. (c)

20. (A → s); (B → r); (C → q), (D → p)

21. (A → r); (B → r); (C → q,s), (D → p)

22. (A → q); (B → s); (C → p), (D → p)

MATHS MUSING Contd. from page no. 6

COMPREHENSION

Let $y = f(x)$ such that $xy = x + y + 1$, $x \in R - \{1\}$ and $g(x) = x f(x)$

7. The minimum value of $g(x) =$

(a) $3-\sqrt{2}$ (b) $3+\sqrt{2}$ (c) $3-2\sqrt{2}$ (d) $3+2\sqrt{2}$

8. The area bounded by the curve $y = g(x)$ and the x-axis is

(a) $3/2 + \ln 4$ (b) $3/2 - \ln 4$

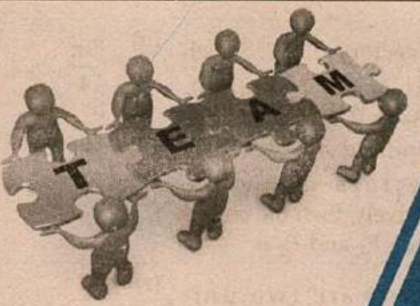
(c) $1/2 + \ln 4$ (d) $\ln 4 - 1$

INTEGER MATCH

9. In a right angled triangle with integer sides the smallest side is of length 2003. The number of prime factors of its area is

10. $A_1 A_2 A_3 \ldots A_{10}$ is a regular decagon. Let n be the number of quadrilaterals using then vertices of the decagon such that no side of the decagon is a side of the quadrilateral. The sum of the digits of n is

See solution set of Math Musing 124 on page no. 83



IMO Problems From TEAM SELECTION TESTS

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PROBLEMS

1. Let O be the centre of the circle inscribed in a triangle ABC, A_0, B_0, C_0 the points of tangency of this circle with the sides BC, CA, AB, respectively. Taken on the rays OA_0, OB_0, OC_0 are points L, M, K, respectively, equidistant from the point O. (a) Prove that the lines AL, BM, and CK meet in the same point. (b) Let A_1, B_1, C_1 be the projections of A, B, C, respectively, on an arbitrary line l passing through O. Prove that the lines A_1L, B_1M and C_1K are concurrent (that is, intersect at a common point).

2. Show that
$$\left(\sqrt[5]{\frac{1}{5}}+\sqrt[5]{\frac{4}{5}}\right)^{\frac{1}{2}} = (1+\sqrt[5]{2}+\sqrt{8})^{\frac{1}{5}}$$
$$= \sqrt[5]{\frac{16}{125}}+\sqrt[5]{\frac{8}{125}}+\sqrt[5]{\frac{2}{125}}+\sqrt[4]{\frac{1}{125}}.$$

3. Prove the following identities:
(a) $\sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{8\pi}{7}} = \sqrt[3]{\frac{1}{2}(5-3\sqrt[3]{7})}$;
(b) $\sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}} = \sqrt[3]{\frac{1}{2}(3\sqrt[3]{9}-6)}$.

4. Let $a_i > 0, b_i > 0\ (i = 1, 2, \ldots, n)$. Prove that
$$\sqrt[n]{(a_1+b_1)(a_2+b_2)\ldots(a_n+b_n)} \ge \sqrt[n]{a_1a_2\ldots a_n}+\sqrt[n]{b_1b_2\ldots b_n}$$

5. Show that there exists a convex hexagon in the plane such that:
(i) all its interior anlges are equal,
(ii) its sides are 1, 2, 3, 4, 5, 6 in some order.

6. Prove that the polynomial
$f(x)=x^4+26x^3+52x^2+78x+1989$
cannot be expressed as product $f(x) = p(x)\,q(x)$ where $p(x)$, $q(x)$ are both polynomials with integral coefficients and with degree not more than 3.

7. Given are five points inside an equilateral triangle of area 1. Prove that there exist three equilateral triangles such that they are homothetic to the given triangle, each of the five points is inside atleast one of these three triangles, and their combined area is atmost 0.64.

8. A tetrahedron is such that a sphere centred at a point O is tangent to all six of its sides. Moreover, four spheres with the vertices as their respective centres are externally tangent to one another, and all are tangent to another sphere with centre O. Prove that the tetrahedron is regular.

9. Given a pair $(\bar{r}, \bar{s})$ of vectors in the plane, a move consists of choosing a nonzero integer k and then changing $(\bar{r}, \bar{s})$ to either (i) $(\bar{r}+2k\bar{s}, \bar{s})$ or (ii) $(\bar{r}, \bar{s}+2k\bar{r})$. A game consists of applying a finite sequence of moves, alternating between moves of types (i) and (ii), to some initial pair of vectors.
(a) Is it possible to obtain the pair ((1, 0), (2, 1)) during a game with initial pair ((1, 0), (0, 1)), if the first move is of type (i)?
(b) Find all pairs $((a, b), (c, d))$ that can be obtained during a game with initial pair ((1, 0), (0, 1)), where the first move can be of either type.

10. We are given $n \ge 4$ points in the plane such that the distance between any two of them is an integer. Prove that atleast $\frac{1}{6}$ of these distances are divisible by 3.

SOLUTION

1. (a) Let the straight line BM intersect AC at a point B', and the line CK intersect AB at a point C'. Through M, we draw a straight line parallel to AC and denote by P and Q the points of its intersection with AB and BC, respectively. Obviously, $\frac{|AB'|}{|B'C|} = \frac{|PM|}{|MQ|}$. Drawing

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through K a straight line parallel to AB and denoting by E and F the points of its intersection with CA and CB, respectively, we have: $\frac{|BC'|}{|C'A|} = \frac{|FK|}{|KE|}$. We carry out a similar construction for the point L. Replacing the ratios entering the expression for R with the aid of that equality we take into account that for each line segment in the numerator there is an equal segment in the denominator, for instance: $|PM| = |KE|$.

(b) Let, for the sake of definiteness, the line l intersect the line segments C_0A, CA_0 and form an acute angle ϕ with OK. The straight line A_1L divides the line segment MK in the ratio $\frac{S_{LMA_1}}{S_{LKA_1}}$ (starting from the point M). The ratios in which the sides KL and LM of the triangle KLM are divided can be found in a similar way. We have to prove that there holds the equality $R = 1$. Let us replace the ratios of the line segments by the ratio of the areas of the corresponding triangles. Then R will contain S_{LMA_1} in the numerator and S_{KMC_1} in the denominator. $\frac{S_{LMA_1}}{S_{KMC_1}} = \frac{\sin C}{\sin A}$, where A and C are angles of the triangle ABC. Obviously $\frac{S_{B_0OA_0}}{S_{B_0OC_0}} = \frac{\sin C}{\sin A}$. In addition,

$$\angle A_1B_0A_0 = \angle C_0B_0A_0 + \angle A_1B_0C_0 = 90° - \frac{\angle B}{2} + \phi$$

(this follows from the fact that the circle of diameter AO passes through B_0, C_0 and A_1) and

$\angle B_0A_1O = \angle B_0AO = \frac{\angle A}{2}$. In similar way

$$\angle B_0C_1O = \frac{\angle C}{2} \text{ and } \angle C_1B_0C_0 = \left(90° - \frac{\angle B}{2}\right) + \angle C_1OL$$

$$= \left(90° - \frac{\angle B}{2}\right) + (180° - \angle C - \angle B_0OC_1)$$

$$= 90° - \frac{\angle B}{2} + (\angle B_0OA_1 - \angle C)$$

$$= 90° - \angle B/2 + (180° - \angle A - \angle C - \phi)$$

$$= 90° + \angle B/2 - \phi_1,$$

i.e., $\sin\angle A_1B_0A_0 = \sin\angle C_1B_0C_0$.

Thus,

$$\frac{S_{A_1B_0A_0}}{S_{C_1B_0C_0}} = \frac{|B_0A_1|\cdot|B_0A_0|}{|B_0C_1|\cdot|B_0C_0|} = \frac{\sin\frac{C}{2}.\cos\frac{C}{2}}{\sin\frac{A}{2}.\cos\frac{A}{2}} = \frac{\sin C}{\sin A}$$

Let r denote the radius of the inscribed circle $|OL| = |OK| = |OM| = a$. We have,

$$\frac{S_{LMA_1}}{S_{KMC_1}} = \frac{S_{LOM} + S_{LOMA_1}}{S_{KOM} + S_{KOMC_1}} = \frac{\frac{a^2}{r^2}S_{A_0OB_0} + \frac{a}{r}S_{A_0OB_0A_1}}{\frac{a^2}{r^2}S_{C_0OB_0} + \frac{a}{r}S_{C_0OB_0C_1}}$$

$$= \frac{\frac{a}{r}S_{A_0OB_0} + (S_{A_0B_0A_1} - S_{A_0OB_0})}{\frac{a}{r}S_{C_0OB_0} + (S_{C_0B_0C_1} - S_{C_0OB_0})}$$

$$= \frac{\left(\frac{a}{r} - 1\right)S_{A_0OB_0} + S_{A_0B_0A_1}}{\left(\frac{a}{r} - 1\right)S_{C_0OB_0} + S_{C_0B_0C_1}} = \frac{\sin C}{\sin A}.$$

(The latter of the equalities follows from the fact that $\frac{S_{A_0OB_0}}{S_{C_0OB_0}} = \frac{S_{A_0B_0A_1}}{S_{C_0B_0C_1}} = \frac{\sin C}{\sin A}$.) In similar way, we single out in the numerator and denominator of the expression for R, two more pairs of magnitudes whose ratios are equal to $\frac{\sin A}{\sin B}$ and $\frac{\sin B}{\sin C}$, respectively. Hence, $R = 1$. It remains only to prove that the number of points of intersection of the straight lines LA_1, KC_1 and MB_1 with the line segments KM, ML and LK, respectively, is odd.

2. Put $\sqrt[5]{2} = \alpha$ and prove the equality which can be rewritten in the following form

$$5(1 + \alpha + \alpha^3)^2 = (1 + \alpha^2)^5.$$

The right member is equal to

$$1 + 5\alpha^2 + 10\alpha^4 + 10\alpha^6 + 5\alpha^8 + \alpha^{10}$$
$$= 5(1 + \alpha^2 + 2\alpha^4 + 2\alpha^6 + \alpha^8)$$

since $\alpha^{10} = 4$.

Further, $\alpha^5 = 2, \alpha^8 = 2\alpha, \alpha^8 = 2\alpha^3$,

and, consequently,

$$(1 + \alpha^2)^5 = 5(1 + \alpha^2 + 2\alpha^4 + 4\alpha + 2\alpha^3).$$

It only remains to prove that

$$(1 + \alpha + \alpha^3)^2 = 1 + 4\alpha + \alpha^2 + 2\alpha^3 + 2\alpha^4.$$

The last equality is readily proved by removing the brackets in the left member and performing simple transformations. To prove the second equality we have to show that

$$\sqrt[5]{\frac{1}{5}} + \sqrt[5]{\frac{4}{5}} = \left(\sqrt[5]{\frac{16}{125}} + \sqrt[5]{\frac{8}{125}} + \sqrt[5]{\frac{2}{125}} - \sqrt[5]{\frac{1}{125}}\right)^2,$$

or $5(1 + \sqrt[5]{4}) = (\sqrt[5]{16} + \sqrt[5]{8} + \sqrt[5]{2} - 1)^2$.

Put $\sqrt[5]{2} = \alpha, \alpha^5 = 2, \alpha^6 = 2\alpha, \alpha^7 = 2\alpha^2, \alpha^8 = 2\alpha^3$.

Then we have to prove that

$$(\alpha^4 + \alpha^3 + \alpha - 1)^2 = 5(1 + \alpha^2).$$

Expanding the left member, we find

$$1+\alpha^2+\alpha^6+\alpha^8+2\alpha^7+2\alpha^5-2\alpha^4+2\alpha^4-2\alpha^3-2\alpha=0$$

Making use of the equalities enabling us to replace high powers of α by lower ones, we find the required identity.

3. The roots of the equation $x^7=1$ are

$$\cos\frac{2k\pi}{7}+i\sin\frac{2k\pi}{7}\ (k=0,1,2,...,6)$$

Therefore, the roots of the equation

$$x^6+x^5+x^4+x^3+x^2+x+1=0 \quad (*)$$

will be

$$x_k=\cos\frac{2k\pi}{7}+i\sin\frac{2k\pi}{7}\ (k=1,2,3,4,5,6).$$

Put $x+\frac{1}{x}=y$, then

$$x^2+\frac{1}{x^2}=y^2-2, x^3+\frac{1}{x^3}=y^3-3y$$

Equation (*) may be rewritten in the following way

$$\left(x^3+\frac{1}{x^3}\right)+\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1=0.$$

It is evident that

$$x_1=\bar{x}_6, x_2=\bar{x}_5, x_3=\bar{x}_4, x_k+\frac{1}{x_k}=x_k+\bar{x}_k=2\cos\frac{2k\pi}{7}$$

Hence, we may conclude that the quantities

$$2\cos\frac{2\pi}{7}, 2\cos\frac{4\pi}{7}, 2\cos\frac{8\pi}{7}$$

are the roots of the following equation

$$y^3+y^2-2y-1=0.$$

Let us set up an equation with the following roots

$$\sqrt[3]{2\cos\frac{2\pi}{7}}, \sqrt[3]{2\cos\frac{4\pi}{7}}, \sqrt[3]{2\cos\frac{8\pi}{7}}.$$

Let the roots of a certain cubic equation $x^3-ax^2+bx-c=0$ be α, β, γ.

We then have

$$\alpha+\beta+\gamma=a, \alpha\beta+\alpha\gamma+\beta\gamma=b, \alpha\beta\gamma=c.$$

Let the equation, whose roots are the quantities $\sqrt[3]{\alpha}, \sqrt[3]{\beta}, \sqrt[3]{\gamma}$ be $x^3-Ax^2+Bx-C=0$

Then $\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}=A$.

$\sqrt[3]{\alpha}\sqrt[3]{\beta}+\sqrt[3]{\alpha}\sqrt[3]{\gamma}+\sqrt[3]{\beta}\sqrt[3]{\gamma}=B, \sqrt[3]{\alpha\beta\gamma}=C$

Let us make use of the following identity

$$(m+p+q)^3=m^3+p^3+q^3+3(m+p+q)(mp+mq+pq)-3mpq$$

Putting here instead of m, p and q first $\sqrt[3]{\alpha}, \sqrt[3]{\beta}, \sqrt[3]{\gamma}$ and then $\sqrt[3]{\alpha\beta}, \sqrt[3]{\alpha\gamma}, \sqrt[3]{\beta\gamma}$, we find

$$A^3=a+3AB-3C, B^3=b+3BCA-3C^2.$$

In our case we have $a=-1, b=-2, c=1, C=1$.

Hence, $A^3=3AB-4, B^3=3AB-5$.

Multiplying these equations and putting $AB=z$, we find

$$z^3-9z^2+27z-20=0, (z-3)^3+7=0, z=3-\sqrt[3]{7}.$$

But $A^3=3z-4=5-3\sqrt[3]{7}, A=\sqrt[3]{5-3\sqrt[3]{7}}$

Therefore, indeed,

$$\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}=\sqrt[3]{2\cos\frac{2\pi}{7}}+\sqrt[3]{2\cos\frac{4\pi}{7}}+\sqrt[3]{2\cos\frac{8\pi}{7}}=\sqrt[3]{5-3\sqrt[3]{7}}$$

The second identity is proved in the same way.

4. Put $\frac{b_i}{a_i}=x_i\ (i=1,2,...,n)$. Then we have to prove the inequality

$$\sqrt[n]{(1+x_1)(1+x_2).....(1+x_n)}\ge 1+\sqrt[n]{x_1x_2....x_n}.$$

The theorem is valid at $n=1, 2, 3$. Suppose it is true at $n=m$ and let us prove that it also holds at $n=2m$.

We have

$$\sqrt[2m]{(1+x_1)(1+x_2)...(1+x_{2m-1})(1+x_{2m})}$$
$$=\sqrt[m]{\sqrt{(1+x_1)(1+x_2)}\cdot\sqrt{(1+x_3)(1+x_4)}......}$$
$$\sqrt[m]{\sqrt{(1+x_{2m-1})(1+x_{2m})}}$$
$$\ge\sqrt[m]{(1+\sqrt{x_1x_2})(1+\sqrt{x_3x_4})....(1+\sqrt{x_{2m-1}x_{2m}})}$$
$$\ge 1+\sqrt[m]{\sqrt{x_1x_2}\sqrt{x_3x_4}......\sqrt{x_{2m-1}x_{2m}}}=1+\sqrt[2m]{x_1x_2....x_{2m}}$$

Thus, the theorem is valid for all indices equal to any power of two. Let us now prove that it is true for any whole n. Let $n+q=2^m$. Then

$$\sqrt[n+q]{(1+x_1)(1+x_2)....(1+x_n)(1+y_1)(1+y_2)...(1+y_q)}$$
$$\ge 1+\sqrt[n+q]{x_1x_2....x_ny_1y_2....y_q}$$

Put $1+y_1=1+y_2=.....=1+y_q$
$$=\sqrt[n]{(1+x_1)(1+x_2)....(1+x_n)}=Y$$

We have

$$\sqrt[n+q]{(1+x_1)(1+x_2)...(1+x_n)\cdot Y^q}\ge 1+\sqrt[n+q]{x_1x_2...x_n(Y-1)^q}$$

But $(1+x_1)(1+x_2).....(1+x_n)=Y^n$

Therefore, $\sqrt[n+q]{Y^nY^q}\ge 1+\sqrt[n+q]{x_1....x_n(Y-1)^q}$.

i.e., $Y\ge 1+\sqrt[n+q]{x_1x_2...x_n(Y-1)^q}$

or $(Y-1)^{n+q}\ge x_1x_2....x_n(Y-1)^q$.

Hence, $(Y-1)^n\ge x_1x_2....x_n$, $Y-1\ge\sqrt[n]{x_1x_2....x_n}$.

Finally

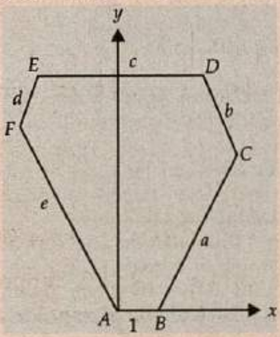
$$Y=\sqrt[n]{(1+x_1)(1+x_2).....(1+x_n)} \geq +\sqrt[n]{x_1x_2....x_n},$$

and the theorem is proved.

The equality sign is possible only if $x_1 = x_2 = ... = x_n = 1$.

5. **1st solution**

Let $ABCDEF$ be an equilangular hexagon with side-lengths as 1, 2, 3, 4, 5, 6 in some order. We may assume without loss of generality that $AB = 1$. Let $BC = a$, $CD = b$, $DE = c$, $EF = d$, $FA = e$.



Since the sum of all the angles of a hexagon is equal to $(6 - 2) \times 180° = 720°$ it follows that each (interior) angle must be equal to $720°/6 = 120°$. Let us take A as the origin, the positive x-axis along AB and the perpendicular at A to AB as the y-axis, as shown in the figure. We use the vector method. If the vector is denoted by (x, y), we then have

$\overline{AB} = (1, 0)$, $\overrightarrow{BC} = (a\cos 60°, a\sin 60°)$,

$\overline{CD} = (b\cos 120°, b\sin 120°)$,

$\overline{DE} = (c\cos 180°, c\sin 180°) = (-c, 0)$.

$\overline{EF} = (d\cos 240°, d\sin 240°)$,

$\overline{FA} = (e\cos 300°, e\sin 300°)$.

This is because these vectors are inclined to the positive x-axis at angles 0, 60°, 120°, 180°, 240°, 300° respectively.

Since the sum of all these 6 vectors is $\vec{0}$, it follows that

$$1+\frac{a}{2}-\frac{b}{2}-c-\frac{d}{2}+\frac{e}{2}=0 \text{ and } (a+b-d-e)\frac{\sqrt{3}}{2}=0.$$

That is

$$a - b - 2c - d + e + 2 = 0 \qquad ... (1)$$

and

$$a + b - d - e = 0 \qquad ... (2)$$

Since $\{a, b, c, d, e\} = \{2, 3, 4, 5, 6\}$, in view of (2) we have

(i) $\{a, b\} = \{2, 5\}$; $\{d, e\} = \{3, 4\}$; $c = 6$,

(ii) $\{a, b\} = \{3, 6\}$; $\{d, e\} = \{4, 5\}$; $c = 2$,

(iii) $\{a, b\} = \{2, 6\}$; $\{d, e\} = \{3, 5\}$; $c = 4$.

[The possibility that $\{a, b\} = \{3, 4\}$; $\{c, d\} = \{2, 5\}$ in (i), for instance, need not be considered separately, because we can reflect the figure about $x = \frac{1}{2}$ and interchange these two sets.]

Case-I: Here $(a - b) - (d - e) = 2c - 2 = 10$.

Since $a = b = \pm 3$, $d - e = \pm 1$, this is not possible.

Case-II: Here $(a - b) - (d - e) = 2c - 2 = 2$.

This is satisfied by $(a, b, d, e) = (6, 3, 5, 4)$.

Case-III: Here $(a - b) - (d - c) = 2c - 2 = 6$.

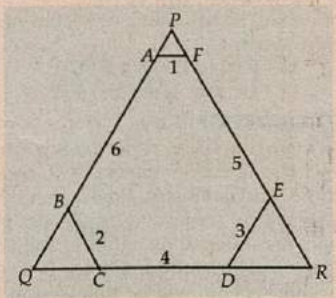
This is satisfied by $(a, b, d, e) = (6, 2, 3, 5)$

Hence we have (essentially) two different solutions:

(1, 6, 3, 2, 5, 4) and (1, 6, 2, 4, 3, 5).

It may be verified that (1) and (2) are both satisfied by these sets of values.

2nd solution:



Consider an equilateral trianlge, of side 9 units. Remove from the three corners equilateral triangles of sides 1 unit, 2 units and 3 units respectively. The remaining portion is now an equiangular hexagon $ABCDEF$ with sides 1, 6, 2, 4, 3, 5 as required.

6. Assume, if possible,

$$f(x) = (x + a)(x^3 + ax^2 + bx + c).$$

Comparing the coefficients of like powers of x, we get

$$a + b = 26, \quad ab + c = 52,$$
$$ac + d = 78, \quad ad = 1989$$

But $1989 = 3^2 \cdot 13 \cdot 17$. Thus 13 divides ad and hence 13 divides a or d but not both. If 13 divides a then 13 divides $d = 78 - ac$ which is not possible. Suppose 13 divides d. Then 13 divides ac. But since 13 does not divide a, 13 divides c which implies 13 divides $ab = 52 - c$ and so b is divisible by 13 which in turn implies 13 divides $a = 26 - b$, a contradiction. Therefore $f(x)$ has no linear factors.

If $f(x) = (x^2 + ax + b)(x^2 + cx + d)$, then again,

$$a = c = 26, \quad b + ac + d = 52,$$
$$ad + bc = 78, \quad bd = 1989.$$

Since $1989 = 3^2 \cdot 13 \cdot 17$, 13 divides bd. This implies that 13 divides b or d but not both. If 13 divides b, then 13 divides ad $(= 78 - bc)$ and hence 13 divides a. But then 13 divides d $(= 52 - b - ac)$ a contradiction. Similar argument shows that 13 divides d is also not possible.

Contd. on page no. 90

Engineering Aspirants 2013 – Are you Prepared for JEE Advanced?

(Here are the few Tips to get your optimum)

The JEE Advanced (formerly known as IIT-JEE) is an annual entrance examination to get Admission in IITs. It is also one of the toughest engineering entrance exams in the world. Only 1.5 lac students will be short listed from JEE Main 2013 to appear for the JEE Advanced 2013 on June 2nd, 2013. A serious aspirant ideally must have completed the syllabus by now.

Schedule of JEE (Advanced), 2013

The examination will be held on Sunday, June 02, 2013 as per the schedule given below:

Paper 1	9:00 to 12:00 hrs. (IST)
Paper 2	14:00 to 17:00 hrs. (IST)

EXAMINATION PATTERN:

There will be two question papers, each of three hours duration. Both the question papers will consist of three separate sections on Chemistry, Physics and Mathematics. Questions will be of objective type, designed to test comprehension, reasoning and analytical ability of Students. All the questions will be Multiple Choice Type (MCQ)

Negative marking scheme will be followed in the checking of examinations.

A Student can opt for question paper in any of the language viz. English or Hindi.

SYLLABUS COVERAGE:

JEE Syllabus of Class XI & XII contributes about 45% and 55% of IIT-JEE question-papers respectively. While preparing all the chapters of Physics, Chemistry and Mathematics, based on our past experience stress may be given in particular on the following topics:

Mathematics: Quadratic Equations & Expressions, Complex Numbers, Probability, Vectors & 3D Geometry, Matrices in Algebra; Circle, Parabola, Hyperbola in Coordinate Geometry; Functions, Limits, Continuity and Differentiability, Application of Derivatives, Definite Integral in Calculus.

Physics: Mechanics, Fluids, Heat & Thermodynamics, Waves and Sound, Capacitors & Electrostatics, Magnetics, Electromagnetic Induction, Optics and Modern Physics.

Chemistry: Qualitative Analysis, Coordination Chemistry & Chemical Bonding in Inorganic Chemistry, Electrochemistry, Thermodynamics, Chemical Equilibrium in Physical Chemistry and Organic Chemistry Complete as a topic.

TIPS FOR JEE Advanced, 2013:

MATHEMATICS (please see Tips on Physics in 'Physics For You' & on Chemistry in 'Chemistry Today')

1. Previous JEE papers suggest that more attention should be paid to topics like Vectors and 3-D than Probability or Indefinite integration as vectors and 3-D offers very less scope to examiner, as far as variety in problem is concerned. One more chapter is Complex Number, every year 2-3 problems (of purely complex no.) are asked. Hence mastering complex numbers, vectors, 3-D and Definite integral must be on top priority.

2. Algebra can be made easier if you have the ability to picture functions as graphs and are good at applying vertical and horizontal origin shifts carefully as zeroes of functions and other specific values can be done in much less time using these techniques.

3. Differential calculus again relates well to roots of equations, especially if you use the Rolle's and Lagrange's theorems.

4. Complex numbers can be used to solve questions in co- ordinate geometry too. Trigonometric questions require applications of De Moivre's theorem.

5. Permutation - Combination and Probability is another very important topic in algebra .You need to be thorough with the basics of Bayes theorem, dearrangements and various ways of distribution, taking care of cases where objects are identical and when they are not.

6. Matrices can be related to equations, hence a 3×3 matrix can actually be visualized as being three-planed in 3D geometry. Determinants have some very nice properties, for instance, the ability to break them into two using a common summand from a row/ column, which should be made use of in tougher questions.

7. Integral calculus can be simplified using tricks and keeping in mind some basic varieties of integrable functions. Remembering the properties and applying them wisely saves lot of time.

8. Coordinate geometry requires a good working knowledge of the parametric forms of various conic sections and an ability to convert the other, tougher ones to these basic forms and then interpret the solutions accordingly.

9. Finally the only one way to do well in Mathematics is to practice problems keeping in mind the pattern of questions in previous years' JEE papers.

Cracking the JEE (ADVANCED) 2013

*Stay focussed and maintain a positive attitude

*Develop speed. Refer to reputed mock-test series to build a winning exam temperament. Solve the past year's IIT-JEE papers. Focus on your weak areas and improve upon your concepts.

*Practise of JEE level questions is necessary as it improves your reasoning and analytical ability.

*Remember it is quality of time spent and not the quantity alone. Hence give short breaks of 5 to 10 minutes every 1-2 hours of serious study. Completely relax when you take a break. Practice meditation to develop inner calm, poise, confidence and power of concentration.

*Don't overstress yourself. Five to six hours of sleep every night is a must, especially three-four days before IIT-JEE to keep you physically and mentally fit. While short naps may help to regain freshness, avoid over-sleeping during the day.

*Finally, don't be nervous if you find the paper tough since it is the relative performance that counts. Put your best analytical mind to work, and believe in your preparation.

Authored by Ramesh Batlish, FIITJEE Expert

22. (d): Let (h, k) is the circumcentre of ΔPQR.

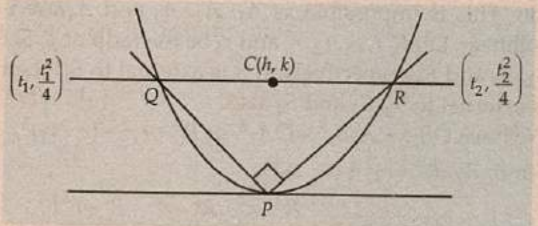
$$h = \frac{t_1 + t_2}{2} \Rightarrow t_1 + t_2 = 2h \quad \ldots(i)$$

$$k = \frac{t_1^2 + t_2^2}{8} \quad \ldots(ii)$$

$$m_{PR} \times m_{PQ} = -1$$

$$\frac{(t_2^2/4) - 0}{t_2} \times \frac{(t_1^2/4) - 0}{t_1} = -1$$

$$\Rightarrow t_1 t_2 = -16$$



Now $t_1^2 + t_2^2 = 8k \Rightarrow (t_1 + t_2)^2 - 2t_1 t_2 = 8k$

$\Rightarrow 4h^2 + 32 = 8k$

Locus of (h, k) is $x^2 = 2y - 8$

23. (b): Equation of directrix $y - 8 = -\frac{1}{4}$

$\Rightarrow 4y - 31 = 0$

24. (d): Circle drawn with diameter as a focal chord will always touch the directrix of the parabola which is $4y - 31 = 0$

$\Rightarrow a + b - c = 0 + 4 + 31 = 35$

25. (a): $P(A \cup B) = 1 - \frac{1}{12} = \frac{11}{12}$

$$\Rightarrow \frac{11}{12} = \frac{3}{4} + \frac{2}{3} - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = \frac{3}{4} + \frac{2}{3} - \frac{11}{12} = \frac{9 + 8 - 11}{12} = \frac{1}{2}$$

$$\therefore P(A \cap B) = \frac{1}{2} = P(A)P(B)$$

So A and B are independent

$$\Rightarrow \left(P\left(\frac{A}{B}\right)\right) = P\left(\frac{A}{\overline{B}}\right)$$

26. (b): $P(A \cap B) \geq \max.\{P(A), P(B)\}$

$$P(A \cup B) \geq \frac{1}{2} \Rightarrow 1 - P(\overline{A} \cap \overline{B}) \geq \frac{1}{2}$$

$$\Rightarrow P(\overline{A} \cap \overline{B}) \leq \frac{1}{2}$$

Obviously $P(\overline{A} \cap \overline{B}) \geq 0$

27. (b): $\frac{3}{4} \leq P(A \cup B) \leq 1$

$$\Rightarrow \frac{3}{4} \leq \frac{3}{4} + \frac{1}{3} - P(A \cap B) \leq 1$$

$$\Rightarrow -\frac{1}{3} \leq -P(A \cap B) \leq -\frac{1}{12}$$

$$\Rightarrow P(A) - \frac{1}{3} \leq P(A) - P(A \cap B) \leq P(A) - \frac{1}{12}$$

$$\Rightarrow \frac{1}{3} - \frac{1}{3} \leq P(A \cap \overline{B}) \leq \frac{1}{3} - \frac{1}{12}$$

$$\Rightarrow 0 \leq P(A \cap \overline{B}) \leq \frac{1}{4}$$

Contd. from page no. 27

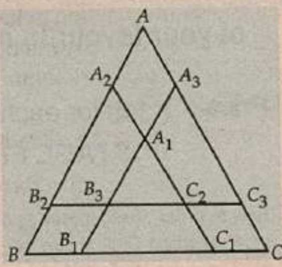
We conclude that $f(x)$ cannot be written as a product of two polynomials with integral coefficients, each of degree < 4.

7. Divide the triangle ABC into seven regions as shown in the diagram. The three lines are parallel to the sides of ABC and at the same distance from them, which is slightly greater than $\frac{1}{5}$ of the altitude of ABC. Suppose one of AB_1C_1, A_2BC_2 or A_3B_3C contains at least 3 of the 5 points. We have to add atmost 2 triangles of negligible area to cover the remaining points. The total area will be under 0.64. We may hence forth assume that this is not the case.

Suppose $AA_2A_1A_3$ does not contain any of the 5 points. By the Pigeonhole principle, either A_2BC_2 or A_3B_3C will contain atleast 3 of them. Hence we may assume that each of the corner regions contains atleast 1 point.

Similar reasoning shows that the central region must be empty, and the three edge regions contain atmost 1 point among them. If all 5 points are in the corner regions, we can draw lines through A_1, B_1 and C_1 parallel to BC, CA and AB respectively. These cut off three corner triangles which will contain the 5 points, with total area slightly greater than 0.48.

Suppose the bottom side region contains 1 point. Then the top corner region must contain 2 points. Let M be the midpoint of B_2C_3. Draw lines through A_1, M and A_1, parallel to BC, CA and AB respectively. The top triangle will cover the 2 points in the top corner region.



One of the two bottom triangles will cover the point in the bottom side region as well as one in a bottom corner region. A third triangle of negligible area will cover the remaining point. The total area of these three triangles is slightly greater than 0.52.

8. Denote by T the sphere tangent to all 6 sides of the tetrahedron $A_1A_2A_3A_4$, S the other sphere with centre O and S_i the sphere with centre A_i, $1 \le i \le 4$. Consider the cross-section along the plane $A_1A_2A_3$. Let B_1, B_2 and B_3 be the points of tangency of S_2 with S_3, S_3 with S_1 and S_1 with S_2, respectively.

Let C_1, C_2 and C_3 be the points of tangency of T with A_2A_3, A_3A_1 and A_1A_2 respectively. Then $A_1B_2 = A_1B_3$, $A_2B_3 = A_2B_1$, $A_3B_1 = A_3B_2$, $A_1C_2 = A_1C_3 = A_2C_3 = A_2C_1$ and $A_3C_1 = A_3C_2$. It follows that $B_i = C_i$ for $1 \le i \le 3$.

Suppose S_1 is inside S and S_2 is outside S. Then S_1 and S_2 must be tangent to each other as well as to S at the same point, and hence to S_3 and S_4 at that point too. This is impossible as A_1, A_2, A_3 and A_4 are not collinear. Let R, r, r_1, r_2, r_3 and r_4 be the radii of T, S, S_1, S_2, S_3 and S_4 respectively. If S is external to S_1, then it is external to S_2, S_3 and S_4 also.

We have $OB_3^2 + A_1B_3^2 = OA_1^2$ or $R^2 + r_1^2 = (r + r_1)^2$.

Similarly, $R^2 + r_2^2 = (r + r_2)^2$.

Hence, $A_1A_2 = r_1 + r_2 = \dfrac{R^2 - r^2}{2r} + \dfrac{R^2 - r^2}{2r} = \dfrac{R^2 - r^2}{r}$.

Similarly, $A_iA_j = \dfrac{R^2 - r^2}{r}$

for $1 \le i < j \le 4$. Hence $A_1A_2A_3A_4$ is a regular tetrahedron.

If S contains S_1, then it contains S_2, S_3 and S_4 also. We have $R^2 + r_i^2 = (r - r_i)^2$ for $1 \le i \le 4$. Hence $A_1A_2A_3A_4$ is a regular tetrahedron of side length $\dfrac{r^2 - R^2}{r}$.

9. Let $||\vec{z}||$ denote the length of vector $\vec{z}$, and let $|z|$ denote the absolute value of the real number z.

(a) Let $(\vec{r}, \vec{s})$ be the pair of vectors, where $\vec{r}$ and $\vec{s}$ change throughout the game. Observe that if $\vec{x}, \vec{y}$ are vectors such that $||\vec{x}|| < ||\vec{y}||$, then

$$||\vec{x} + 2k\vec{y}|| \ge ||2k\vec{y}|| - ||\vec{x}|| > 2||\vec{y}|| - ||\vec{y}|| = ||\vec{y}||$$

After the first move of type (i), we have $\vec{r} = (1, 2k)$ and $\vec{s} = (0,1)$ for some nonzero k so that $||\vec{r}|| > ||\vec{s}||$. Applying the above result with $\vec{x} = \vec{s}$ and $\vec{y} = \vec{r}$, we see that after the next move (of type (ii)), the magnitude of $\vec{r}$ does not change while that of $\vec{s}$ increases to over $||\vec{r}||$. Applying the above result again with $\vec{x} = \vec{r}$ and $\vec{y} = \vec{s}$, we see that after the next move (of type(i)), the magnitude of $\vec{s}$ remains the same while that of $\vec{r}$ increases to over $||\vec{s}||$. Continuing in this manner, we find that $||\vec{r}||$ and $||\vec{s}||$ never decrease as a result of a move. Because after the very first move, the first vector has magnitude greater than 1, we can never obtain $((1, 0), (2, 1))$.

(b) We modify the game slightly by not requiring that moves alternate between types (i) and (ii) and by allowing the choice $k = 0$. Of course, any pair that can be obtained under the original rules can be obtained

under these new rules as well. The converse is true as well: by repeatedly discarding any moves under the new rules with $k = 0$ and combining any adjacent moves of the same type into one move, we obtain a sequence of moves valid under the original rules that yields the same pair.

Let $((w, x), (y, z))$ represent the pair of vectors, where w, x, y, and z change throughout the game. It is easy to verify that the value of $wz - xy$, and the parity of x and y, are invariant under any move in the game. In a game that starts with $((w, x), (y, z)) = ((1, 0), (0, 1))$, we must always have $wz - xy = 1$ and $x \equiv y \equiv 0 \pmod 2$. Because x and y are always even, w and z remain constant modulo 4 as well; specifically, we must have $w \equiv z \equiv 1 \pmod 4$ throughout the game.

Call a pair $((a, b), (c, d))$ desirable when $ad - bc = 1$, $a \equiv d \equiv 1 \pmod 4$, and $b \equiv c \equiv 0 \pmod 2$. Above we showed that any pair obtainable during a game with initial pair $((1, 0), (0, 1))$ must be desirable; we now prove the converse. Assume, for the sake of contradiction, that there are desirable pairs $((a, b), (c, d))$ that are not obtainable; let $((e, f), (g, h))$ be such a pair for which $|ac|$ is minimal.

If $g=0$, then $eh=1+fg=1$; because $e \equiv h \equiv 1 \pmod 4$, $e=h=1$. If $f = 0$, the pair is clearly obtainable. Otherwise, by performing a move of types (i) with $k = \frac{f}{2}$, we can transform $((1, 0), (0, 1))$ into the pair $((e, f), (g, h))$, a contradiction.

Thus, $g \neq 0$. Because g is even and e is odd, either $|e| > |g|$ or $|g| > |e|$. In the former case, $e - 2k_0 g$ is in the interval $(-|e|, |e|)$ for some $k_0 \in \{1, -1\}$. Performing a type-(i) move on $((e, f), (g, h))$ with $k = -k_0$ yields another desirable pair $((e', f'), (g, h))$. Because $|e'| < |e|$ and $g \neq 0$, we have $|e'g| < |eg|$. Therefore, because $|ac| = |ag|$ is minimal among unobtainable desirable pairs, the new desirable pair $((e', f'), (g, h))$ can be obtained from $((1, 0), (0, 1))$ through some sequence of moves S. We can then obtain $((e, f), (g, h))$ from $((1, 0), (0, 1))$ as well, by first applying the moves in S to $((1, 0), (0, 1))$, then applying one additional move of type (i) $k = k_0$. Thus, our minimal pair is obtainable – a contradiction.

A similar proof holds if $|e| < |g|$, where we instead choose k_0 such that $g - 2k_0e \in (-|g|, |g|)$ and perform type-(ii) moves. Thus, in all cases, we get a contradiction. Therefore, we can conclude that every obtainable pair of vectors indeed desirable. This completes the proof.

10. In this solution, all congruences are taken modulo 3. We first show that if $n = 4$, then atleast two points are separated by a distance divisible by 3. Denote the points by A, B, C, D. We show that atleast one of the six distances AB, BC, CD, DA, AC, BD is divisible by 3. We approach indirectly by assuming that all these distances are not divisible by 3.

Without loss of generality, we assume that $\angle BAD = \angle BAC + \angle CAD$. Let $\angle BAC = x$ and $\angle CAD = y$. Also, let $\alpha = 2AB \cdot AC \cos x$, $\beta = 2AD \cdot AC \cos y$ and $\gamma = 2AB \cdot AD \cdot \cos(x + y)$. Applying the Law of Cosines in triangles ABC, ACD, ABD gives

$$BC^2 = AB^2 + AC^2 - \alpha,$$
$$CD^2 = AD^2 + AC^2 - \beta,$$
$$BD^2 = AB^2 + AD^2 - \gamma$$

Because the square of each distance is an integer congruent to 1, it follows from the above equations that α, β and γ are also integers congruent to 1. Also,

$$\begin{aligned} 2AC^2\gamma &= 4AC^2 \cdot AB \cdot AD \cos(x + y) \\ &= 4AC^2 \cdot AB \cdot AD (\cos x \cos y - \sin x \sin y) \\ &= \alpha\beta - 4AC^2 \cdot AB \cdot AD \cdot \sin x \sin y \end{aligned}$$

implying that $4AC^2 \cdot AB \cdot AD \cdot \sin x \sin y$ is an integer congruent to 2.

Thus, $\sin x \sin y = \sqrt{(1 - \cos^2 x)(1 - \cos^2 y)}$ is a rational number which, when written in lowest terms, has a numerator that is not divisible by 3.

Let $p = 2AB \cdot AC$ and $q = 2AD \cdot AC$, so that

$$\cos x = \frac{\alpha}{p} \text{ and } \cos y = \frac{\beta}{q}.$$

Because $\sin x \sin y = \dfrac{\sqrt{(p^2 - \alpha^2)(q^2 - \beta^2)}}{pq}$

is rational, the numerator on the right-hand side must be an integer. This numerator is divisible by 3 because $p^2 \equiv \alpha^2 \equiv 1$, but the denominator is not divisible by 3. Therefore, when $\sin x \sin y$ is written in lowest terms, its numerator is divisible by 3, a contraction. Therefore, our assumption was wrong and there is atleast one distance divisible by 3 for $n = 4$.

Now assume that $n \geq 4$. From the set of n given points, there exist $\binom{n}{4}$ four-element subsets $\{A, B, C, D\}$. Atleast two points in each subset are separated by a distance divisible by 3, and each such distance is counted in atmost $\binom{n-2}{2}$ subsets. Hence, atleast

$$\frac{\binom{n}{4}}{\binom{n-2}{2}} = \frac{\binom{n}{2}}{6}$$ distances are divisible by 3.